Algorithm Analysis III: Recursive

CSE 373 Winter 2020

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Announcements

- No lecture on Monday; MLK Day of Service
  - Please go out there and volunteer to improve your communities
  - DITs cancelled for Monday
- HW1 feedback form emailed to your @uw
  - Help us make your homeworks better!
- Homeworks:
  - The score you get in Gradescope is your “final” score. No style grading; late deductions already factored in.
  - HW2 due Tuesday; check your Gradle now while the staff is still around!

Questions from Reading Quiz

- What's the base for log N?
- What do you mean by "unrolling the recurrence"? How do you get T(N) = T(N / 2) + c?
- What is T(N)?

Lecture Outline

- **Recursion**
- Pattern #1: (Almost) Doubling the Input
- Pattern #2: Halving the Input
- Pattern #3: Constant-size Input *and* Doing Work

Recursion

- **Recursion**
  - An algorithm or a data structure that is defined in terms of itself
  - Usually has a **base case** that doesn't use recursion to terminate

- Examples:
  - Fibonacci:
    - fib(n) = fib(n-1) + fib(n-2)
    - fib(1) = 1
  - An ancestor is a person who is:
    - Your parent
    - Your parent's ancestor
  - Sourdough starter
    - A colony of microorganisms
    - Your last loaf's starter

Lecture Outline

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Asymptotic Analysis for ~~Iterative~~ Recursive Problems

- Case Analysis != Asymptotic Analysis

- Memorize these summations since they're common:

$$1 + 2 + 3 + 4 + \ldots + (N-1) = N(N-1)/2 \in \Theta(N^2)$$

$$1 + 2 + 4 + 8 + \ldots + 2^{\text{floor}(\log_2 N)} = 2N - 1 \in \Theta(N)$$

- Strategies for finding an asymptotic bound:
  - Use a geometric argument / visualizations
  - Find an expression for the exact step count
  - Write out examples

Poll Everywhere

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Find a tight bound for f's runtime

A. 1

B. log N

C. N

D. N^2

E. 2^N

```
int f(int n) {
  if (n <= 1)
    return 1;
  return f(n-1) + f(n-1);
}
```

(Almost) Doubling the Input: f

```
int f(int n) {
  if (n <= 1)
    return 1;
  return f(n-1) + f(n-1);
}
```

Constant time "d"

$T(0) = d$

$T(1) = d$

$\vdots$

$T(10) = T(9) + T(9) + c$

$\vdots$

$T(N) = 2\,T(N-1) + c$

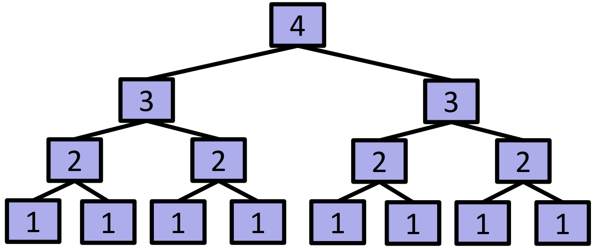
Runtime is recursively defined!

(Almost) Doubling the Input, Geometrically

- Draw one node for each function call
  - Why are we counting function calls and not operations?

```
int f(int n) {
  if (n <= 1)
    return 1;
  return f(n-1) + f(n-1);
}
```

$f(4) =$

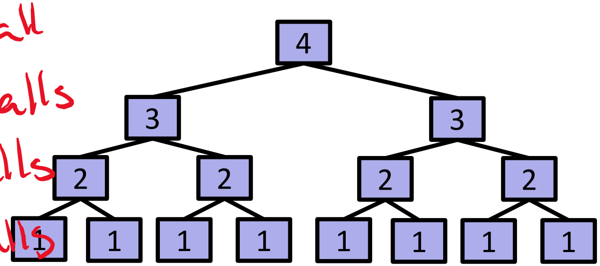


(Almost) Doubling the Input, Examples

N=	Total Function Calls
1	1
2	1 + 2
3	1 + 2 + 4
4	1 + 2 + 4 + 8

```
int f(int n) {
  if (n <= 1)
    return 1;
  return f(n-1) + f(n-1);
}
```

Level 1 = 1 call
Level 2 = 2 calls
Level 3 = 4 calls
Level 4 = 8 calls



(Almost) Doubling the Input, Counting

- How many "levels" are in the function call tree? N

- How much work occurs at each level L? 2^{L-1}

```
int f(int n) {
  if (n <= 1)
    return 1;
  return f(n-1) + f(n-1);
}
```

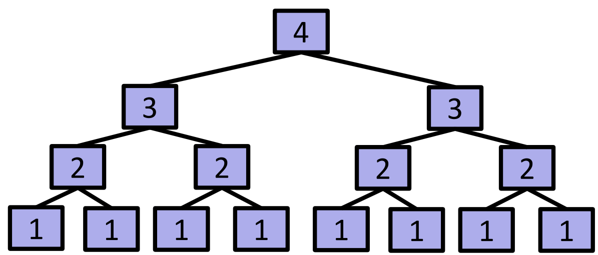
calls per level

$2^{1-1} =$ **1**

$2^{2-1} =$ **2**

$2^{3-1} =$ **4**

$2^{4-1} =$ **8**



(Almost) Doubling the Input, Counting

We know that f's runtime looks like:

$$T(4) = 1 + 2 + 4 + 8$$

$$T(N) = 1 + 2 + 4 + 8 + \ldots + 2^{N-1}$$

And we've previously seen:

$$1 + 2 + 4 + 8 + \ldots + 2^{\text{floor}(\log_2 Q)} = 2Q - 1$$

If we let $Q = 2^{\mathbf{N-1}}$

$$T(N) = 1 + 2 + 4 + 8 + \ldots + 2^{N-1} = 2 * (2^{N-1}) - 1$$

$$T(N) = 2^N - 1 \in \Theta(2^N)$$

Out-of-Scope: Recurrence Solution

$$
\begin{aligned}
C(1) &= 1 \\
C(N) &= 2C(N-1)+1 \\
&= 2(2C(N-2)+1)+1 \\
&= 2(2(2C(N-2)+1)+1)+1 \\
&= 2(\cdots 2 \cdot 1 + 1) + 1) + \cdots 1 \\
&= \underbrace{2(\cdots 2)}_{N-1} \cdot 1 + 1) + \cdots 1 \\
&= 2^{N-1} + 2^{N-2} + \cdots + 1 = 2^N - 1 \in \Theta(2^N)
\end{aligned}
$$

Lecture Outline

- Recursion
- Pattern #1:(Almost) Doubling the Input
- **Pattern #2: Halving the Input**
- Pattern #3: Halving the Input *and* Doing Work

Halving the Input: Binary Search

```
public static boolean binarySearch(int[] sorted, int findMe) {
  if (sorted.length == 0)
    return false;

  int mid = sorted.length / 2;
  if (findMe < sorted[mid])
    int[] subrange = Arrays.copyOfRange(sorted, 0, mid);
    return binarySearch(subrange, findMe);
  else if (x > sorted[mid])
    int[] subrange = Arrays.copyOfRange(sorted, mid, sorted.length);
    return binarySearch(subrange, findMe);
  else
    return true;
}
```

Halving the Input, Geometrically

- Draw one node for each call to binarySearch. If there is more than one case, draw a tree for each case

Halving the Input, Counting

- $T(1) = d$
  $T(2) = T(1) + c$
  …
  $T(N) = T(N/2) + c$

- *Worst case*: keep halving the input size until you reach 0, which takes $\log_2 N$ "halve" operations
  - In other words, there are $\log_2 N$ function calls or $\log_2 N$ layers in our function call tree

- Worst case $T(N) \in \Theta(\log_2 N)$

Halving the Input, Counting

- *Worst case*: keep halving the input size until you reach 0, which takes $\log_2 N$ "halve" operations
- Similar to f, there is a constant amount of work at each recursive step

Case	Big-Theta
Best	$\Theta(1)$
Worst	$\Theta(\log_2 N)$
Overall	$\emptyset$

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MergeSort: The Merge Operation

Given **two sorted arrays**, the merge operation combines them into a single sorted array by successively copying the smallest item from the two arrays into a target array.

N

2	3	6	10	11

4	5	7	8

Poll Everywhere

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- What is the runtime of merge, specified in terms of N, the total number of items?

1. $\Theta(1)$
2. $\Theta(\log_2 N)$
3. $\Theta(N)$
4. $\Theta(N \log_2 N)$
5. $\Theta(N^2)$

2	3	**6**	10	11

4	5	**7**	8

2	3	4	**5**					

MergeSort

- MergeSort algorithm is recursive
  - If array is of size 1, return
  - MergeSort the left half
  - MergeSort the right half
  - Merge the two sorted halves

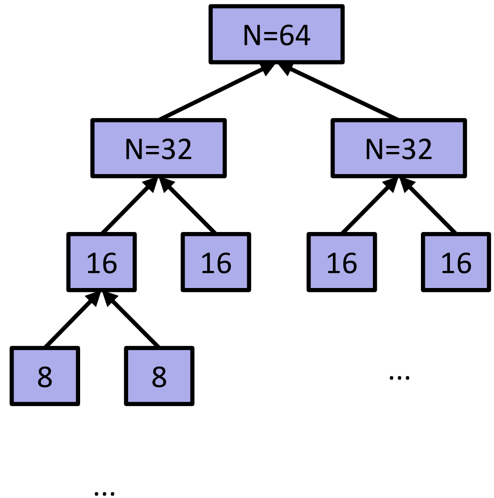
```
void mergeSort(int[] arr) {
  if (arr.length == 1)
    return;

  int mid = arr.length / 2;
  int[] left = Arrays.copyOfRange(arr, 0, mid);
  int[] right = Arrays.copyOfRange(arr, mid, unsorted.length);

  mergeSort(left);
  mergeSort(right);
  System.arraycopy(left, 0, arr, 0, left.length);
  System.arraycopy(right, 0, arr, left.length, right.length);
}
```

MergeSort

- Geometrically:
  - How many *layers* are in our function call tree?
    - Compare with BinarySearch

      7 layers, or $\log_2 N$
  - How much work is done at each layer?
    - Compare with f

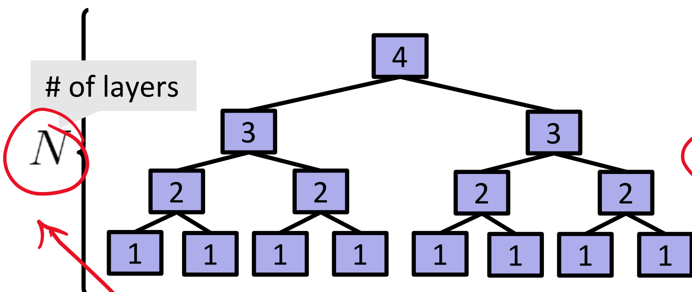
      64, or N
- Counting:
  - T(1) = d
  - T(2) = T(1) + T(1) + c
  - …
  - T(N) = 2T(N/2) + c



Counting Calls vs. Work-per-Layer

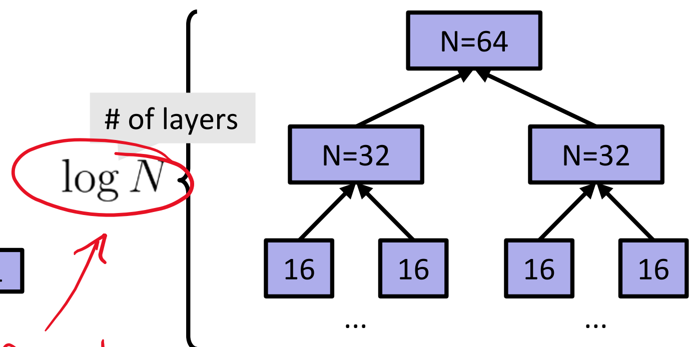
f: work for each call is constant, so can count the number of calls

$$R(N) \in \Theta(2^N)$$



MergeSort: work for each call is *variable*. However, the *work per layer* is the same

$$R(N) = \Theta(N \cdot \log N)$$



heights are different

Linear vs Linearithmic (N log N) vs Quadratic

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Algorithm Design (Jon Kleinberg, Éva Tardos/Pearson Education)

asymptotic analysis deals with infinities not specific inputs (eg |1|2|7|4|) or classes of input (eg best case)

tl;dr Asymptotic Analysis ~~for Iterative Recursive Problems~~

- Case Analysis != Asymptotic Analysis

- Memorize these summations since they're common:

$$1 + 2 + 3 + 4 + \ldots + (N-1) = N(N-1)/2 \in \Theta(N^2)$$
$$1 + 2 + 4 + 8 + \ldots + 2^{\text{floor}(\log_2 N)} = 2N - 1 \in \Theta(N)$$
$$1 + 2 + 4 + 8 + \ldots + 2^N = 2^{N+1} - 1 \in \Theta(2^N)$$

- Strategies for finding an asymptotic bound:
  - Use a geometric argument / visualizations
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