QuickSort

CSE 373 Winter 2020

Instructor: Hannah C. Tang

Teaching Assistants:

Aaron Johnston	Ethan Knutson	Nathan Lipiarski
Amanda Park	Farrell Fileas	Sam Long
Anish Velagapudi	Howard Xiao	Yifan Bai
Brian Chan	Jade Watkins	Yuma Tou
Elena Spasova	Lea Quan	

Poll Everywhere

pollev.com/uwcse373

- Approximately how long did HW7: HuskyMaps take?

A. 0-4 hours
B. 5-9 hours
C. 10-14 hours
D. 15-19 hours
E. 20-24 hours
F. 25-29 hours
G. 29+ hours
H. I'm not done / I don't want to say …

Announcements

- HW8 (Seam Carving) has been released!
  - 🚨 NOTE 🚨: We are NOT offering late days for this homework. If you think you'll need extra time, pretend it's due on Tuesday instead of Friday

- You can always make appointments with staff (TAs or me) to discuss anything: homework, concepts, imposter syndrome, and more

- Your health is more important than this class!
  - COVID-19 announcements/updates: https://uw.edu/coronavirus
  - Will adjust the in-class participation (PollEverywhere) policy so that you can remain at home for the rest of the quarter

Lecture Outline

- **Comparison Sorts Review**
- Partitioning
- QuickSort Intro
- Analyzing QuickSort's Runtime
- Avoiding QuickSort's Worst Case
- QuickSort in Practice

An (Oversimplified) Summary of Sorting Algorithms So Far

- **SelectionSort**: find the smallest item and put it in the front
- **HeapSort**: SelectionSort, but use a heap to find the smallest item
- **MergeSort**: Merge two sorted halves into one sorted whole

- **QuickSort**:
  - Much stranger core idea: *Partitioning*
  - Invented by Sir Tony Hoare in 1960, at the time a novice programmer
  - Interview: https://www.bl.uk/voices-of-science/interviewees/tony-hoare/audio/tony-hoare-inventing-quicksort
  - "I thought, that's a nice exercise: how would I program sorting?"

Lecture Outline

- Comparison Sorts Review
- **Partitioning**
- QuickSort Intro
- Analyzing QuickSort's Runtime
- Avoiding QuickSort's Worst Case
- QuickSort in Practice

Partitioning Definition

- **Partitioning** an array a[] on a **pivot** x=a[i] rearranges a[] so that:
  - x moves to position j (may be the same as i)
  - All entries to the left of x are <= x
  - All entries to the right of x are >= x
- Which of these are valid partitions?

		i				
5	550	10	4	10	9	330

A.

			j			
4	5	9	10	10	330	550

B.

			j			
5	4	9	10	10	550	330

C.

				j		
5	9	10	4	10	550	330

D.

		j				
5	9	10	4	10	550	330

Your Turn! Implement Partitioning

- Write pseudocode to implement the following:
  - Given an array of elements, rearrange the array so that all the less-than-0^{th}-value elements are to the left of the 0^{th} value and all greater-than-0^{th}-value elements are to the right

- Constraints:
  - Your algorithm must complete in O(N log N) time, but ideally Θ(N)
  - Your algorithm must use O(N) space, but ideally Θ(1)
  - You may use any data structure (eg, BSTs, stacks/deques/queues, etc)
    - Please don't copy the two solutions discussed in the reading: sort and copy-lessthan-then-copy-greaterthan
  - Relative order does NOT need to stay the same

Poll Everywhere

pollev.com/uwcse373

- Describe your implementation in a sentence or two
- Constraints:
  - Your algorithm must complete in O(N log N) time, but ideally Θ(N)
  - Your algorithm must use O(N) space, but ideally Θ(1)
  - You may use any data structure (eg, BSTs, stacks/deques/queues, etc)
    - Please don't copy the two solutions discussed in the reading: sort and copy-lessthan-then-copy-greaterthan
  - Relative order does NOT need to stay the same

Input:

6	8	3	1	2	7	4

Valid outputs:

3	1	2	4	6	8	7

3	4	2	1	6	7	8

Partitioning Implementations

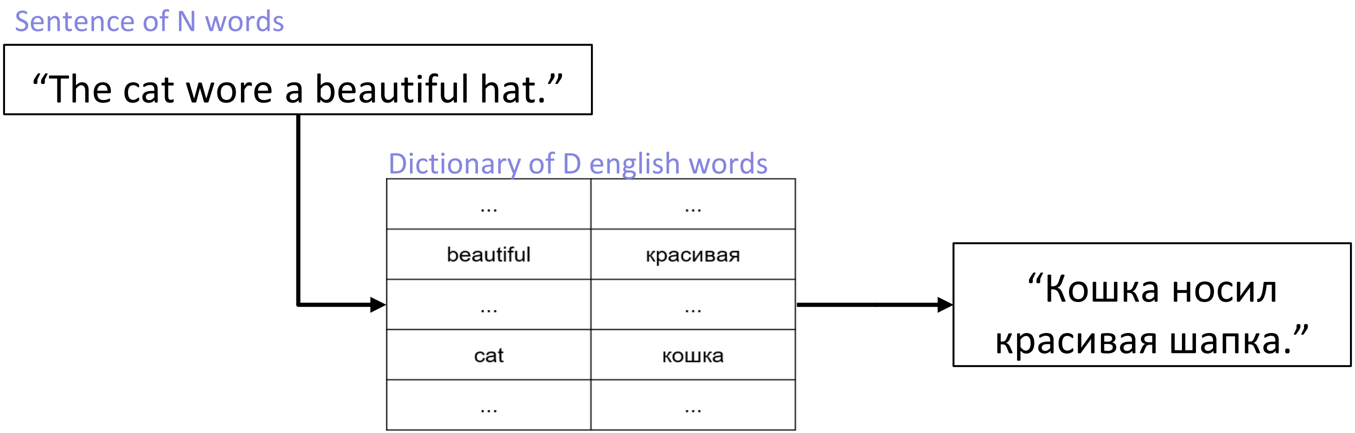
- Sort the elements (described in the reading)
  - Note: this implies that **partitioning reduces to sorting**
- Three-pass: copy “less than”s, then copy pivots, finally copy “greater-than”s
  - Described in reading
  - Demo: https://docs.google.com/presentation/d/16pOLboxhtJlaDxF7iRT5XcltDKmwab_wbvjZ4wPmJYk/edit

Lecture Outline

- Comparison Sorts Review
- Partitioning
- **QuickSort Intro**
- Analyzing QuickSort's Runtime
- Avoiding QuickSort's Worst Case
- QuickSort in Practice

Context for QuickSort's Invention

- In 1960, exchange student (!!) Tony Hoare worked on a translation program between Russian and English



- O(N log D) if we binary search the dictionary: not bad!
- … alas, the dictionary was stored on magnetic tape. Seeks require very slow physical movement of a tape head

Constants Matter (Sometimes)

- Iterating through an in-memory array != iterating through a magnetic tape

Question 5:

[3 pts] Let's map the latency of common computer operations to the human-scale operations required for studying for the 333 final. You may use the following:

A. Reading a sticky note on your monitor (0.5 secs)
B. Finding the right page/paragraph in the textbook kept next to your monitor (2 mins)
C. Asking on Piazza (36 mins)
D. Texting another 333 student for the answer (1 hour)
E. Requesting a scanned article from UW Libraries (2 days)
F. Buying the physical textbook *without Amazon Prime* (1 week)
G. Re-taking CSE 351 and then re-taking CSE 333 (20 weeks)
H. Buying the physical textbook *currently on Jupiter* (6 years)
I. Buying the physical textbook *currently in the Alpha Centauri system* (78,000 years)

Computer Operation	Human Analogue
L1 cache reference	A
Main memory reference	B
Packet round trip within same datacenter	F
Disk seek	G
Packet round trip across a submarine cable	H

Context for QuickSort's Invention

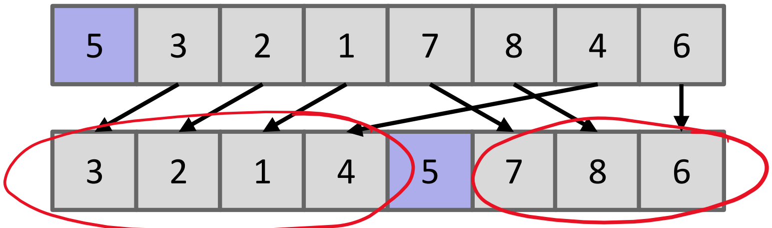
- O(N log D) if we binary search the dictionary: not bad!
  - … alas, the dictionary was stored on magnetic tape. Seeks require very slow physical movement of a tape head. Moving the head N times was too slow

- Better solution: sort the sentence and scan the dictionary (ie, the tape) in a single pass

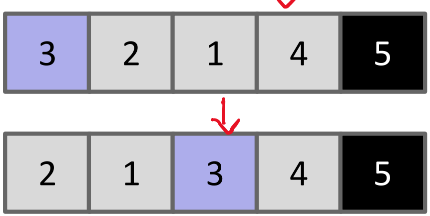
- Named the resultant algorithm "QuickSort", although "PartitionSort" may be clearer

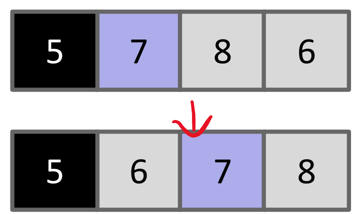
QuickSort is Partitioning

- After partitioning on 5:
  - 5 is in its "correct place" (ie, where it'd be if the array were sorted)



  - Can now sort two halves separately (eg, through recursive use of partitioning)





QuickSort is Partitioning

32	15	2	17	19	26	41	17	17

```
QuickSort(a[]):
  p = SelectPivot(a)
  a1, a2 = Partition(a, p)
  QuickSort(a1)
  QuickSort(a2)
```

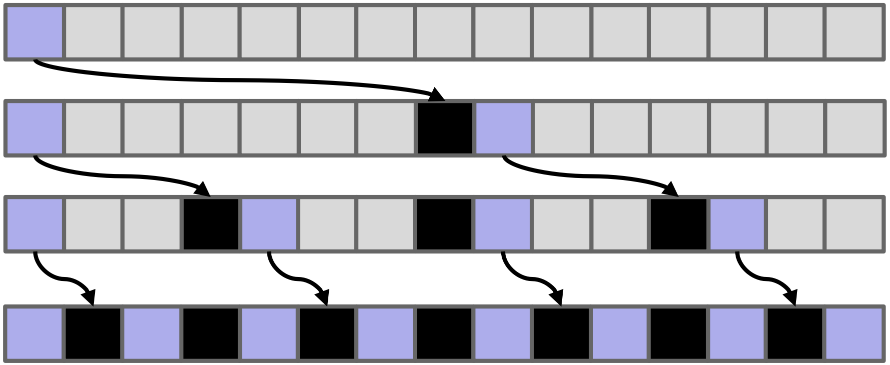
- For Naïve QuickSort:
  - **SelectPivot**() selects the 0th element
  - **Partition**() copies into a new array using three-pass method (see reading)

- Demo: https://docs.google.com/presentation/d/1QjAs-zx1i0_XWlLqsKtexb-iueao9jNLkN-gW9QxAD0/present?ueb=true&slide=id.g463de7561_042

Lecture Outline

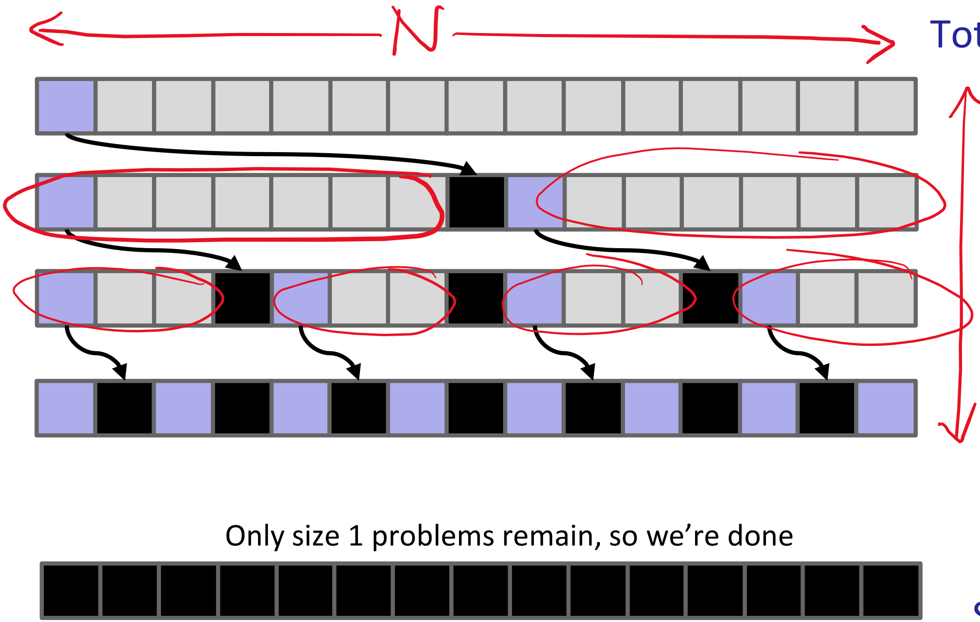
- Comparison Sorts Review
- Partitioning
- QuickSort Intro
- **Analyzing QuickSort's Runtime**
- Avoiding QuickSort's Worst Case
- QuickSort in Practice

Best Case: Pivot Always Lands in the Middle



Only size 1 problems remain, so we're done.

Best Case: Runtime



Total work at each level:

≈ N

≈N/2 + ≈N/2 = ≈N

≈N/4 * 4 = ≈N

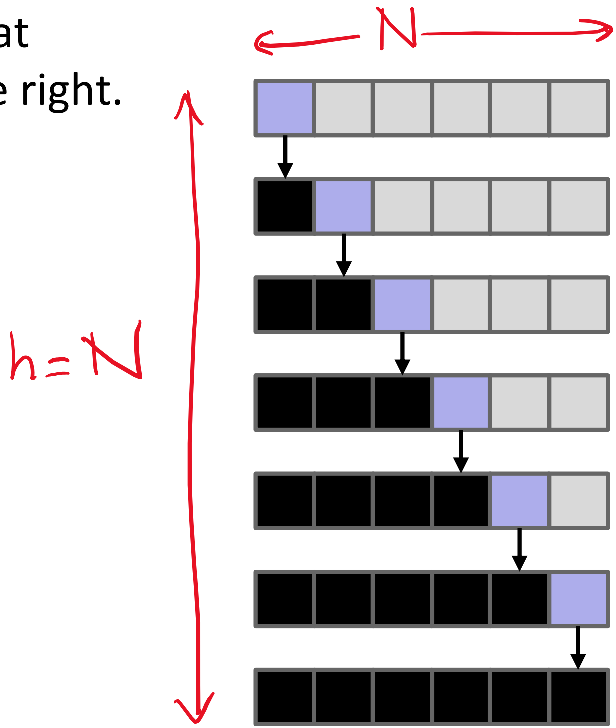
h = Θ(log N)

So best-case runtime is Θ(N log N)

Worst Case: Pivot Always Lands at Beginning

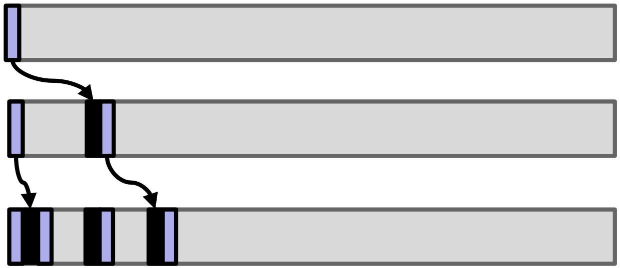
- Give an example of an array that would follow the pattern to the right.
  - 1 2 3 4 5 6

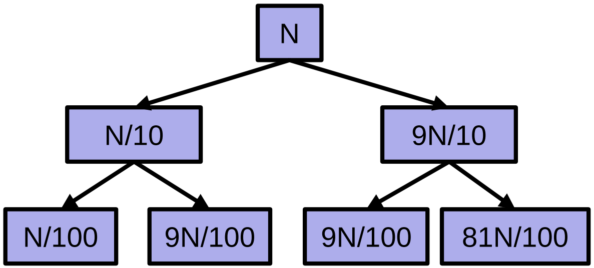
- What is the runtime Θ(·)?
  - N^2



Randomized Case

- Suppose pivot always ends up *at least 10% from either edge*





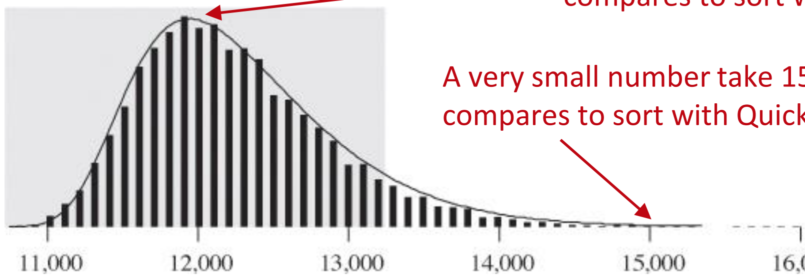
- Work at each level: O(N) and Runtime is O(NH)
  - H is approximately $\log_{10/9} N$ = O(log N)
- Randomized Case: O(N log N)
  - Even if you're unlucky enough to have a pivot that never lands anywhere near the middle but is at least 10% from one edge, runtime is still O(N log N)

QuickSort Runtime, Empiracally

- For N items:
  - Mean number of compares to complete Quicksort: ~2N ln N
  - Standard deviation: $\sqrt{(21 - 2\pi^2)/3}N \approx 0.6482776N$

Lots of arrays take 12,000ish compares to sort with Quicksort

A very small number take 15,000ish compares to sort with Quicksort.



Empirical histogram for quicksort compare counts (10,000 trials with N = 1000)

Chance of taking 1,000,000ish compares is effectively zero.

- For more, see: http://www.informit.com/articles/article.aspx?p=2017754&seqNum=7

QuickSort Performance

- Theoretical analysis:
  - Best case: $\Theta(N \log N)$
  - Worst case: $\Theta(N^2)$
  - **Randomized case:** $\Theta(N \log N)$ expected

- Compare this to Mergesort
  - Best case: $\Theta(N \log N)$
  - Worst case: $\Theta(N \log N)$

- Why is QuickSort empirically faster than MergeSort in the best and randomized cases?
  - No obvious reason why, just need to run experiments to show that constants are better

Lecture Outline

- Comparison Sorts Review
- Partitioning
- QuickSort Intro
- Analyzing QuickSort's Runtime
- **Avoiding QuickSort's Worst Case**
- QuickSort in Practice

Avoiding QuickSort's Worst Case

- If pivot lands "somewhere good", Quicksort is Θ(N log N)

- However, the very rare $\Theta(N^2)$ cases do happen in practice
  - **Bad ordering**: Array already in (almost-)sorted order
  - **Bad elements**: Array with all duplicates

- What can we do to avoid worst case behavior?

- Three philosophies:
  1. **Randomness**: pick a random pivot; shuffle before sorting
  2. **Smarter Pivot Selection**: calculate or approximate the median
  3. **Introspection**: switch to safer sort if recursion goes too deep

#1: Randomness

- Dealing with Bad Ordering:
  - Strategy 1: Pick pivots randomly
  - Strategy 2: Shuffle before you sort

- Dealing with Bad Elements (ie, duplicates):
  - 😭

#2a: Smarter Pivot Selection (Constant Time)

- Any algorithm for picking a pivot which requires constant time *and* determinism (ie, not random) has a corresponding family of dangerous inputs
  - "A Killer Adversary for QuickSort": https://www.cs.dartmouth.edu/~doug/mdmspe.pdf
- Dealing with Bad Elements (ie, duplicates):
  - 😭

#2b: Smarter Pivot Selection (Linear Time)

- Dealing With Bad Ordering:
  - We can calculate the actual median in linear time!
  - Worst-case is Θ(N log N), but constants make it slower than MergeSort 😭
  - Note: we can adapt QuickSort into QuickSelect
    - Selects the k-th element in Θ(N) time; we can use it to find the N/2 aka median element
- Dealing with Bad Elements (ie, duplicates):
  - 😭

#3: Introspection

- If recursion depth exceeds some threshold (eg, 10 log N), switch to MergeSort
  - Reasonable, but not common in practice

- Dealing With Bad Ordering:
  - ¯_(ツ)_/¯

- Dealing with Bad Elements (ie, duplicates):
  - ¯_(ツ)_/¯

Ultimately …

- As we saw with LLRB trees and B-trees, having a "100% guarantee" against worst-case input came with a cost
  - Here, our "100% guarantee" changed QuickSort's constants so that it became slower than MergeSort in the cases where it used to be faster: best-case and randomized-case

- Ultimately, most QuickSort implementations choose a few "reasonable protections" against pessimal input to maintain its performance against MergeSort in best-case and randomized-case
  - If you, the implementer, need a "100% guarantee" against worst-case input you should choose MergeSort instead. You should also recognize that you're paying for that guarantee with a slower runtime in most other cases

Lecture Outline

- Comparison Sorts Review
- Partitioning
- QuickSort Intro
- Analyzing QuickSort's Runtime
- Avoiding QuickSort's Worst Case
- **QuickSort in Practice**

Decisions When Implementing QuickSort

- How to select pivot?
  - Naïve QuickSort uses 0th element
  - Dual-pivot QuickSort uses 1/3rd-2/3rd

```
QuickSort(a[]):
  p = SelectPivot(a)
  a1, a2 = Partition(a, p)
  QuickSort(a1)
  QuickSort(a2)
```

- How to partition?
  - Naïve QuickSort uses a stable three-pass partition
  - Dual-pivot QuickSort uses three-way partition

Pivot Selection: Median-of-Three

- Median-of-Three *approximates* the true median in Θ(1) time
  - Pick 3 items and take the median of the sample

- Options for picking 3:
  - Randomly choose 3 indices
  - Pick first, middle, last
  - … ?

- "Good enough" for protecting against bad ordering
  - Intuitively: it's not-that-hard to one bad pivot, but it's pretty-hard to pick three bad pivots simultaneously

```
if (a < b)
  if      (b < c) return b;
  else if (a < c) return c;
  else            return a;
else
  if      (a < c) return a;
  else if (b < c) return c;
  else            return b;
```

Partitioning: Hoare Partitioning

- This is the original QuickSort partitioning algorithm
  - Good constants: single-pass and in-place
  - Yields an unstable sort

- Idea: initialize two pointers, L and R
  - L loves small items < pivot
  - R loves large items > pivot
  - Walk towards eachother, swapping anything they don't like

- Demo: https://docs.google.com/presentation/d/1DOnWS59PJOa-LaBfttPRseIpwLGefZkn450TMSSUiQY/pub?start=false&loop=false&delayms=3000&slide=id.g463de7561_042

Partitioning: Three-Way Partition

- Pick *two* pivots
  - Same intuition as median-of-three: it's hard to pick two bad pivots simultaneously

- Like Hoare Partitioning, use two pointers walking to the middle
  - But split array into three pieces, not two
  - Good constants: single-pass and in-place; $\log_3 N$ vs $\log_2 N$
  - Still results in an unstable sort

Case Study: Dual-Pivot QuickSort

- In 2009, Dual-Pivot QuickSort was introduced to the world by a previously-unknown guy in a Java developers' forum
  - Link: https://web.archive.org/web/20100428064017/http:/permalink.gmane.org/gmane.comp.java.openjdk.core-libs.devel/2628

- It is now the de-facto QuickSort implementation for many languages, including Java's Arrays.sort(), Python's unstable sort, etc

Case Study: Dual-Pivot QuickSort

- Dual-Pivot QuickSort combines several ideas:
  - InsertionSort when array length < 48 elements
    - Provides some protection against bad ordering and bad elements
  - Three-way partition
    - Good constants: single-pass and in-place; $\log_3 N$ vs $\log_2 N$
    - Dual "middle pivots" provides some protection against bad ordering
      - 1/3rd and 2/3rd elements instead of "the end elements" (first and last)

tl;dr

- Constants matter in the real world, even if they don't matter asymptotically!

	Best-Case Time	Worst-Case Time	Space	Stable?	Notes
SelectionSort	$\Theta(N^2)$	$\Theta(N^2)$	$\Theta(1)$	No	
In-Place HeapSort	$\Theta(N)$	$\Theta(N \log N)$	$\Theta(1)$	No	Slow in practice
MergeSort	$\Theta(N \log N)$	$\Theta(N \log N)$	$\Theta(N)$	Yes	Fastest stable sort
In-Place InsertionSort	$\Theta(N)$	$\Theta(N^2)$	$\Theta(1)$	Yes	Best for small or partially-sorted input
Naïve QuickSort	$\Theta(N \log N)$	$\Theta(N^2)$	$\Theta(N)$	Yes	>=2x slower than MergeSort
Dual-Pivot QuickSort	$\Omega(N)$	$O(N^2)$	$\Theta(1)$	No	Fastest comparison sort