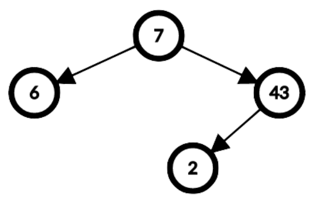
Section 04: Solutions

Section Problems

1. Valid BSTs and AVL Trees

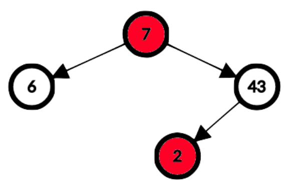
For each of the following trees, state whether the tree is (i) a valid BST and (ii) a valid AVL tree. Justify your answer.

(a)



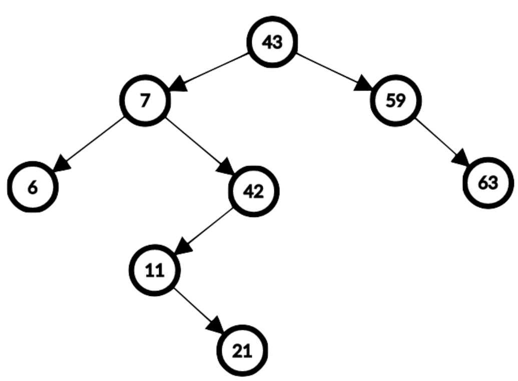
Solution:

This is not a valid BST! The 2 is located in the right sub-tree of 7, which breaks the BST property. Remember that the BST property applies to **every** node in the left and right sub-trees, not just the immediate child!



All AVL trees are BSTs. Because of this, this tree can't be a valid AVL tree either.

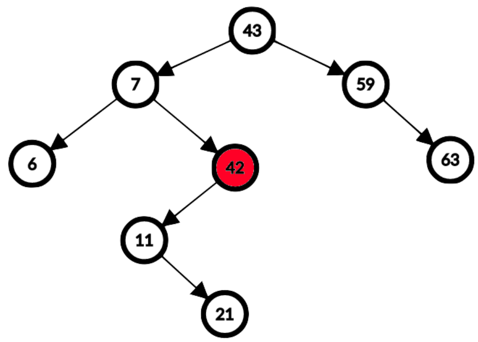
(b)



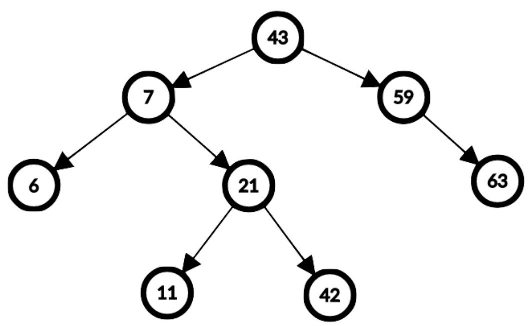
Solution:

This tree is a valid BST! If we check every node, we see that the BST property holds at each of them.

However, this is not a valid AVL tree. We see that some nodes (for example, the 42) violate the balance condition, which is an extra requirement compared to BSTs. Because the heights of 42's left and right sub-trees differ by more than one, this violates the condition.



(c)



Solution:

This tree is a valid BST! If we check every node, we see that the BST property holds at each of them.

This tree is also a valid AVL tree! If we check every node, we see that the balance condition also holds at each of them.

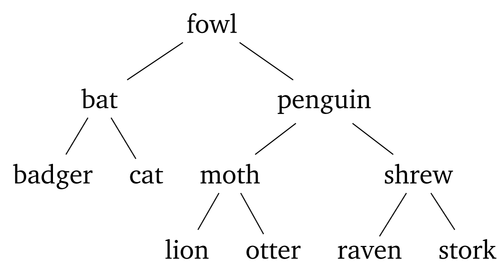
2. Constructing AVL trees

Draw an AVL Tree as each of the following keys are added in the order given. Show intermediate steps.

(a)

{"penguin", "stork", "cat", "fowl", "moth", "badger", "otter", "shrew", "lion", "raven", "bat"}

Solution:

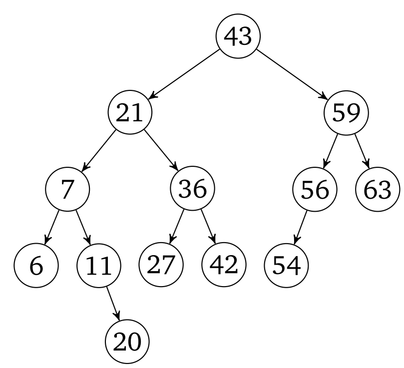


(b)

$$\{6, 43, 7, 42, 59, 63, 11, 21, 56, 54, 27, 20, 36\}$$

Solution:

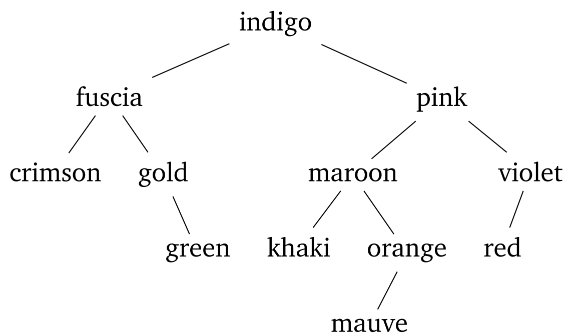
Note: The Section slides have a step-by-step walkthough of this one!



(c)

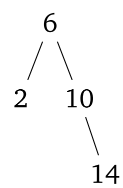
{“indigo”, “fuscia”, “pink”, “goldenrod”, “violet”, “khaki”, “red”, “orange”, “maroon”, “crimson”, “green”, “mauve”}

Solution:



3. AVL tree rotations

Consider this AVL tree:



Give an example of a value you could insert to cause:

(a) A single rotation

Solution:

Any value greater than 14 will cause a single rotation around 10 (since 10 will become unbalanced, but we’ll be in the line case).

(b) A double rotation

Solution:

Any value between 10 and 14 will cause a double rotation around 10 (since 10 will be unbalanced, and we’ll be in the kink case).

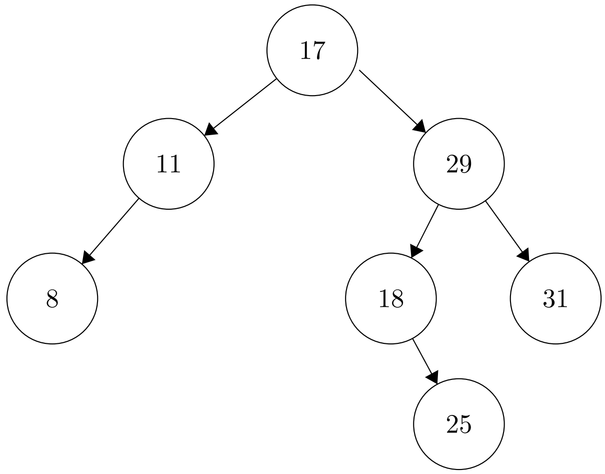
(c) No rotation

Solution:

Any value less than 10 will cause no rotation (since we can’t cause any node to become unbalanced with those values).

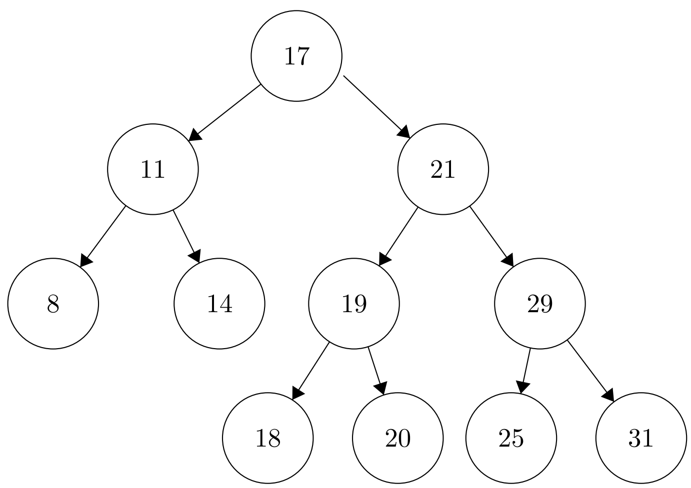
4. Inserting keys and computing statistics

In this problem, we will see how to compute certain statistics of the data, namely, the minimum and the median of a collection of integers stored in an AVL tree. Before we get to that let us recall insertion of keys in an AVL tree. Consider the following AVL tree:



(a) We now add the keys $\{21, 14, 20, 19\}$ (in that order). Show where these keys are added to the AVL tree. Show your intermediate steps.

Solution:



(b) Recall that if we use an unsorted array to store n integers, it will take us $O(n)$ runtime in order to compute the minimum element in the array. This can be done by running a loop that scans the array from the first index to the last index, which keeps track of the minimum element that it has seen so far. Now we will see how to compute the minimum element of a set of integers stored in an AVL tree which runs *much* faster than the procedure described above.

i. Given an AVL tree storing numbers, like the one above, describe a procedure that will return the minimum element stored in the tree.

Solution:

Remember that an AVL tree satisfies the BST property, i.e. for any node, all keys in the left sub-tree below that node must be smaller than all the keys in the right sub-tree. Since the minimum is the smallest element in the tree, it must lie in the left sub-tree below the root. By the same reasoning, the minimum must also lie in the left sub-tree below the left node connected to the root and so on and so forth.

Proceeding this way, we can set l_0 to be the root of the tree and for all $i \geq 1$, we can set l_i to the left node connected to l_{i-1}. By our reasoning above, the minimum lies in the subtree below l_i for every i. Hence, we can simply start at the root i.e. l_0 and keep following the edge towards the nodes $l_1, l_2, \ldots$ until we hit a leaf! The leaf must be the minimum, as there is no subtree rooted below it.

ii. Supposing an AVL tree has n elements, what is the runtime of the above procedure in terms of n? How does this runtime compare with the $O(n)$ runtime of the linear scan of the array?

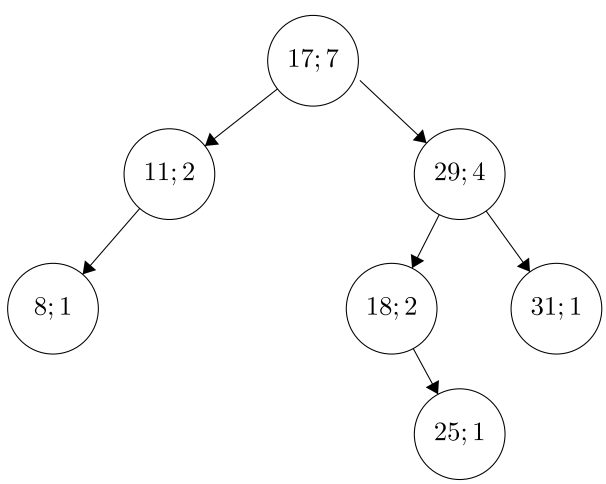
Solution:

The above procedure, is essentially a loop that starts at the root and stops when it reaches a leaf. The length of any path from the root to a leaf in an AVL tree with n elements is a most $O(\log n)$. Hence, the above procedure has runtime $O(\log n)$. This runtime is exponentially better than the linear scan which takes $O(n)$ time!

(c) In the next few problems, we will see how to compute the median element of the set of elements stored in the AVL tree. The median of a set of n numbers is the element that appears in the $\lceil n/2 \rceil$-th position, when this set is written in sorted order. When n is even, $\lceil n/2 \rceil = n/2$ and when n is odd, $\lceil n/2 \rceil = (n+1)/2$. For example, if the set is $\{3, 2, 1, 4, 6\}$ then the set in sorted order is $\{1, 2, 3, 4, 6\}$, and the median is 3.

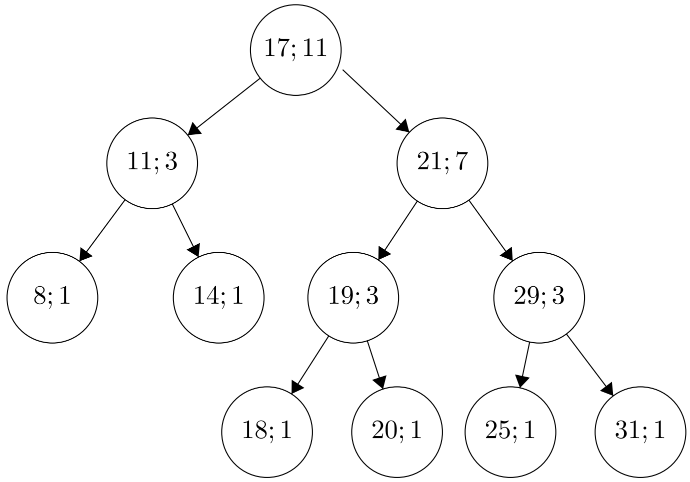
If we were to simply store n integers in an array, one way to compute the median element would be to first sort the array and then look up the element at the $\lceil n/2 \rceil$-th position in the sorted array. This procedure has a runtime of $O(n \log n)$, even when we use a clever sorting algorithm like Mergesort. We will now see how to compute the median, when the data is stored in a rather modified AVL tree *much* faster.

For the time being, assume that we have a modified version of the AVL tree that lets us maintain, not just the key but also the number of elements that occur below the node at which the key is stored plus one (for that node). The use of this will become apparent very soon. As an example, the modified version of the AVL tree above, would like so (the number after the semi-colon in each node accounts for the number of nodes below that node plus one).



i. We now again add the keys $\{21, 14, 20, 19\}$ (in that order) to the modified AVL tree. How does the modified AVL tree look after the insertions are done?

Solution:



ii. Given a modified AVL tree, like the one above, describe a procedure that will return the median element stored in the tree. Note that in the modified tree, you can access the number of elements lying below a node in addition to the number stored in that node. Can you use this extra information to find the median more quickly?

Solution:

We will actually show that using a modified AVL tree, we can compute the k-th smallest element for any k. The k-th smallest element of a set of n numbers is the number at index k when the set is written in sorted order. Note that this problem is more general than computing the median! If we plug $k = \lceil n/2 \rceil$, we can compute the median!

Similar to the strategy that we used to compute the minimum, we start by setting l_0 to be the root of the tree. At this point, we check the number of nodes that lie below the left and right nodes connected to the root. Let these numbers be $x_{l_0,0}$ and $x_{l_0,1}$ respectively. We consider three cases below.

I. Let us suppose for the moment that $x_{l_0,0} = k - 1$. We observe that if the elements in the AVL tree were to be written in sorted order, all the elements in the left subtree below root would appear before the root, which itself would appear before the elements in the right sub-tree. Since there are $k-1$ elements in the left subtree, the index of the root is k, which is the desired element.

II. Now suppose $x_{l_0,0} < k - 1$. Again, if we were to write the elements in the AVL tree in sorted order, the k-th smallest element would now lie in the subtree below the right node.

III. Finally, if $x_{l_0,0} > k - 1$, the k-th smallest number would lie in subtree below the left node.

The upshot of this is that by checking the number of nodes in the left and right subtrees below a given node, we were able to find out which subtree the k-th smallest element lies in! We can repeat this, recursing in the appropriate subtree. For example, if $x_{l_0,0} < k - 1$ then we recurse in the right subtree. However, the k-th smallest element in the entire tree may not be the k-th smallest element in the right subtree!

We want to find out what element we need to look for in the right subtree in order to find the k-th smallest element in the entire tree. Let us suppose that the k-th smallest element in the entire subtree is in fact the k'-th smallest element in the right subtree. If we were to write out the elements in the AVL tree in sorted order, it follows that the k'-th smallest element in the right subtree is at position $(x_{l_0,0} + 1) + k'$ in the entire tree. But this position is also the position of the k-th smallest element. It follows that

$$(x_{l_0,0} + 1) + k' = k \implies k' = k - (x_{l_0,0} + 1).$$

Therefore, in order to locate the k-th smallest element in the entire tree, it suffices to locate the $(k - (x_{l_0,0} + 1))$-th smallest element in the right subtree, which we can do as detailed above.

We repeat the procedure until, we either find the k-th smallest element to be a node, like in Case **I**, or we hit a leaf.

iii. Supposing a modified AVL tree has n elements, what is the runtime of the above procedure in terms of n? How does this runtime compare with the $O(n \log n)$ runtime described earlier?

Solution:

The above procedure, is essentially a loop that starts at the root and stops when it reaches a leaf. The length of any path from the root to a leaf in an AVL tree with n elements is a most $O(\log n)$. Hence, the above procedure has runtime $O(\log n)$. This runtime is far far better than the solution based on sorting which takes $O(n \log n)$ time; we managed a shave off the linear term in the latter expression!

iv. Bonus: After every insertion, the number of nodes that lie below a given node need not remain the same. For example, after four insertions, the number of nodes below the root increased and the number of nodes below the node where the key ”29” was stored, decreased. Describe a procedure that takes as input a modified AVL tree T with n nodes, an integer key k and, returns the modified AVL T', that has the key k inserted in T. What is the runtime of this procedure?

5. Hash table insertion

For each problem, insert the given elements into the described hash table. Do not worry about resizing the internal array.

(a) Suppose we have a hash table that uses separate chaining and has an internal capacity of 12. Assume that each bucket is a linked list where new elements are added to the front of the list.

Insert the following elements in the EXACT order given using the hash function $h(x) = 4x$:

0, 4, 7, 1, 2, 3, 6, 11, 16

Solution:

To make the problem easier for ourselves, we first start by computing the hash values and initial indices:

key	hash	index (pre probing)
0	0	0
4	16	4
7	28	4
1	4	4
2	8	8
3	12	0
6	24	0
11	44	8
16	64	4

The state of the internal array will be

$6 \to 3 \to 0$	/	/	/	$16 \to 1 \to 7 \to 4$	/	/	/	$11 \to 2$	/	/	/

(b) Suppose we have a hash table that uses linear probing and has an internal capacity of 13.

Insert the following elements in the EXACT order given using the hash function $h(x) = 3x$:

2, 4, 6, 7, 15, 13, 19

Solution:

Again, we start by forming the table:

key	hash	index (before probing)
2	6	6
4	12	12
6	18	5
7	21	8
15	45	6
13	39	0
19	57	5

Next, we insert each element into the internal array, one-by-one using linear probing to resolve collisions. The state of the internal array will be:

13	/	/	/	/	6	2	15	7	19	/	/	4

(c) Suppose we have a hash table that uses quadratic probing and has an internal capacity of 10.

Insert the following elements in the EXACT order given using the hash function $h(x) = x$:

0, 1, 2, 5, 15, 25, 35

Solution:

The state of the internal array will be:

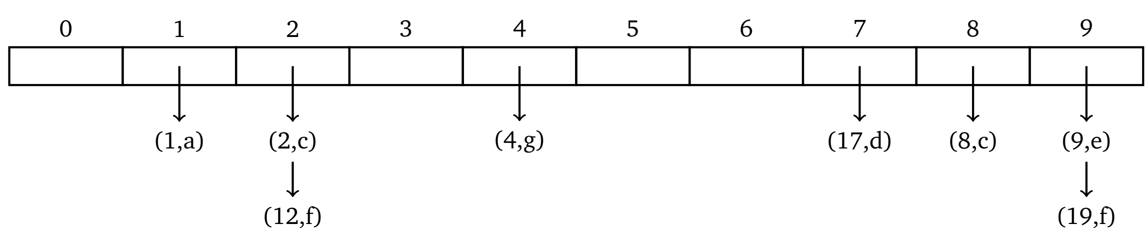
0	1	2	/	35	5	15	/	/	25

(d) Suppose we have a hash table implemented using separate chaining. This hash table has an internal capacity of 10. Its buckets are implemented using a linked list where new elements are appended to the end. Do not worry about resizing.

Show what this hash table internally looks like after inserting the following key-value pairs in the order given using the hash function $h(x) = x$:

$$(1, a), (4, b), (2, c), (17, d), (12, e), (9, e), (19, f), (4, g), (8, c), (12, f)$$

Solution:



6. Evaluating hash functions

Consider the following scenarios.

(a) Suppose we have a hash table with an initial capacity of 12. We resize the hash table by doubling the capacity. Suppose we insert integer keys into this table using the hash function $h(x) = 4x$.

Why is this choice of hash function and initial capacity suboptimal? How can we fix it?

Solution:

Notice that the hash function will initially always cause the keys to be hashed to at most one of three spots: 12 is evenly divided by 4.

This means that the likelihood of a key colliding with another one dramatically increases, decreasing performance.

This situation does not improve as we resize, since the hash function will continue to map to only a fourth of the available indices.

We can fix this by either picking a new hash function that's relatively prime to 12 (e.g. $h(x) = 5x$), by picking a different initial table capacity, or by resizing the table using a strategy other then doubling (such as picking the next prime that's roughly double the initial size).

(b) Suppose we have a hash table with an initial capacity of 8 using quadratic probing. We resize the hash table by doubling the capacity.

Suppose we insert the integer keys $2^{20}, 2 \cdot 2^{20}, 3 \cdot 2^{20}, 4 \cdot 2^{20}, \ldots$ using the hash function $h(x) = x$.

Describe what the runtime of the dictionary operations will over time as you keep adding these keys to the table.

Solution:

Initially, for the first few keys, the performance of the table will be fairly reasonable.

However, as we insert each key, they will keep colliding with each other: the keys will all initially mod to index 0.

This means that as we keep inserting, each key ends up colliding with every other previously inserted key, causing all of our dictionary operations to take $\mathcal{O}(n)$ time.

However, once we resize enough times, the capacity of our table will be larger then 2^{20}, which means that our keys no longer necessarily map to the same array index. The performance will suddenly improve at that cutoff point then.

7. Hash tables

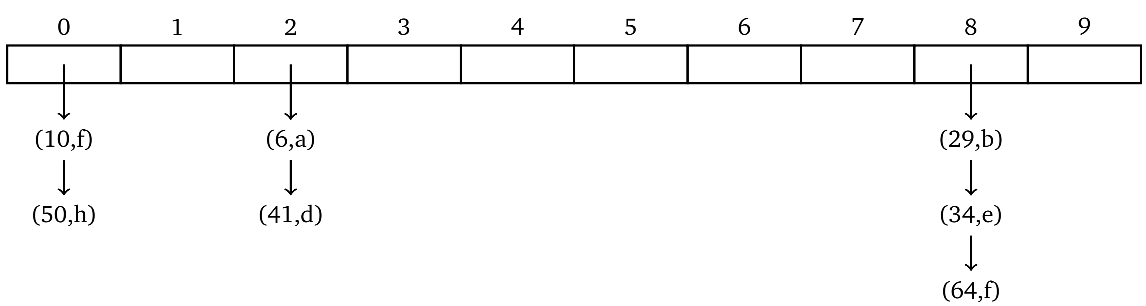
(a) Consider the following key-value pairs.

(6, a), (29, b), (41, d). (34, e), (10, f), (64, g), (50, h)

Suppose each key has a hash function $h(k) = 2k$. So, the key 6 would have a hash code of 12. Insert each key-value pair into the following hash tables and draw what their internal state looks like:

(i) A hash table that uses separate chaining. The table has an internal capacity of 10. Assume each bucket is a linked list, where new pairs are appended to the end. Do not worry about resizing.

Solution:



(ii) A hash table that uses linear probing, with internal capacity 10. Do not worry about resizing.

Solution:

0	1	2	3	4	5	6	7	8	9
(10, f)	(64, g)	(6, a)	(41, d)	(50, h)				(29, b)	(34, e)

(iii) A hash table that uses quadratic probing, with internal capacity 10. Do not worry about resizing.

Solution:

0	1	2	3	4	5	6	7	8	9
(10,f)	(50, h)	(6,a)	(41,d)				(64, g)	(29,b)	(34,e)

8. Analyzing dictionaries

(a) What are the constraints on the data types you can store in an AVL tree?

Solution:

The keys need to be orderable because AVL trees (and BSTs too) need to compare keys with each other to decide whether to go left or right at each node. (In Java, this means they need to implement `Comparable`). Unlike a hash table, the keys do *not* need to be hashable. (Note that in Java, every object is technically hashable, but it may not hash to something based on the object's value. The default hash function is based on reference equality.)

The values can be any type because AVL trees are only ordered by keys, not values.

(b) When is using an AVL tree preferred over a hash table?

Solution:

(i) You can iterate over an AVL tree in sorted order in $\mathcal{O}(n)$ time.

(ii) AVL trees never need to resize, so you don't have to worry about insertions occasionally being very slow when the hash table needs to resize.

(iii) In some cases, comparing keys may be faster than hashing them. (But note that AVL trees need to make $\mathcal{O}(\log n)$ comparisons while hash tables only need to hash each key once.)

(iv) AVL trees *may* be faster than hash tables in the worst case since they guarantee $\mathcal{O}(\log n)$, compared to a hash table's $\mathcal{O}(n)$ if every key is added to the same bucket. But remember that this only applies to pathological hash functions. In most cases, hash tables have better asymptotic runtime ($\mathcal{O}(1)$) than AVL trees, and in practice $\mathcal{O}(1)$ and $\mathcal{O}(\log n)$ have roughly the same performance.

(c) When is using a BST preferred over an AVL tree?

Solution:

One of AVL tree's advantages over BST is that it has an asymptotically efficient `find` even in the worst case.

However, if you know that `insert` will be called more often than `find`, or if you know the keys will be inserted in a random enough order that the BST will stay balanced, you may prefer a BST since it avoids the small runtime overhead of checking tree balance properties and performing rotations. (Note that this overhead is a constant factor, so it doesn't matter asymptotically, but may still affect performance in practice.)

BSTs are also easier to implement and debug than AVL trees.

(d) Consider an AVL tree with n nodes and a height of h. Now, consider a single call to `get(...)`. What's the maximum possible number of nodes `get(...)` ends up visiting? The minimum possible?

Solution:

The max number is $h + 1$ (remember that height is the number of edges, so we visit $h + 1$ nodes going from the root to the farthest away leaf); the min number is 1 (when the element we're looking for is just the root).

(e) **Challenge Problem:** Consider an AVL tree with n nodes and a height of h. Now, consider a single call to `insert(...)`. What's the maximum possible of nodes `insert(...)` ends up visiting? The minimum possible? Don't count the new node you create or the nodes visited during rotation(s).

Solution:

The max number is $h + 1$. Just like a `get`, we may have to traverse to a leaf to do an insertion.

To find the minimum number, we need to understand which elements of AVL trees we can do an insertion at, i.e. which ones have at least one `null` child.

In a tree of height 0, the root is such a node, so we need only visit the one node.
In an AVL tree of height 1, the root can still have a (single) `null` child, so again, we may be able to do an insertion visiting only one node.

On taller trees, we always start by visiting the root, then we continue the insertion process in either a tree of height $h - 1$ or a tree of height $h - 2$ (this must be the case since the the overall tree is height h and the root is balanced). Let $M(h)$ be the minimum number of nodes we need to visit on an insertion into an AVL tree of height h. The previous sentence lets us write the following recurrence

$$M(h) = 1 + \min\{M(h-1), M(h-2)\}$$

The 1 corresponds to the root, and since we want to describe the minimum needed to visit, we should take the minimum of the two subtrees.

We could simplify this recurrence and try to unroll it, but it's easier to see the pattern if we just look at the first few values:

$$M(0) = 1, M(1) = 1, M(2) = 1 + \min\{1, 1\} = 2, M(3) = 1 + \min\{1, 2\} = 2, M(4) = 1 + \min\{2, 2\} = 3$$

In general, $M()$ increases by one every other time h increases, thus we should guess the closed-form has an $h/2$ in it. Checking against small values, we can get an exactly correct closed-form of:

$$M(h) = \lfloor h/2 \rfloor + 1$$

which is our final answer.

Note that we need a very special (as empty as possible) AVL tree to have a possible insertion visiting only $\lfloor h/2 \rfloor + 1$ nodes. In general, an AVL of height h might not have an element we could insert that visits only $\lfloor h/2 \rfloor + 1$. For example, a tree where all the leaves are at depth h is still a valid AVL tree, but any insertion would need to visit $h + 1$ nodes.