Asymptotic Bounds on Comparison Sorts

CSE 373 Winter 2020

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Announcements

- Remember : no late days for HW8 (Seam Carving)!

- You can always make appointments with staff (TAs or me) to discuss anything: homework, concepts, imposter syndrome, and more
  - Crucially, if you need health accommodations for the final please reach out!

Lecture Outline

- **Reading Review: Twiddling with Constants!**
- Asymptotic Analysis Practice
- Theoretical Lower Bound for Comparison Sorts
- Comparison Sorts Summary

Feedback from the Reading Quiz

- How were 47 and 67 decided upon?
- What is a run (in the sorting context)?
- If QuickSort looks for runs, doesn't that mean it's doing extra work?
- Why is the best possible sorting algorithm $\Omega(N)$?

Reference Types vs Primitive Types

- Java uses MergeSort* for reference types because MergeSort is stable

- Java uses adaptive QuickSort for primitive types because stability doesn't matter for these types, and QuickSort's constants are better than MergeSort's
  - However: InsertionSort's constants are better for small arrays
  - However: MergeSort's constants are better for partially-sorted arrays

* technically, this is a MergeSort variant known as TimSort

Java's QuickSort Adapts to its Input

- At the beginning of the sort (ie, only once), sort() checks whether the input is partially sorted by looking for *runs*
  - 9 7 1 2 4 8 6 3 5 (a run!)
  - This is Θ(N) work, so still dominated by our N log N runtimes
  - If there are long-enough runs, switches to MergeSort*. Done.
- If not, it picks two pivots, partitions the input, and recursively sorts each partition
  - When the partition is small enough, switches to InsertionSort. Done.
- Why is "small enough" defined as <47?
  - Performance testing on InsertionSort and QuickSort using randomized input; the "break point" happened at 47

* technically, this is a MergeSort variant known as TimSort

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Problem #1

- Consider the functions N! and $(N/2)^{N/2}$. Is $N! \in \Omega((N/2)^{N/2})$? Prove your answer.

$$N! = N * (N-1) * \ldots * (N/2 + 1) * N/2 * \ldots * 2 * 1$$
$$(N/2)^{N/2} = N/2 * N/2 * \ldots * N/2 * N/2$$

- $N! > (N/2)^{N/2}$ for large N, therefore $N! \in \Omega((N/2)^{N/2})$

- Demo: https://www.desmos.com/calculator/7lahriir6s

Problem #2

- Now, let's consider the functions log N! and N log N. Show N log N ∈ Ω(log N!)

```
log N!    = log(N   *     (N-1)  * … *     1)
          = log(N)  + log(N-1)   + … + log 1
N log N   = log(N)  + log(N)     + … + log(N)
```

- log N! < N log N for large N, therefore N log N ∈ Ω(log N!)
- Demo: https://www.desmos.com/calculator/4jeakr9vvb

Problem #3

- Is log N! ∈ Ω(N log N)? Prove your answer.

- From problem #1, we know $N! > (N/2)^{N/2}$ for large N
  - Taking the log of both sides: $\log N! > \log (N/2)^{N/2}$
  - log N! > N/2 log (N/2)
  - log N! > N/2 (log N – log 2)
    - … for large N

- Therefore, log N! ∈ Ω(N log N)

- Demo: https://www.desmos.com/calculator/4ptk1kcmss

Poll Everywhere

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- Given that N log N ∈ Ω(log N!) and log N! ∈ Ω(N log N), which of the following statements are true?

A. N log N ∈ Θ(log N!)
B. log N! ∈ Θ(N log N)
C. Both A and B
D. Neither A nor B
E. I'm not sure …

Intuitively:
N log N ≥ log N!
log N! ≥ N log N
∴ N log N = log N!

Formally:
- We've shown Ω(N log N) = Ω(log N!)
- we can use similar logic to prove O(N log N) = O(log N!)

∴ Both A & B are true

(note: you can also show Θ(N log N) = Θ(log N!))

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Comparison Sorts Review

	Best-Case Time	Worst-Case Time	Space	Stable?	Notes
SelectionSort	Θ(N²)	Θ(N²)	Θ(1)	No	
In-Place HeapSort	Θ(N)	Θ(N log N)	Θ(1)	No	Slow in practice
MergeSort	Θ(N log N)	Θ(N log N)	Θ(N)	Yes	Fastest stable sort
In-Place InsertionSort	Θ(N)	Θ(N²)	Θ(1)	Yes	Best for small or partially-sorted input
Naïve QuickSort	Θ(N log N)	Θ(N²)	Θ(N)	Yes	>=2x slower than MergeSort
Dual-Pivot QuickSort	Ω(N)	O(N²)	Θ(1)	No	Fastest comparison sort

Best Case != Worst Case

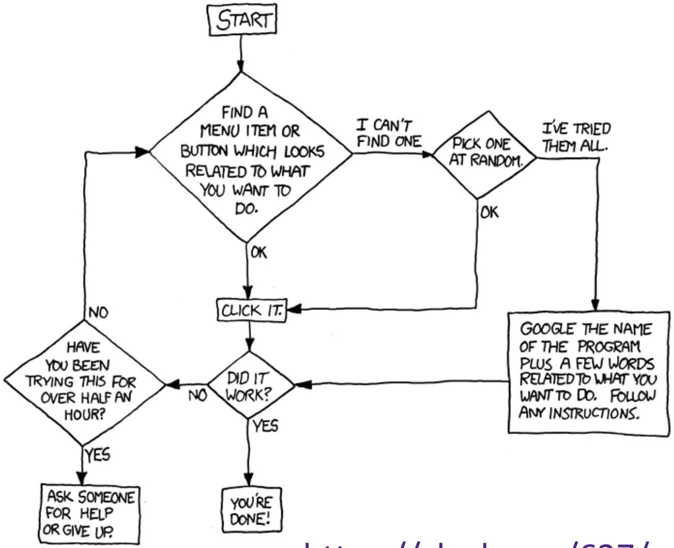
- Our best-cases are linear, but our worst-cases are N log N
- We spent all of last lecture *and* the reading twiddling with real-world constants to speed up N log N, but we didn't ask ourselves:

  *Does there exist a **comparison-based sorting** algorithm whose worst-case is asymptotically faster than N log N?*

- Let's ask that now. Call this theoretical algorithm "OptimalSort"
  - Next, we will describe the constraints on OptimalSort and then try to derive its worst-case runtime

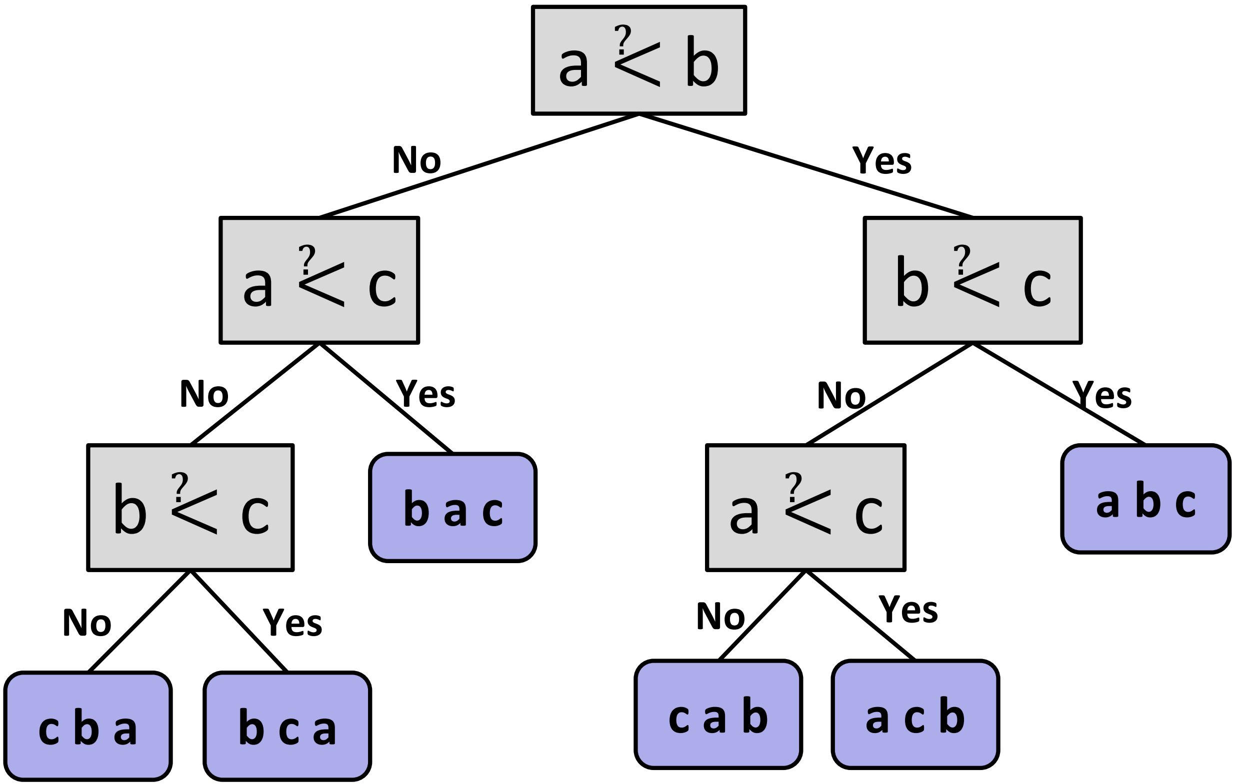
Modeling OptimalSort as a Decision Tree

- Given 3 items a, b, c, what is the minimum number of *comparisons* OptimalSort needs to order them?
- We don't know in what order OptimalSort would do the comparisons, but we can *model* those comparisons as a decision-tree



https://xkcd.com/627/

OptimalSort Decision Tree: N=3



Poll Everywhere

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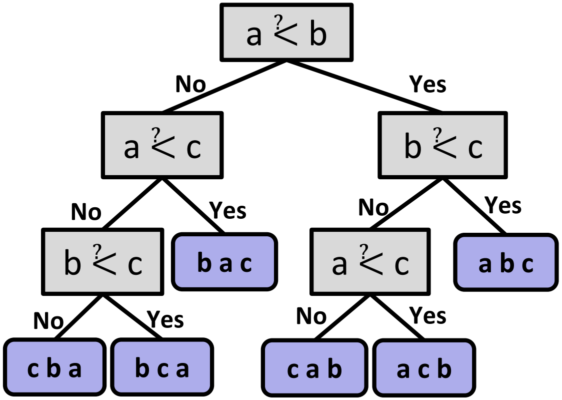
- How many possible *permutations* exist for a list of N=4 elements?

N!

A. 16
B. 24 = 4!
C. 32
D. 36
E. I'm not sure …



Poll Everywhere

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- In the worst case, how many *comparisons* would OptimalSort make for a list of N=4 elements? $\lceil \log_2 N! \rceil$

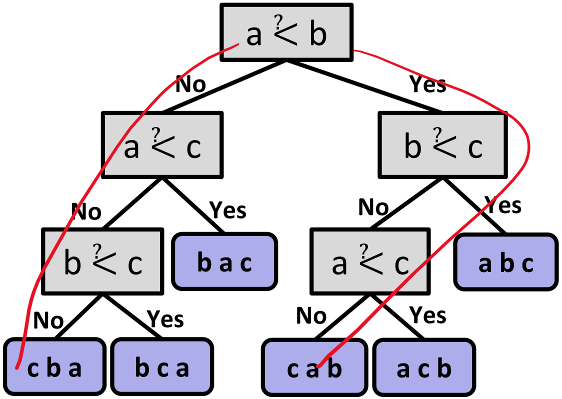
A. 3

B. 4

C. 5 $= \lceil \log_2 4! \rceil = \lceil 4.58 \rceil$

D. 6

E. I'm not sure …



OptimalSort for all N

- OptimalSort needs to decide between N! possible permutations (ie, N! leaves) in a list of N elements
- The height of OptimalSort's decision tree is $\log_2 N!$, rounded up
- Therefore, OptimalSort's worst-case requires Ω(log N!) comparisons
  - So its total runtime must also be Ω(log N!)
    - (because we still need to do swaps, merges, partitions, etc)
  - … which is equivalent to Ω(N log N)
  - … which means that OptimalSort's worst-case runtime is Ω(N log N)

Comparison Sorts Review

	Best-Case Time	Worst-Case Time	Space	Stable?	Notes
SelectionSort	$\Theta(N^2)$	$\Theta(N^2)$	$\Theta(1)$	No	
In-Place HeapSort	$\Theta(N)$	$\Theta(N \log N)$	$\Theta(1)$	No	Slow in practice
MergeSort	$\Theta(N \log N)$	$\Theta(N \log N)$	$\Theta(N)$	Yes	Fastest stable sort
In-Place InsertionSort	$\Theta(N)$	$\Theta(N^2)$	$\Theta(1)$	Yes	Best for small or partially-sorted input
Naïve QuickSort	$\Theta(N \log N)$	$\Theta(N^2)$	$\Theta(N)$	Yes	>=2x slower than MergeSort
Dual-Pivot QuickSort	$\Omega(N)$	$O(N^2)$	$\Theta(1)$	No	Fastest comparison sort

- HeapSort, MergeSort, and Dual-Pivot QuickSort are asymptotically optimal
  - Mathematically impossible to make *asymptotically fewer* comparisons
  - That's why we focus on optimizing their constants

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Why Did We Just Spend a Week on Sorting?

- Sorting is a great case study for algorithm-design techniques
  1. "Start Simple" / "Do the Straightforward Thing First"
     - SelectionSort
     - MergeSort
  2. Data structures can improve your algorithm
     - BFS vs Dijkstra's
     - HeapSort
  3. Pay attention to your asymptotes first; pay attention to your constants *afterwards*
     - Noticed that MergeSort "equals" HeapSort "equals" QuickSort
     - Spent an entire lecture optimizing QuickSort's constants (in-place, single-pass, $\log_3 N$ vs $\log_2 N$, switching to InsertionSort)
     - Realized that there's nothing "asymptotically better" than N log N
     - Asymptotes require analysis; constants require performance benchmarks
  4. Question your assumptions (see next lecture)