Priority Queues and Heaps
CSE 373 Winter 2020

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About how long did Homework 3 take?

A. 0-2 Hours  
B. 2-4 Hours  
C. 4-6 Hours  
D. 6-10 Hours  
E. 10-14 Hours  
F. 14+ Hours  
G. I haven’t finished yet / I don’t want to say
Announcements

❖ Homework 4: Heap is released and due *Wednesday*
  ▪ Hint: you will need an additional data structure to improve the runtime for changePriority(). This data structure may or may not be a (classic) Red-Black tree.

❖ Workshop this Friday @ 11:30am, CSE 203
  ▪ Topics include 2-3 Trees and LLRBs

❖ Please attend 373 DITs, not other classes’!
Questions from Reading Quiz

❖ When do we use Priority Queues?

❖ How is a Queue and Priority Queue different?

❖ How do we handle duplicate values?
Lecture Outline

❖ Priority Queues and Review: Binary Trees

❖ Binary Heaps

❖ Binary Heap Representation
**Set ADT.** A collection of values.
- A set has a size defined as the number of elements in the set.
- You can add and remove values.
- Each value is accessible via a “get” or “contains” operation.

**Map ADT.** A collection of keys, each associated with a value.
- A map has a size defined as the number of elements in the map.
- You can add and remove (key, value) pairs.
- Each value is accessible by its key via a “get” or “contains” operation.
ADTs So Far (2 of 3)

**List ADT.** A collection storing an ordered sequence of elements.

- Each element is accessible by a zero-based index.
- A list has a size defined as the number of elements in the list.
- Elements can be added to the front, back, *or any index in the list*.
- Optionally, elements can be removed from the front, back, *or any index in the list*.
Deque ADT. A collection storing an ordered sequence of elements.
• Each element is accessible by a zero-based index.
• A deque has a size defined as the number of elements in the deque.
• Elements can be added to the front or back.
• Optionally, elements can be removed from the front or back.

Stack ADT. A collection storing an ordered sequence of elements.
• A stack has a size defined as the number of elements in the stack.
• Elements can only be added and removed from the top (“LIFO”)

Queue ADT. A collection storing an ordered sequence of elements.
• A queue has a size defined as the number of elements in the queue.
• Elements can only be added to one end and removed from the other (“FIFO”)

We found more-performant data structures to implement the Queue ADT when we took advantage of its more-limited-than-list functionality.
ADTs To Come

**Priority Queue ADT.** A collection of values.
- A PQ has a size defined as the number of elements in the set.
- You can add values.
- You cannot access or remove arbitrary values, only the max value.

**Disjoint Set ADT.** Coming Soon!
- After the midterm

**Graph ADT.** Coming Soon!
- After the midterm

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Today’s Topic!

Can we find a more-performant data structure to implement the Priority Queue ADT when we take advantage of its more-limited-than-queue functionality?
Priority Queues

- In lecture, we will study **max priority queues** but **min priority queues** are also common
  - Same as max-PQs, but invert the priority

- In a PQ, the only item that matters is the max (or min)
Priority Queue: Applications

- Used heavily in **greedy algorithms**, where each phase of the algorithm picks the locally optimum solution

- Example: route finding
  - Represent a map as a series of *segments*
  - At each intersection, ask which segment gets you closest to the destination (i.e., has max priority or min distance)
Lecture Outline

❖ Priority Queues and Review: Binary Trees

❖ Binary Heaps

❖ Binary Heap Representation
Review: Binary Search Trees

- A **Binary Search Tree** is a binary tree with the following invariant: for every node with value \( k \) in the BST:
  - The left subtree only contains values \( <k \)
  - The right subtree only contains values \( >k \)

```plaintext
class BSTNode<Value> {
    Value v;
    BSTNode left;
    BSTNode right;
}
```

Reminder: the BST ordering applies **recursively** to the entire subtree
**Priority Queue: Possible Data Structures**

- We have two viable implementations of this ADT (so far):

<table>
<thead>
<tr>
<th></th>
<th>Sorted LinkedList PQ (worst case)</th>
<th>Balanced Search Tree PQ (worst case)</th>
</tr>
</thead>
<tbody>
<tr>
<td>add</td>
<td>O(N)</td>
<td>O(log N)</td>
</tr>
<tr>
<td>max</td>
<td>O(1)</td>
<td>O(1)*</td>
</tr>
<tr>
<td>removeMax</td>
<td>O(1)</td>
<td>O(log N)</td>
</tr>
</tbody>
</table>

*If we keep a pointer to the largest element in the BST*
Review: Binary Tree Data Structure

- A **Binary Tree** (not a binary search tree) is a tree where each node has \(0 \leq \text{children} \leq 2\)

```java
class BinaryNode<Value> {
    Value v;
    BinaryNode left;
    BinaryNode right;
}
```
Heaps

- A **Max Heap**: a binary tree where each node’s value is greater than any of its descendents. It implements the Max Priority Queue ADT.
  - This is a *recursive* property!
- A **Min Heap** is the same, but each node is *less than* its descendents.
Which of these are valid max heaps?

1. Valid / Invalid / Valid
2. Valid / Invalid / Invalid
3. Valid / Valid / Invalid
4. Valid / Valid / Valid

Violation of (max) heap invariant
A **Binary Heap** is a heap that is completely filled, with the possible exception of the bottom level which is filled left-to-right.

- Its height is $\Theta(\log N)$

![Binary Heap Diagram]

but is removeMax and add also $\in \Theta(\log N)$?
Binary Heaps: `removeMax`

- Remove the root’s value (but keep the root node)
- Swap in the to-be-deleted leaf’s value
- Recursively `percolateDown()` against each level’s larger child

\[ \Theta(\log N) \]
Binary Heaps: add

- Add the new value at the next valid location in the complete tree
- Recursively percolateUp() ...

0(\log N) but average case is O(1)!

(why?)
Lecture Outline

❖ Priority Queues and Review: Binary Trees

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❖ Binary Heap Representation
A **Binary Heap** is a heap that is completely filled, with the possible exception of the bottom level which is filled left-to-right.

... which makes it easily representable as an array.

- (note: we leave the 0\textsuperscript{th} index empty to make the arithmetic easier)
Binary Heaps as Arrays

- A **Binary-Heap-as-Array**’s node with index i has:
  - Its children at 2*i and 2*i + 1
  - Its parent at i/2
Binary Heaps as Arrays: percolateDown

```java
void percolateDown(int idx) {
    tmp = a[idx];
    for (; idx * 2 <= a.length; ) {
        idx = idx * 2;
        if (a[idx] < a[idx + 1]) idx++;
        if (a[idx] > tmp) {
            a[idx/2] = a[idx];
        } else {
            break;
        }
    }
}
```

We’ve rewritten our recursive algorithm iteratively!

Find our children in the array

Get the index of our larger child

swap if we’re still violating the heap invariant
Other Priority Queue Operations

❖ The two “primary” PQ operations are:
  ▪ removeMax()
  ▪ add()

❖ However, because PQs are used in so many algorithms there are three common-but-nonstandard operations:
  ▪ merge(): merge two PQs into a single PQ
  ▪ buildHeap(): reorder the elements of an array so that its contents can be interpreted as a valid binary heap
  ▪ changePriority(): change the priority of an item already in the heap

we’ll revisit soon!

you will implement in HW 4!
Other Priority Queue data structures

- **D-Heaps**
  - Binary heap, but with a >2 branching factor “d”

- **Leftist Heap**
  - Unbalanced heap that skews “leftward”, optimized for merge()

- **Skew Heap**
  - Leftist Heap variant, also optimized for merge()

- **Binomial Queue**
  - A “forest” of heaps
tl;dr

- **Priority Queue ADT** is designed to find the max (or min) quickly
  - We can implement it with many data structures

- **The Binary Heap** is a data structure which is simple to reason about and implement *and* has constant- to $\Theta(\log N)$ bounds

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* If we keep a pointer to the largest element in the BST
** Average case is constant
BONUS! ADT / Data Structure Taxonomy

Maps and Sets

- Search Trees ("left is less-than, right is greater-than")
  - Binary Search Trees (branching factor == 2)
    - Plain BST (unbalanced)
      - Balanced BSTs: LLRB (other examples: “Classic” Red-Black, AVL, Splay, etc)
  - B-Trees (have a branching factor >2; balanced)
    - 2-3 Trees
    - 2-3-4 Trees

- Hash Tables (will cover later!)