Algorithm Analysis I: Intro

CSE 373 Winter 2020

Instructor: Hannah C. Tang

Teaching Assistants:

Aaron Johnston	Ethan Knutson	Nathan Lipiarski
Amanda Park	Farrell Fileas	Sam Long
Anish Velagapudi	Howard Xiao	Yifan Bai
Brian Chan	Jade Watkins	Yuma Tou
Elena Spasova	Lea Quan	

Announcements

- Schedule for Drop-in Times (finally!!!! 😳) posted

Questions from Reading Quiz

- Do we care about the specific operation counts (eg calculating 2 to 50,015,001)? What if we are off by one?

- What is the conversion between the number of operations and the runtime? I had a little trouble understanding what each "step" was and what counted

- Could we go over how to count the operations for dup1?

Lecture Outline

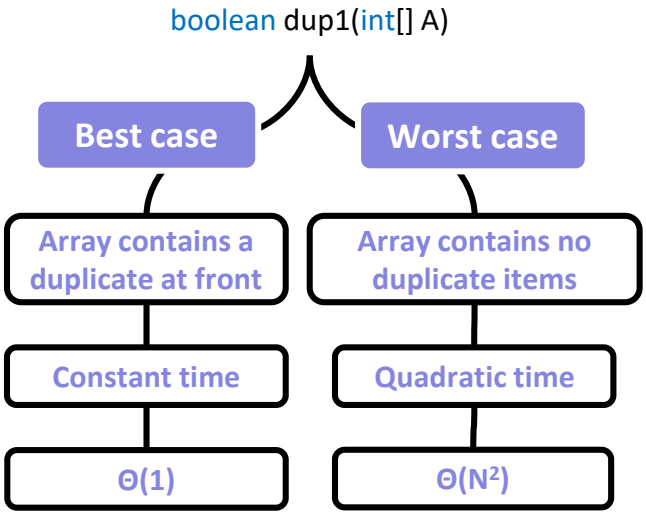
- **Algorithm Analysis Concepts**
- Asymptotic Analysis Case Study: dup1 Worst Case
- Asymptotic Analysis Simplifications
- Case Analysis vs Asymptotic Analysis
- Takeaways

Review: What is Runtime Analysis?

- What does it mean for something to be "slow" or "fast"?
- Let's run it and measure the (wallclock) time! Oh wait …
  - Input can affect time
  - Hardware can affect time
  - Other programs running on the machine can affect time
  - … and so much more!
- Count how many steps a program takes to execute on an input of size N

Algorithm Analysis: Our Destination

- Comprehension: Understanding the implementation
- Cost Model: Calculating the cost in terms of N, the size of the input
- Analysis Types:
  - Case Analysis. How certain conditions affect the program execution
  - Asymptotic Analysis: What happens as $N \to \infty$
- Formalizing. Summarizing the final result in precise English or a mathematical notation



Comprehension

- What is the data structure or algorithm doing?

```
public static boolean dup1(int[] A) {
  for (int i = 0; i < A.length; i += 1) {
    for (int j = i + 1; j < A.length; j += 1) {
      if (A[i] == A[j]) {
        return true;
      }
    }
  }
  return false;
}
```

Cost Model: Time

- An algorithm's runtime is dictated by the number of CPU instructions it executes. This is why we "count steps": each "step" is an idealized instruction
  - Example CPU instructions: comparing integers (<, ==), incrementing numbers (+=, *), accessing an array

```
public static boolean dup1(int[] A) {
  for (int i = 0; i < A.length; i += 1) {
    for (int j = i + 1; j < A.length; j += 1) {
      if (A[i] == A[j]) {
        return true;
      }
    }
  }
  return false;
}
```

"Computers are really dumb. They can only do a few things like shuffling around numbers, but they do them really really fast so that they appear smart."

– Hal Perkins, UW CSE

Cost Model: Space

- We usually talk about time rather than space, but the principles are the same

- Similar to time, we measure "cost" in higher-level abstractions rather than machine-specific bytes
  - Examples: queue elements, graph nodes, etc.

Asymptotic Analysis

- **Asymptotic Analysis** is how the algorithm or data structure behaves as N→∞
  - Simulating billions of particles
  - Social network with billions of users
  - Logging billions of transactions
  - Encoding billions of bytes of video data

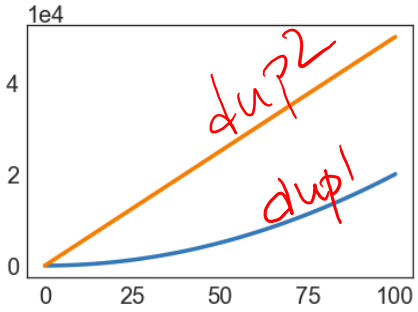
- Why ∞?
  - We need a way to characterize data structures' and algorithms' complexity, which can vary for every finite N

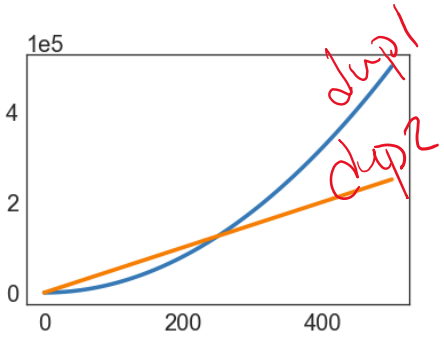
Asymptotic Analysis Means "Infinity": Graphically

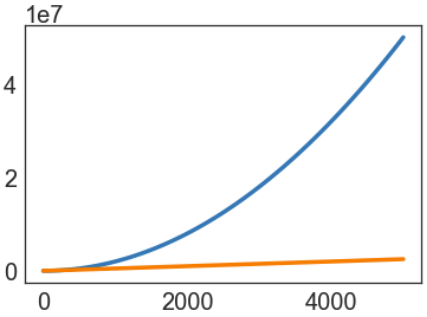
- Asymptotic analysis is non-intuitive because it deals with infinity!
  - What if you know your algorithm will only run on N < 100 elements?
  - Why do we "throw away constants"?
  - How come lower-order terms "don't matter"?

because ∞

any finite value for N means that asymptotic analysis does not apply







- Linear-time algorithms **scale better** than quadratic-time algorithms (parabolas).

Asymptotic Analysis Means "Infinity": Intuitively

- Since we're dealing with infinity, constants and lower-order terms don't meaningfully add to the final result
- The highest-order term is what matters and drives growth

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Algorithm Design (Jon Kleinberg, Éva Tardos/Pearson Education)

Lecture Outline

- Asymptotic Analysis Concepts
- **Asymptotic Analysis Case Study: dup1 Worst Case**
- Asymptotic Analysis Simplifications
- Case Analysis vs Asymptotic Analysis
- Takeaways

Counting Steps: Asymptotic Worst Case

- Assume N = A.length = 6
  - I've broken up the for-loops to make it easier to count
  - Cheatsheet: 1 + 2 + 3 + 4 + 5 + 6 = 21

→ no dups at all!

eg


```
public static boolean dup1(int[] A) {
  int i = 0;
  for ( ; i < A.length; ) {
    int j = i + 1;
    for ( ; j < A.length; ) {
      if (A[i] == A[j]) {
        return true;
      }
      j += 1;
    }
    i += 1;
  }
  return false;
}
```

num iterations + 1

3

6ish

Counting Steps: Asymptotic Worst Case

- Cheatsheet: 1 + 2 + 3 + 4 + 5 + 6 = 21

Operation	Number of Executions (N = 6)	Symbolic Expression
i = 0		
Less-than (<)		
Increment (+=)		
Equals-to (==)		
Array accesses		

Counting Steps: Asymptotic Worst Case

- Cheatsheet: 1 + 2 + 3 + … (N – 1) = ???

Operation	Number of Executions (N = 6)	Symbolic Expression
i = 0	1	
Less-than (<)	28	
Increment (+=)	21	
Equals-to (==)	21	
Array accesses ([])	42	

Counting Steps: Double-checking Our Work

- Demo

Symbolic Expression

$$C = 1 + 2 + 3 + \cdots + (N-3) + (N-2) + (N-1)$$

$$C = (N-1) + (N-2) + (N-3) + \cdots + 3 + 2 + 1$$

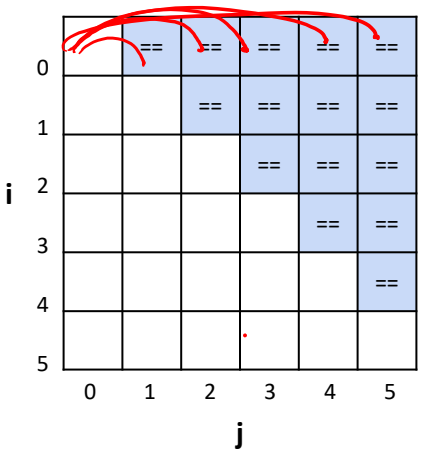
$$2C = N + N + N + \cdots + N + N + N = N(N-1)$$

$$\therefore \quad C = N(N-1)/2$$

$$\frac{N^2 - N}{2}$$

```
public static boolean dup1(int[] A) {
  for (int i = 0; i < A.length; i += 1) {
    for (int j = i + 1; j < A.length; j += 1) {
      if (A[i] == A[j]) {
        return true;
      }
    }
  }
  return false;
}
```

Symbolic Expression, Intuitively



- We want the area of the right triangle
- If its side length is N, its order of growth is N^2.

```
public static boolean dup1(int[] A) {
  for (int i = 0; i < A.length; i += 1) {
    for (int j = i + 1; j < A.length; j += 1) {
      if (A[i] == A[j]) {
         return true;
      }
    }
  }
  return false;
}
```

Counting Steps: Asymptotic Worst Case

- Cheatsheet: $1 + 2 + 3 \ldots N = (N^2 - N)/2$

Operation	Number of Executions (N = 6)	Symbolic Expression
i = 0	1	1
Less-than (<)	28	$(N^2 + 3N + 2)/2$
Increment (+=)	21	$(N^2 + N)/2$
Equals-to (==)	15	$(N^2 - N)/2$
Array accesses	30	$N^2 + N$

+2N

+N

*2

Lecture Outline

- Asymptotic Analysis Concepts
- Asymptotic Analysis Case Study: dup1 Worst Case
- **Asymptotic Analysis Simplifications**
- Case Analysis vs Asymptotic Analysis
- Takeaways

Simplification 1: Eliminate Lower-Order Terms

- Ignore lower-order terms

Operation	Number of Executions (N = 6)	Symbolic Expression
i = 0	1	1
Less-than (<)	28	$(N^2 + 3N + 2)/2$
Increment (+=)	21	$(N^2 + N)/2$
Equals-to (==)	15	$(N^2 - N)/2$
Array accesses	30	$N^2 + N$

- $6 + (N^2 + 3N + 2)/2 + (N^2 + N)/2 + (N^2 - N)/2 + N^2 + N$
- $6 + \frac{1}{2}N^2 + \frac{3}{2}N + 1 + \frac{1}{2}N^2 + \frac{1}{2}N + \frac{1}{2}N^2 - \frac{1}{2}N + N^2 + N$
- $6 + \frac{1}{2}N^2 + \frac{3}{2}N + 1 + \frac{1}{2}N^2 + \frac{1}{2}N + \frac{1}{2}N^2 - \frac{1}{2}N + N^2 + N$
- $\frac{5}{2}N^2$

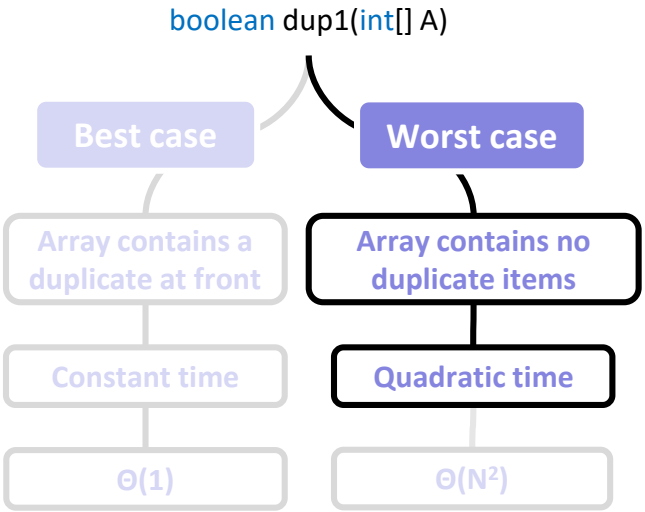
Simplification 2: Eliminate Multiplicative Constants

- Ignore multiplicative constants.
  - We already threw away the meaningful constant when we used a simplified cost model

- $\frac{5}{2} N^2$
-  N^2

Algorithm Analysis: Our Destination

- "The worst-case order-of-growth for dup1's runtime is quadratic"



Order of Growth

- What is the order of growth for each function?
  - (Informally, what is the shape of each function for very large N?)

Function	Order of Growth
$N^3 + 3N^4$	N^4
$(1 / N) + N^3$	N^3
$Ne^N + N$	Ne^N
$40 \sin(N) + 4N^2$	N^2

Lecture Outline

- Asymptotic Analysis Concepts
- Asymptotic Analysis Case Study: dup1 Worst Case
- Asymptotic Analysis Simplifications
- **Case Analysis vs Asymptotic Analysis**
- Takeaways

Poll Everywhere

pollev.com/uwcse373

For a very large array with billions of elements (i.e. asymptotic analysis), is it possible for dup1 to execute exactly 1 equals (==) operation? YES! In its best case: 

Operation	Symbolic Expression (Worst Case)
Equals-to (==)	$(N^2 - N)/2$

```
public static boolean dup1(int[] A) {
  for (int i = 0; i < A.length; i += 1) {
    for (int j = i + 1; j < A.length; j += 1) {
      if (A[i] == A[j]) {
        return true;
      }
    }
  }
  return false;
}
```

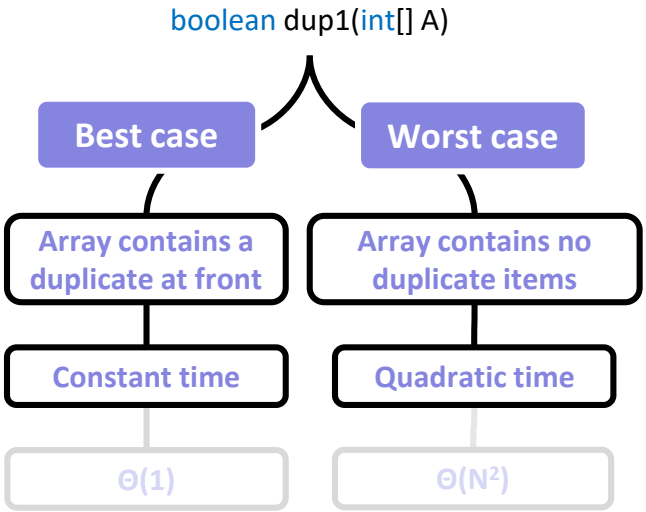
Simplification 3: Consider the Worst Case

- Oftentimes, we only care about worst case
  - If a runtime statement doesn't mention, assume it's a worst-case analysis

- But dup1's best case is significantly better than its worst-case!
  - Demo

Algorithm Analysis: Our Destination

- "The worst-case order-of-growth for dup1's runtime is quadratic (parabolic)"
- "The best-case order-of-growth for dup1's runtime is constant"



Lecture Outline

- Asymptotic Analysis Concepts
- Asymptotic Analysis Case Study: dup1 Worst Case
- Asymptotic Analysis Simplifications
- Case Analysis vs Asymptotic Analysis
- **Takeaways**

tl;dr

- Runtime analysis uses a simplified cost model to represent computation
  - Space analysis also uses a simplified cost model
- Asymptotic analysis can take liberties with mathematical expressions because it deals with infinity
  - Eg, dropping lower-order terms and constants
- Asymptotic analysis gives us a common "frame of reference" with which to compare algorithms
  - "Parabolas grow faster than lines"
- Case analysis is a different axis on which to evaluate runtime and space