LEC 06

CSE 373

Recurrences, Master Theorem

BEFORE WE START

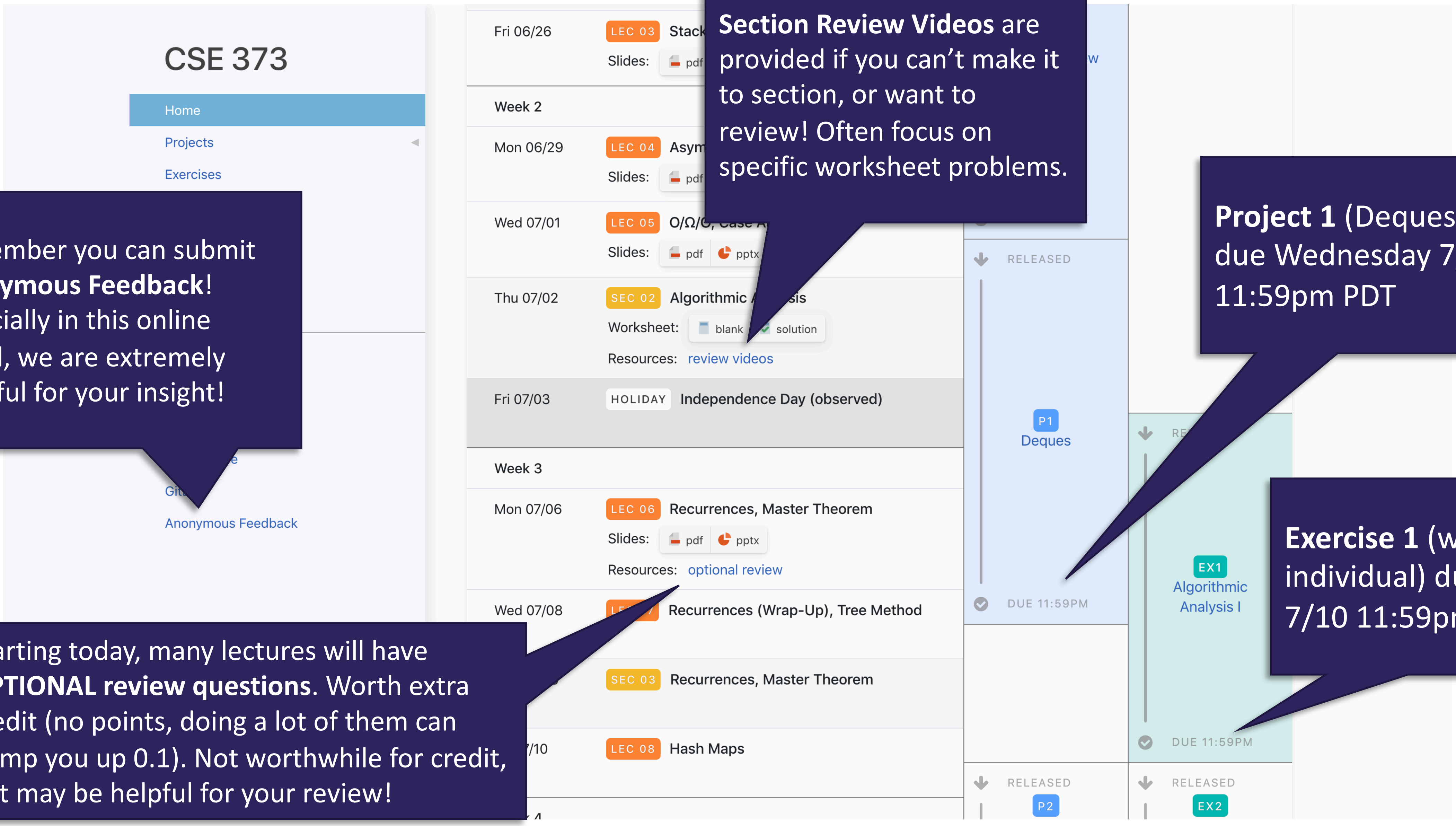
Review: Which of the following are evidence that a Big-Theta exists?

a) Big-Oh == Big-Theta
b) We're analyzing a function that can be fully expressed as a polynomial
c) There aren't extra terms (e.g. $n^2 + n$)
d) Runtime isn't affected by array contents

pollev.com/uwcse373

Instructor	Aaron Johnston	
TAs	Timothy Akintilo Brian Chan Joyce Elauria Eric Fan Siddharth Vaidyanathan	Farrell Fileas Leona Kazi Keanu Vestil Howard Xiao

Announcements



CSE 373

Home

Projects

Exercises

Git

Anonymous Feedback

Fri 06/26 LEC 03 Stack
Slides: pdf

Week 2

Mon 06/29 LEC 04 Asym
Slides: pdf

Wed 07/01 LEC 05 O/Ω/Θ, Case A
Slides: pdf pptx

Thu 07/02 SEC 02 Algorithmic A sis
Worksheet: blank solution
Resources: review videos

Fri 07/03 HOLIDAY Independence Day (observed)

Week 3

Mon 07/06 LEC 06 Recurrences, Master Theorem
Slides: pdf pptx
Resources: optional review

Wed 07/08 LEC 07 Recurrences (Wrap-Up), Tree Method

SEC 03 Recurrences, Master Theorem

/10 LEC 08 Hash Maps

RELEASED
P1 Deques
DUE 11:59PM

EX1 Algorithmic Analysis I
DUE 11:59PM

RELEASED P2

RELEASED EX2

Section Review Videos are provided if you can't make it to section, or want to review! Often focus on specific worksheet problems.

Remember you can submit **Anonymous Feedback**! Especially in this online world, we are extremely grateful for your insight!

Project 1 (Deques) due Wednesday 7/8 11:59pm PDT

Exercise 1 (written, individual) due Friday 7/10 11:59pm PDT

Starting today, many lectures will have **OPTIONAL review questions**. Worth extra credit (no points, doing a lot of them can bump you up 0.1). Not worthwhile for credit, but may be helpful for your review!

Learning Objectives

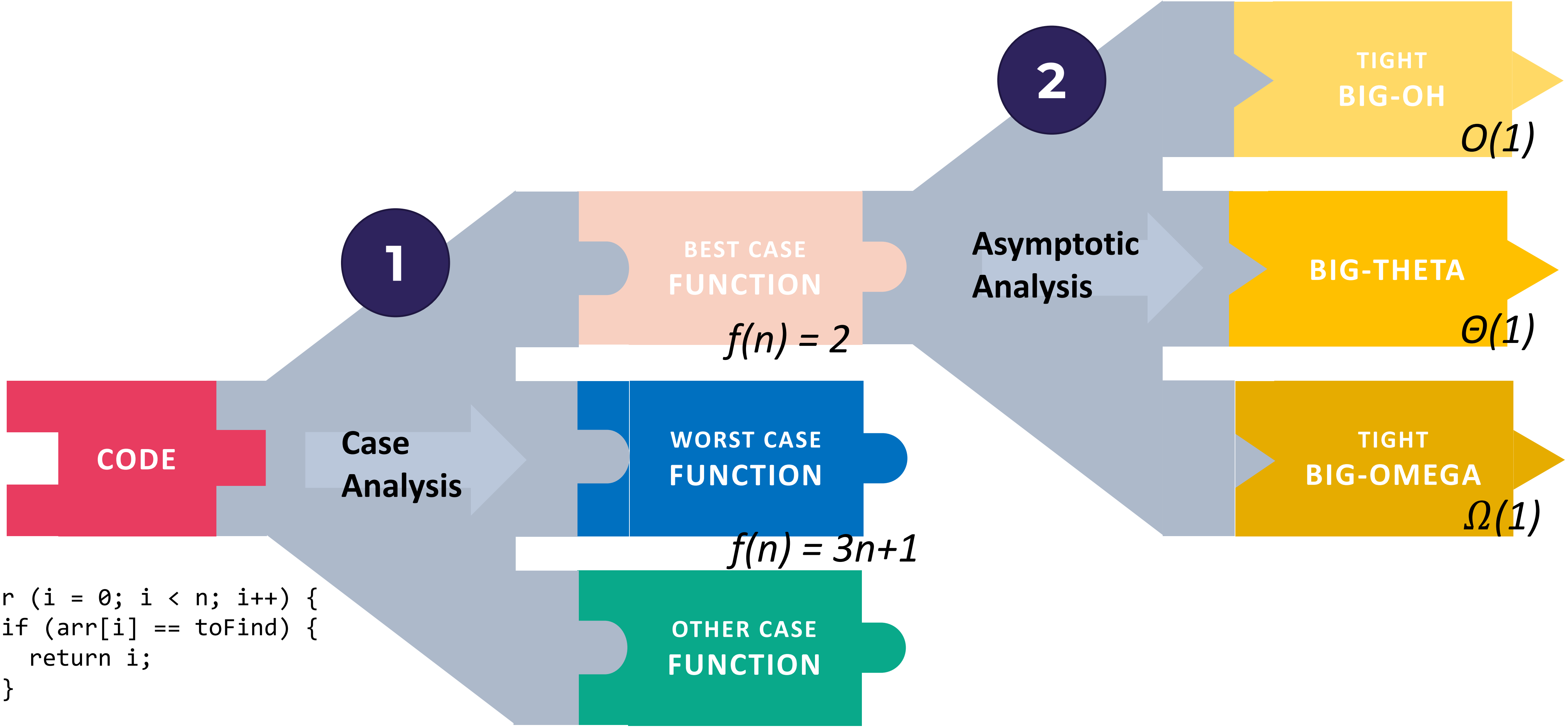
After this lecture, you should be able to...

1. *Review* Distinguish between Asymptotic Analysis & Case Analysis, and apply both to code snippets
2. Describe the 3 most common recursive patterns and identify whether code belongs to one of them
3. Model recursive code using a recurrence relation (Step 1)
4. Use the Master Theorem to characterize a recurrence relation with Big-Oh/Big-Theta/Big-Omega bounds (Step 2)

Lecture Outline

- *Review* Asymptotic Analysis & Case Analysis
- Analyzing Recursive Code

Review Algorithmic Analysis Roadmap



```
for (i = 0; i < n; i++) {
  if (arr[i] == toFind) {
    return i;
  }
}
return -1;
```

Review Oh, and Omega, and Theta, oh my

- Big-Oh is an **upper bound**
  - My code takes at most this long to run
- Big-Omega is a **lower bound**
  - My code takes at least this long to run
- Big Theta is **"equal to"**
  - My code takes "exactly"* this long to run
  - *Except for constant factors and lower order terms
  - Only exists when Big-Oh == Big-Omega!

Big-Oh

$f(n)$ is $O(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \leq c \cdot g(n)$$

Big-Omega

$f(n)$ is $\Omega(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \geq c \cdot g(n)$$

Big-Theta

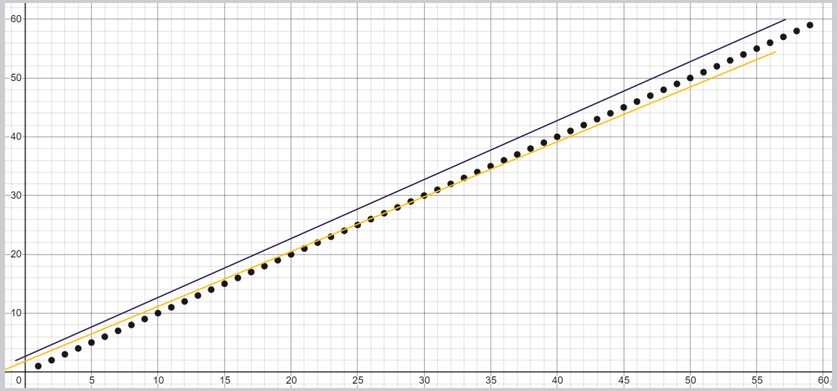
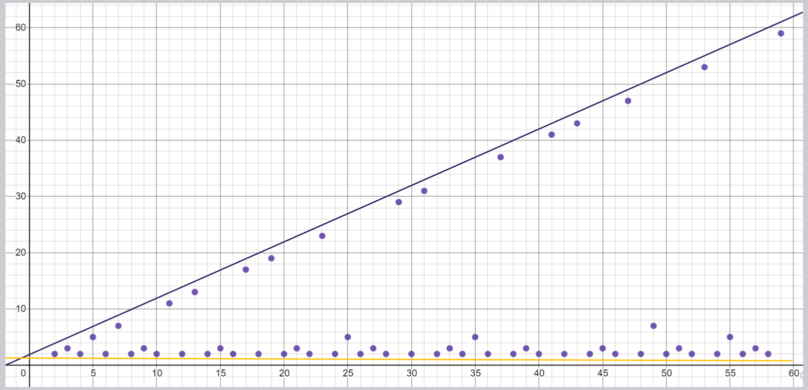
$f(n)$ is $\Theta(g(n))$ if
$f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

(in other words: there exist positive constants $c1, \mathrm{c}2, n_0$ such that for all $n \geq n_0$)

$$\mathrm{c}_1 \cdot g(n) \leq f(n) \leq \mathrm{c}_2 \cdot g(n)$$

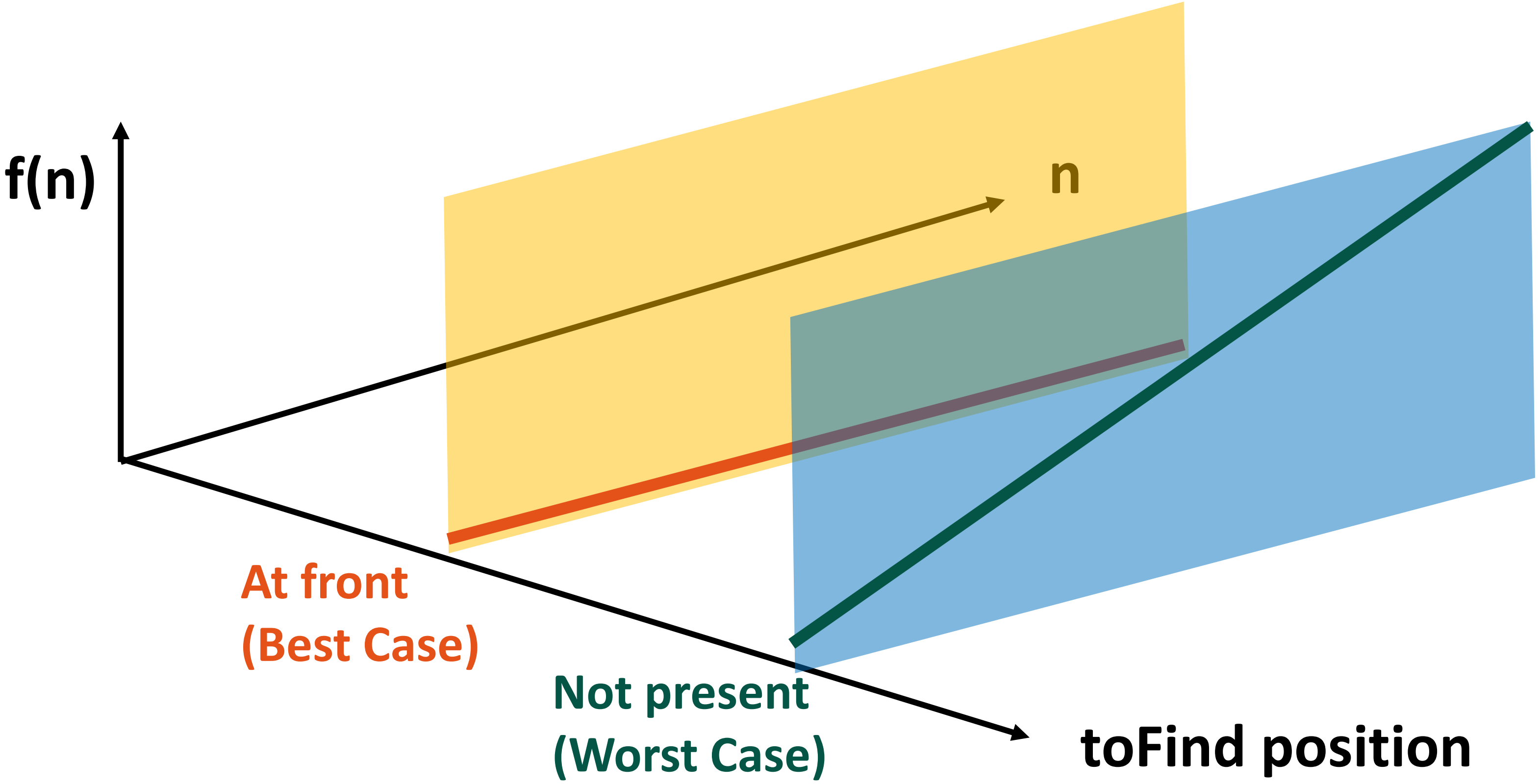
A Note on Asymptotic Analysis Tools

- We’ll generally use Big-Theta from here on out: most specific
- In industry, people often use Big-Oh to mean “Tight Big-Oh” and use it even when a Big-Theta exists

When to use Big-Theta (most of the time):	When you have to use Big-Oh/Big-Omega:
for any function that's just the sum of its terms like f(n) = 2^n + 3n^3 + 4n - 5 we can always just do the approach of dropping constant multipliers / removing the lower order terms to find the big-Theta at a glance.  	f(n) { n if n is prime, 1 otherwise} since in this case, the big-Oh (n) and the big-Omega (1) don't overlap at the same complexity class, there is no reasonable big-Theta and we couldn't use it here. 

Review When to do Case Analysis?

- Imagine a 3-dimensional plot
  - Which case we're considering is one dimension
  - Choosing a case lets us take a "slice" of the other dimensions: n and f(n)
  - We do asymptotic analysis on each slice in step 2

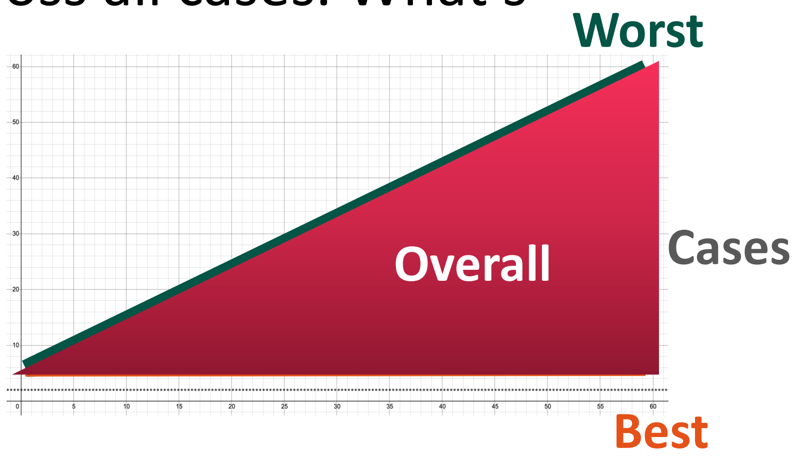


Review How to do Case Analysis

1. Are there significantly different cases?
   - Do other variables/parameters/fields affect the runtime, other than input size? For many algorithms, the answer is no.
2. Figure out how things could change depending on the input (excluding n, the input size)
   - Can you exit loops early?
   - Can you return early?
   - Are some branches much slower than others?
3. Determine what inputs could cause you to hit the best/worst parts of the code.

Other Useful Cases You Might See

- Overall Case:
  - Model code as a "cloud" that covers all *possibilities* across all cases. What's the O/Ω/Θ of that cloud?
- "Assume X Won't Happen Case":
  - E.g. Assume array won't need to resize
- "Average Case":
  - Assume random input
  - Lots of complications – what distribution of random?
- "In-Practice Case":
  - Not a real term, but a useful idea
  - Make reasonable assumptions about how the world will work, then do worst-case analysis under those assumptions.



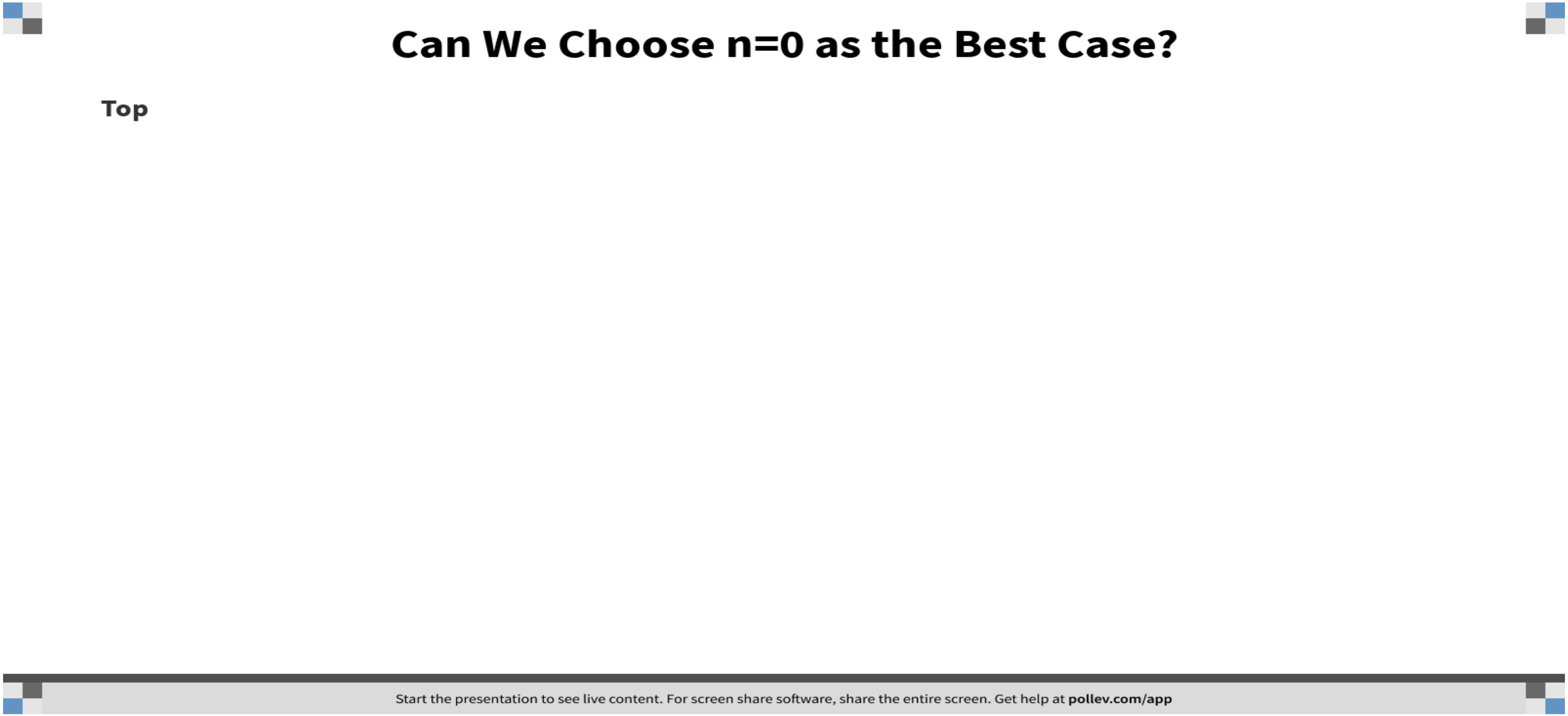
How Can You Tell if Best/Worst Cases Exist?

- Are there other possible models for this code?
- **If n is given, are there still other factors that determine the runtime?**

- Note: sometimes there aren't significantly different cases! Sometimes we just want to model the code with a single function and go straight to asymptotic analysis!

Poll Everywhere

pollev.com/uwcse373



Can We Choose n=0 as the Best Case?

Top

Start the presentation to see live content. For screen share software, share the entire screen. Get help at **pollev.com/app**

Can We Choose n=0 as the Best Case?

- Remember that each case needs to be a “slice”: a function over n
  - The input to asymptotic analysis is a function over all of n, because we’re concerned with growth rate
  - Fixing n doesn’t work with our tools because it wouldn’t let us examine the bound asymptotically

- Think of it as “Best Case as n grows infinitely large”, not “Best Case of all inputs, including n”

Lecture Outline

- *Review* Asymptotic Analysis & Case Analysis
- **Analyzing Recursive Code**

Recursive code usually falls into one of 3 common patterns:

1

Halving the Input

Binary Search

2

Constant size Input

3

Doubling the Input

Case Study: Binary Search

```
public int binarySearch(int[] arr, int toFind, int lo, int hi) {
 if (hi < lo) {
  return -1;
 } else if (hi == lo) {
  if (arr[hi] == toFind) {
   return hi;
  }
  return -1;
 }

 int mid = (lo + hi) / 2;
 if (arr[mid] == toFind) {
  return mid;
 } else if (arr[mid] < toFind) {
  return binarySearch(arr, toFind, mid+1, hi);
 } else {
  return binarySearch(arr, toFind, lo, mid-1);
 }
}
```

Base Cases

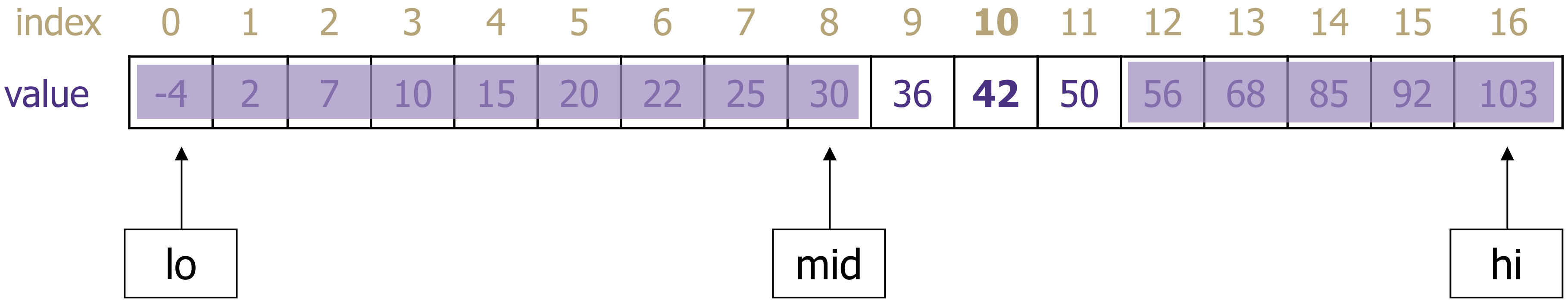
Recursive Cases

Note: the parameters passed to recursive call *reduce* the size of the problem!

Binary Search Runtime

Binary search: An algorithm to find a target value in a *sorted* array or list by successively eliminating half of the array from consideration.

- Example: Searching the array below for the value **42**:



index	0	1	2	3	4	5	6	7	8	9	**10**	11	12	13	14	15	16
value	-4	2	7	10	15	20	22	25	30	36	**42**	50	56	68	85	92	103

Let’s consider the runtime of Binary Search

What’s the first step?

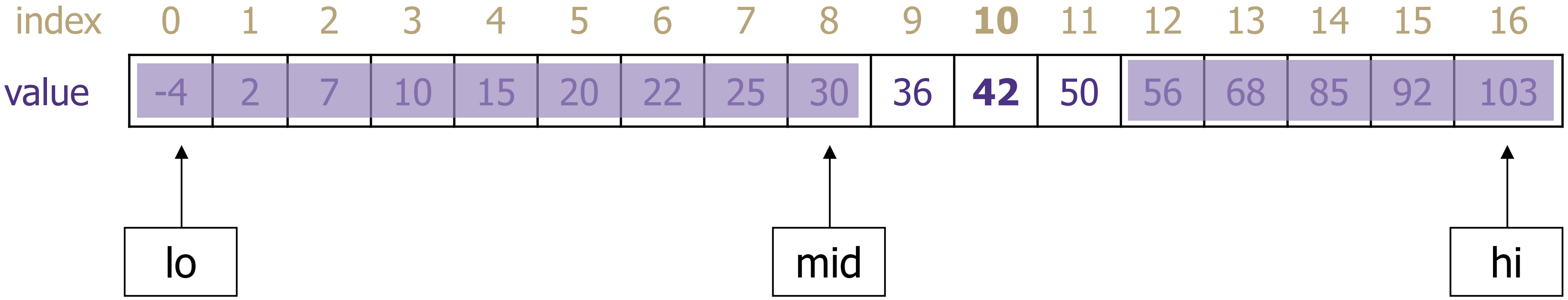
Poll Everywhere

pollev.com/uwcse373

Binary Search Runtime

Binary search: An algorithm to find a target value in a *sorted* array or list by successively eliminating half of the array from consideration.

- Example: Searching the array below for the value **42**:



index	0	1	2	3	4	5	6	7	8	9	**10**	11	12	13	14	15	16
value	-4	2	7	10	15	20	22	25	30	36	**42**	50	56	68	85	92	103

What's the Best Case?

Element found at first index examined (index 8) **Θ(1)**

What's the Worst Case?

Element not found, cut input in half, then in half again... **???**

1 **Halving the Input**

Binary Search Runtime

- For an array of size n, eliminate ½ until 1 element remains.

  n, n/2, n/4, n/8, ..., 4, 2, 1

  - How many divisions does that take?

- Think of it from the other direction:
  - How many times do I have to multiply by 2 to reach n?

    1, 2, 4, 8, ..., n/4, n/2, n
  - Call this number of multiplications "x".

    $2^x = n$

    $\mathbf{x = \log_2 n}$

- Binary search is in the **logarithmic** complexity class.

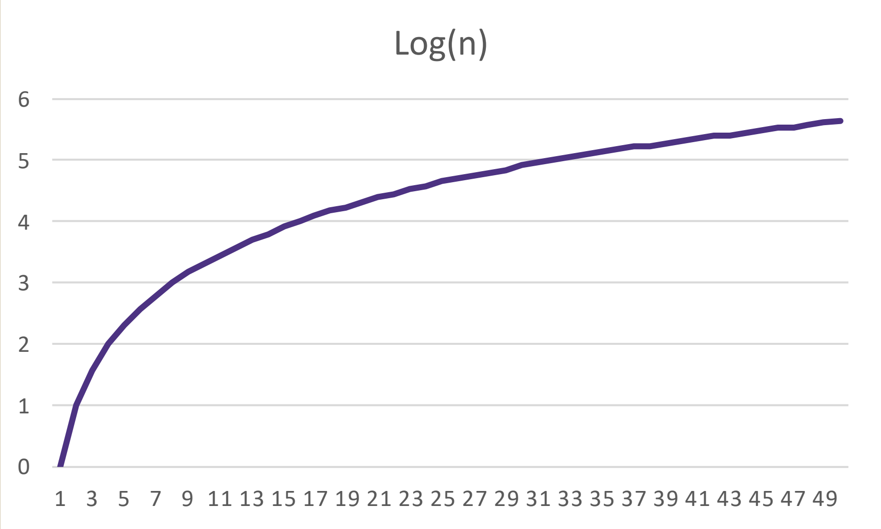
Logarithm – inverse of exponentials

$y = \log_b x \textit{ is equal to } b^y = x$

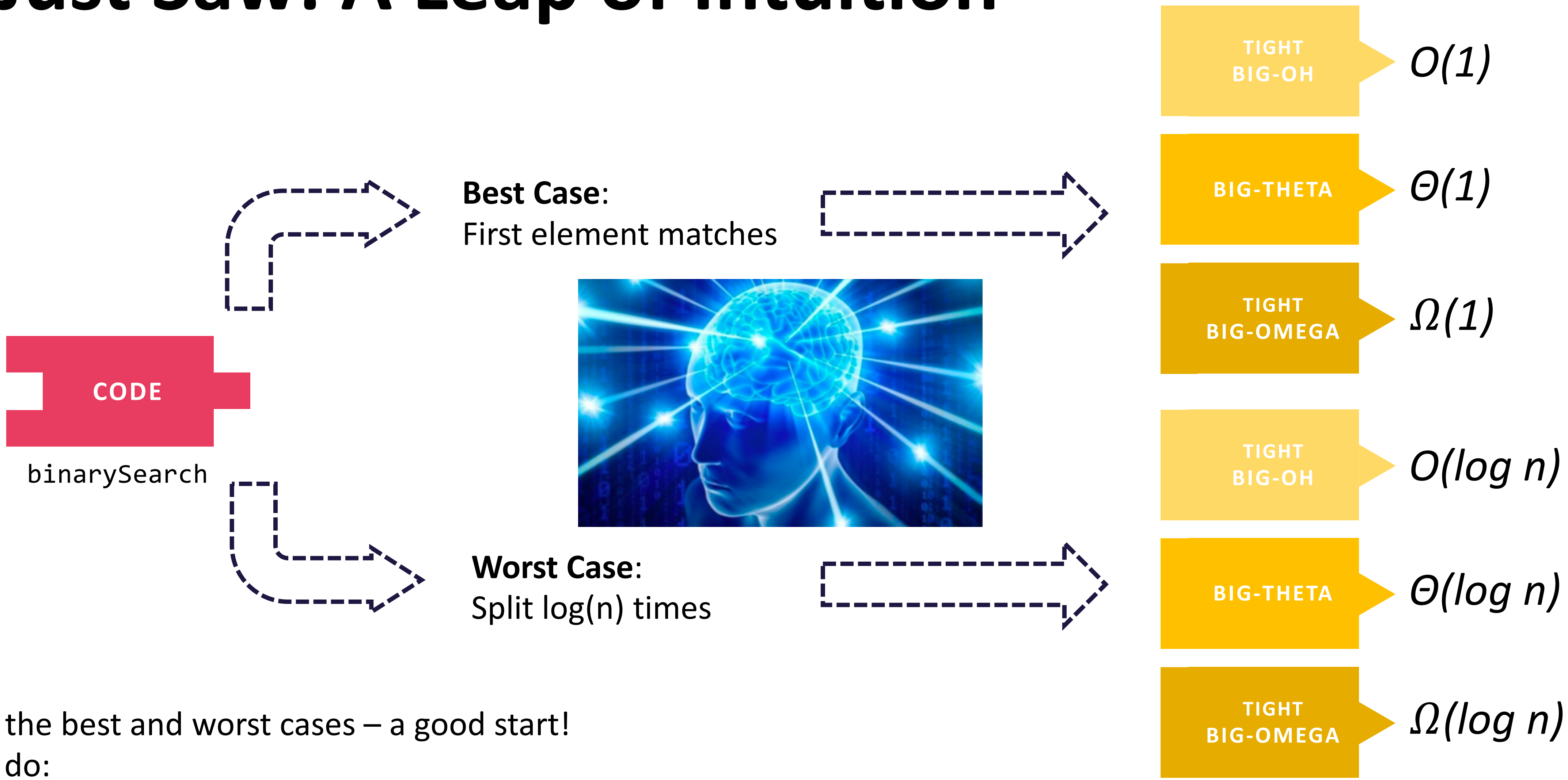
Examples:

$2^2 = 4 \Rightarrow 2 = \log_2 4$

$3^2 = 9 \Rightarrow 2 = \log_3 9$



We Just Saw: A Leap of Intuition



CODE

binarySearch

Best Case:
First element matches

Worst Case:
Split log(n) times

TIGHT BIG-OH $O(1)$

BIG-THETA $\Theta(1)$

TIGHT BIG-OMEGA $\Omega(1)$

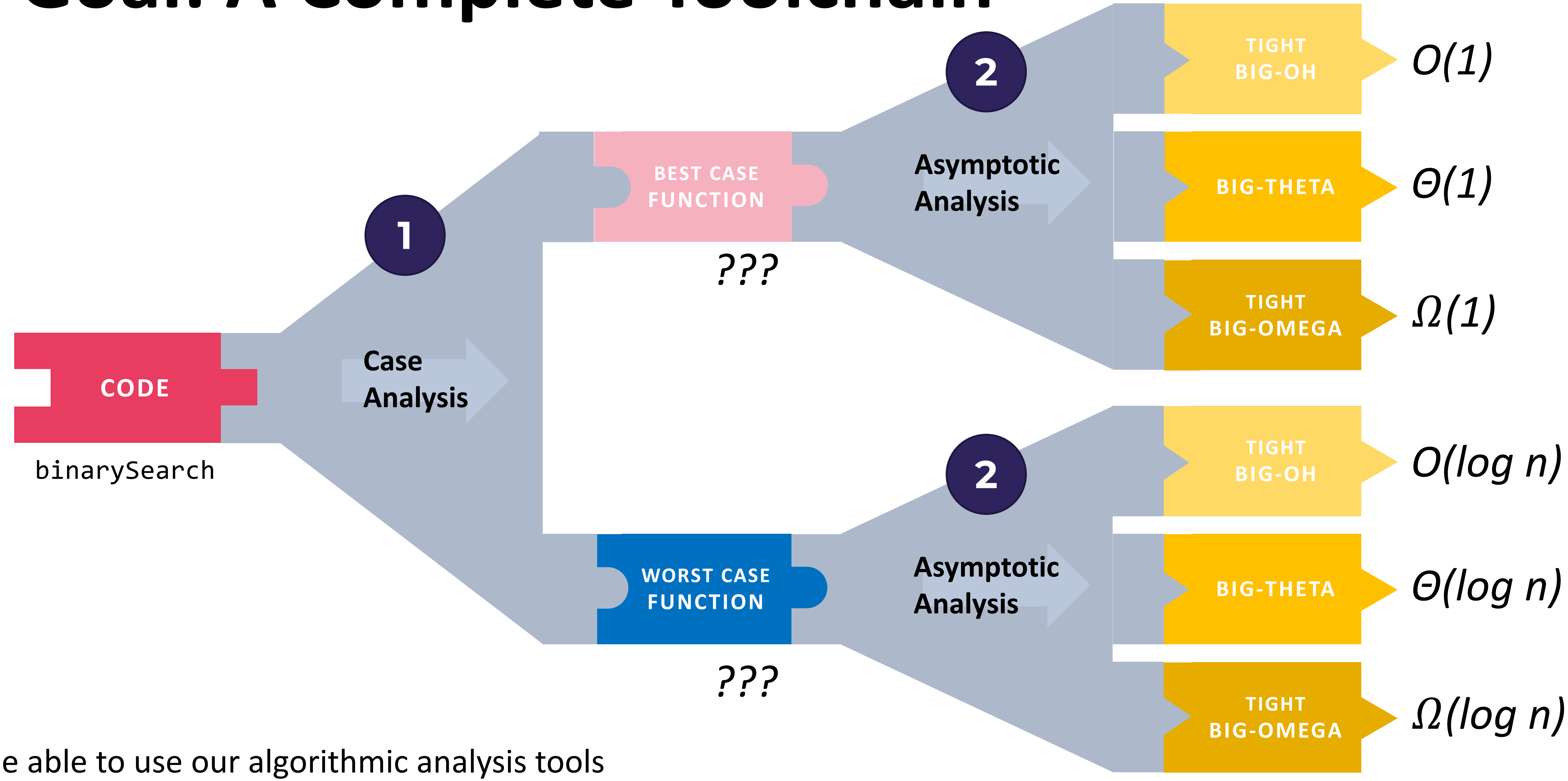
TIGHT BIG-OH $O(\log n)$

BIG-THETA $\Theta(\log n)$

TIGHT BIG-OMEGA $\Omega(\log n)$

- We identified the best and worst cases – a good start!
- But we didn't do:
  - Step 1: model the code as a function
  - Step 2: analyze that function to find its bounds

Our Goal: A Complete Toolchain



- We want to be able to use our algorithmic analysis tools
- To do that, we need an essential intermediate: to model the code with runtime functions

Modeling Binary Search

```
public int binarySearch(int[] arr, int toFind, int lo, int hi) {
  if (hi < lo) {
    return -1;
  } else if (hi == lo) {
    if (arr[hi] == toFind) {
      return hi;
    }
    return -1;
  }

  int mid = (lo + hi) / 2;
  if (arr[mid] == toFind) {
    return mid;
  } else if (arr[mid] < toFind) {
    return binarySearch(arr, toFind, mid+1, hi);
  } else {
    return binarySearch(arr, toFind, lo, mid-1);
  }
}
```

+2 Base Case

+4 Base Case

+6

Recursive Case

???

How do we model a recursive call?

Fortunately, we have a tool for this!

Meet the Recurrence

A **recurrence** relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term(s)

It’s a lot like recursive code:

- At least one base case and at least one recursive case
- Each case should include the values for n to which it corresponds
- The recursive case should reduce the input size in a way that eventually triggers the base case
- The cases of your recurrence usually correspond exactly to the cases of the code

A generic example of a recurrence:

$$T(n) = \begin{cases} 5 & \text{if } n < 3 \\ 2T\left(\frac{n}{2}\right) + 10 & \text{otherwise} \end{cases}$$

Writing Recurrences: Example 1

```
public int recurse(int n) {
  if (n < 3) {
    return 80;
  }

  int a = n * 2;

  int val1 = recurse(n / 3);
  int val2 = recurse(n / 3);

  return val1 + val2;
}
```

+2 Base Case

+2

+2

Recursive Case

Non-recursive Work: **+4**

Recursive Work: **+ 2*T(n/3)**

$$T(n) = \begin{cases} 2 & \text{if } n < 3 \\ 2T\left(\frac{n}{3}\right) + 4 & \text{otherwise} \end{cases}$$

Writing Recurrences: Example 2

```
public int recurse(int n) {
  if (n < 3) {
    return 80;
  }

  for (int i = 0; i < n; i++) {
    System.out.println(i);
  }

  int val1 = recurse(n / 3);
  int val2 = recurse(n / 3);

  return val1 + val2;
}
```

+2 Base Case

+n

+2

Recursive Case

Non-recursive Work: **+ n + 2**

Recursive Work: **+ 2*T(n/3)**

$$T(n) = \begin{cases} 2 & \text{if } n < 3 \\ 2T\left(\frac{n}{3}\right) + n & \text{otherwise} \end{cases}$$

Writing Recurrences: Example 3

```
public int recurse(int n) {
  if (n < 3) {
    return 80;
  }

  for (int i = 0; i < n; i++) {
    System.out.println(i);
  }

  int val1 = recurse(n / 3);
  int val2 = recurse(n / 3);
  int val3 = recurse(n / 3);


  return val1 + val2 + val3;
}
```

+2 Base Case

+n

+3

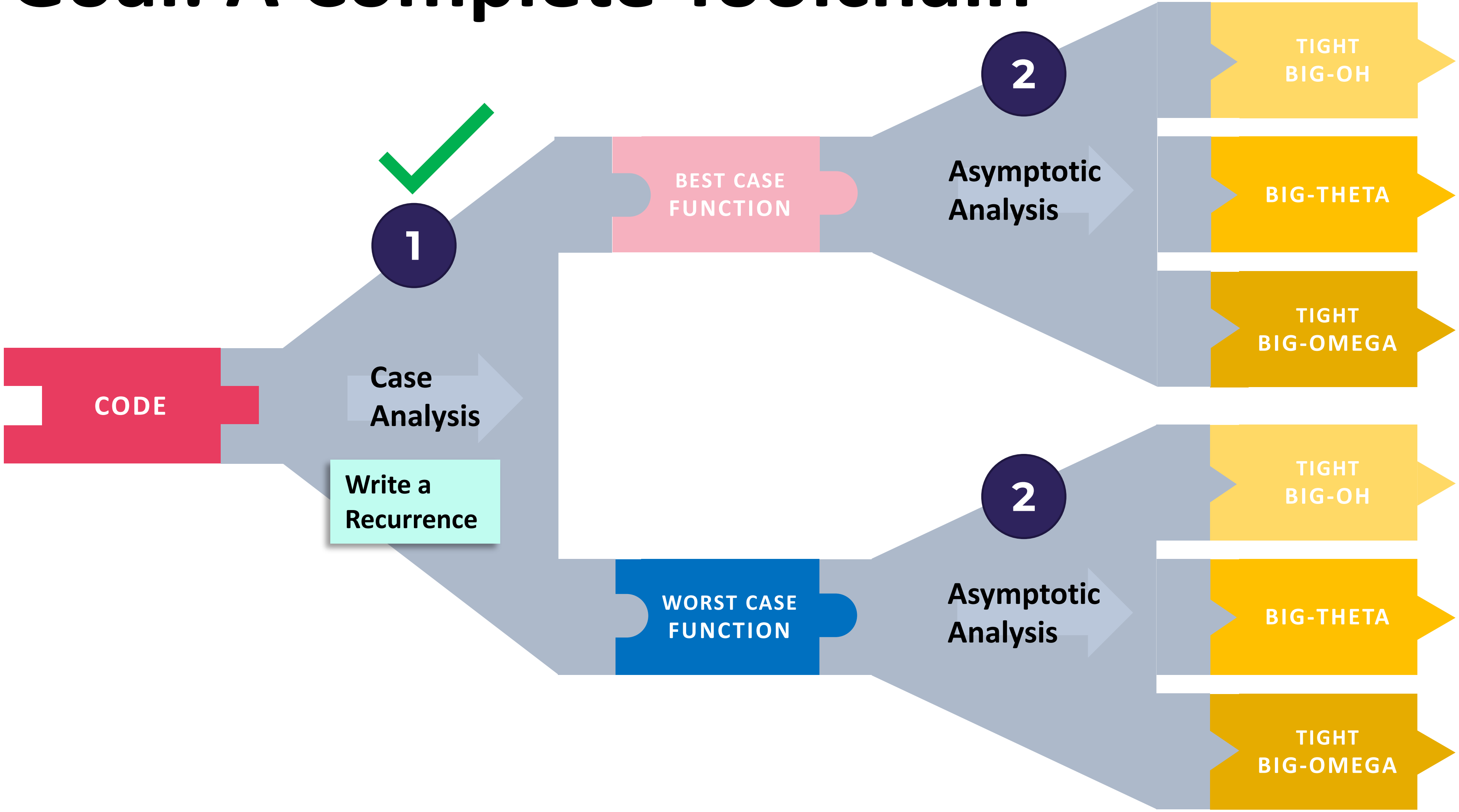
Recursive Case

Non-recursive Work: **+ n + 3**

Recursive Work: **+ 3*T(n/3)**

$$T(n) = \begin{cases} 2 & \text{if } n < 3 \\ 3T\left(\frac{n}{3}\right) + n & \text{otherwise} \end{cases}$$

Our Goal: A Complete Toolchain



Recurrence to Big-Θ

$$T(n) = \begin{cases} 2 & \text{if } n < 3 \\ 2T\left(\frac{n}{3}\right) + n & \text{otherwise} \end{cases}$$

- It's still really hard to tell what the big-O is just by looking at it.
- But fancy mathematicians have a formula for us to use!

MASTER THEOREM

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a < c$ then $T(n) \in \Theta(n^c)$

If $\log_b a = c$ then $T(n) \in \Theta(n^c \log n)$

If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

a=2 b=3 and c=1

$y = \log_b x$ *is equal to* $b^y = x$

$\log_3 2 = x \ (3^x = 2 \Rightarrow x \ \cong 0.63)$

$\log_3 2 < 1$

We're in case 1

$T(n) \in \Theta(n)$

Aside Understanding the Master Theorem

MASTER THEOREM

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If	$\log_b a < c$	then	$T(n) \in \Theta(n^c)$
If	$\log_b a = c$	then	$T(n) \in \Theta(n^c \log n)$
If	$\log_b a > c$	then	$T(n) \in \Theta(n^{\log_b a})$

- A measures how many recursive calls are triggered by each method instance
- B measures the rate of change for input
- C measures the dominating term of the non recursive work within the recursive method
- D measures the work done in the base case

- The $\log_b a < c$ case
  - Recursive case does a lot of non recursive work in comparison to how quickly it divides the input size
  - Most work happens in beginning of call stack
  - Non recursive work in recursive case dominates growth, n^c term
- The $\log_b a = c$ case
  - Recursive case evenly splits work between non recursive work and passing along inputs to subsequent recursive calls
  - Work is distributed across call stack
- The $\log_b a > c$ case
  - Recursive case breaks inputs apart quickly and doesn’t do much non recursive work
  - Most work happens near bottom of call stack

Lecture Outline

- *Review* Asymptotic Analysis & Case Analysis
- **Analyzing Recursive Code**

Recursive code usually falls into one of 3 common patterns:

1

Halving the Input

Binary Search

Θ (log n)

2

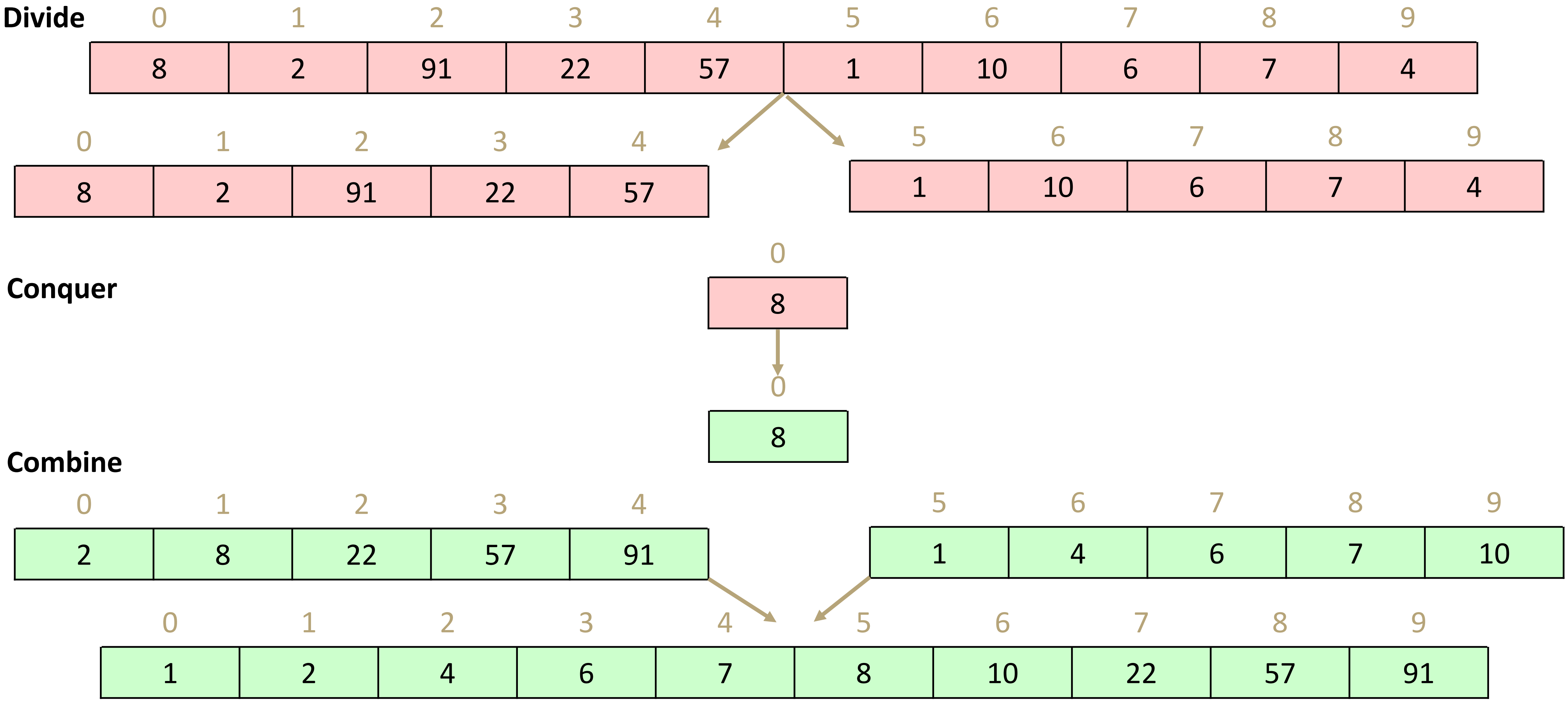
Constant size Input

Merge Sort

3

Doubling the Input

Merge Sort



Divide

0	1	2	3	4	5	6	7	8	9
8	2	91	22	57	1	10	6	7	4

0	1	2	3	4
8	2	91	22	57

5	6	7	8	9
1	10	6	7	4

Conquer

0
8

0
8

Combine

0	1	2	3	4
2	8	22	57	91

5	6	7	8	9
1	4	6	7	10

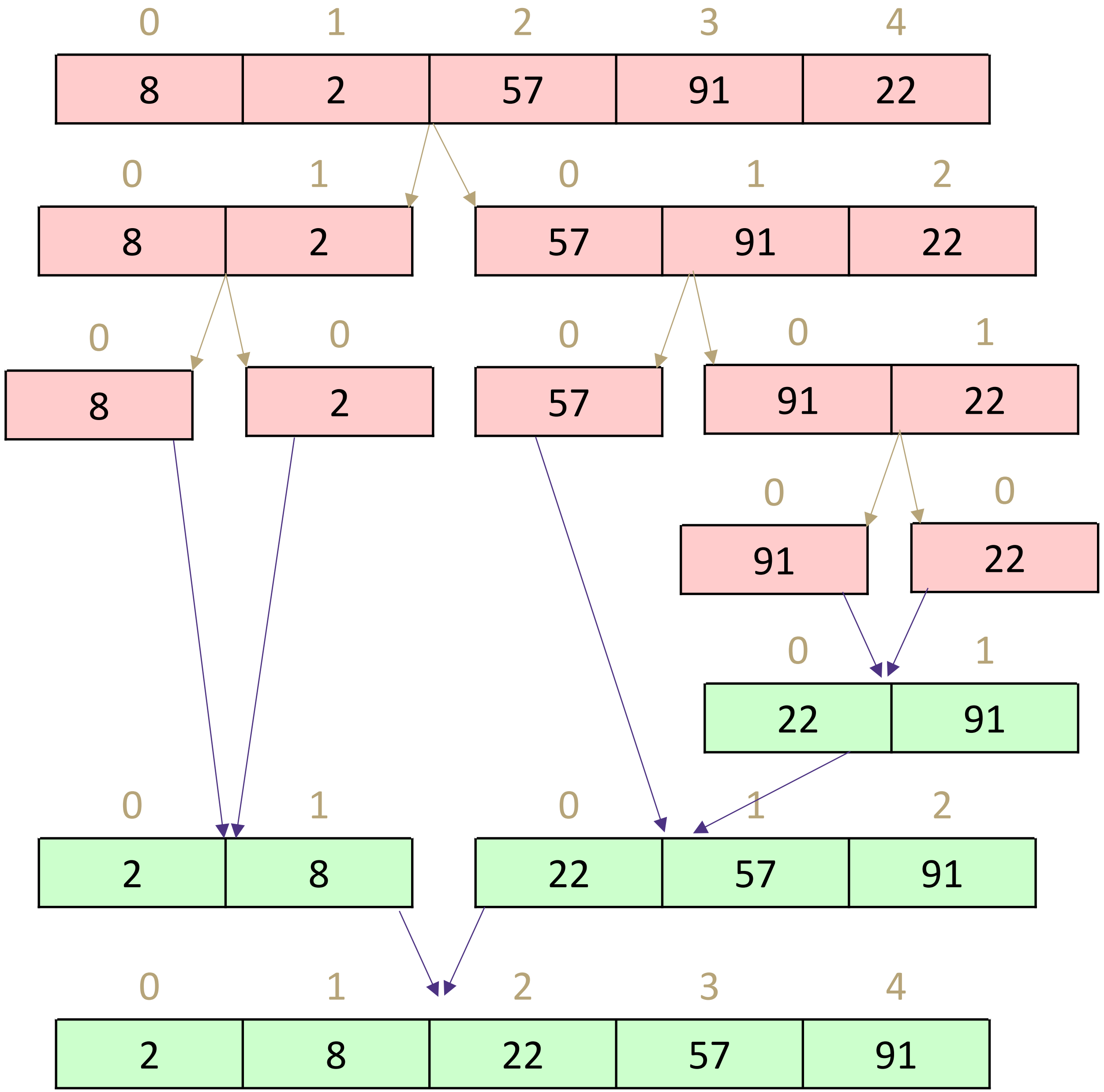
0	1	2	3	4	5	6	7	8	9
1	2	4	6	7	8	10	22	57	91

Merge Sort

```
mergeSort(input) {
   if (input.length == 1)
      return
   else
      smallerHalf = mergeSort(new [0, ..., mid])
      largerHalf = mergeSort(new [mid + 1, ...])
      return merge(smallerHalf, largerHalf)
}
```

$$T(n) = \begin{cases} 1 & \text{if } n \leq 3 \\ 2T\left(\frac{n}{2}\right) + n & \text{otherwise} \end{cases}$$

2 **Constant size Input**

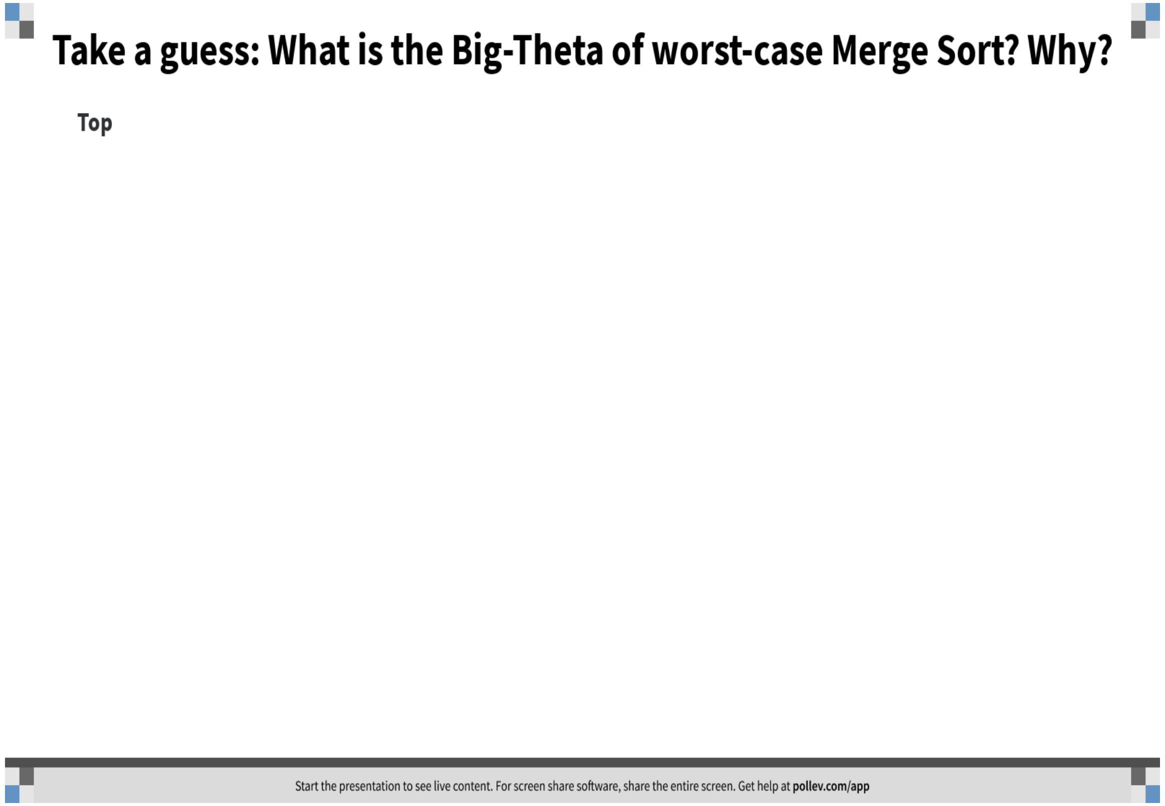


Poll Everywhere

pollev.com/uwcse373

Take a guess: What is the Big-Theta of worst-case Merge Sort?

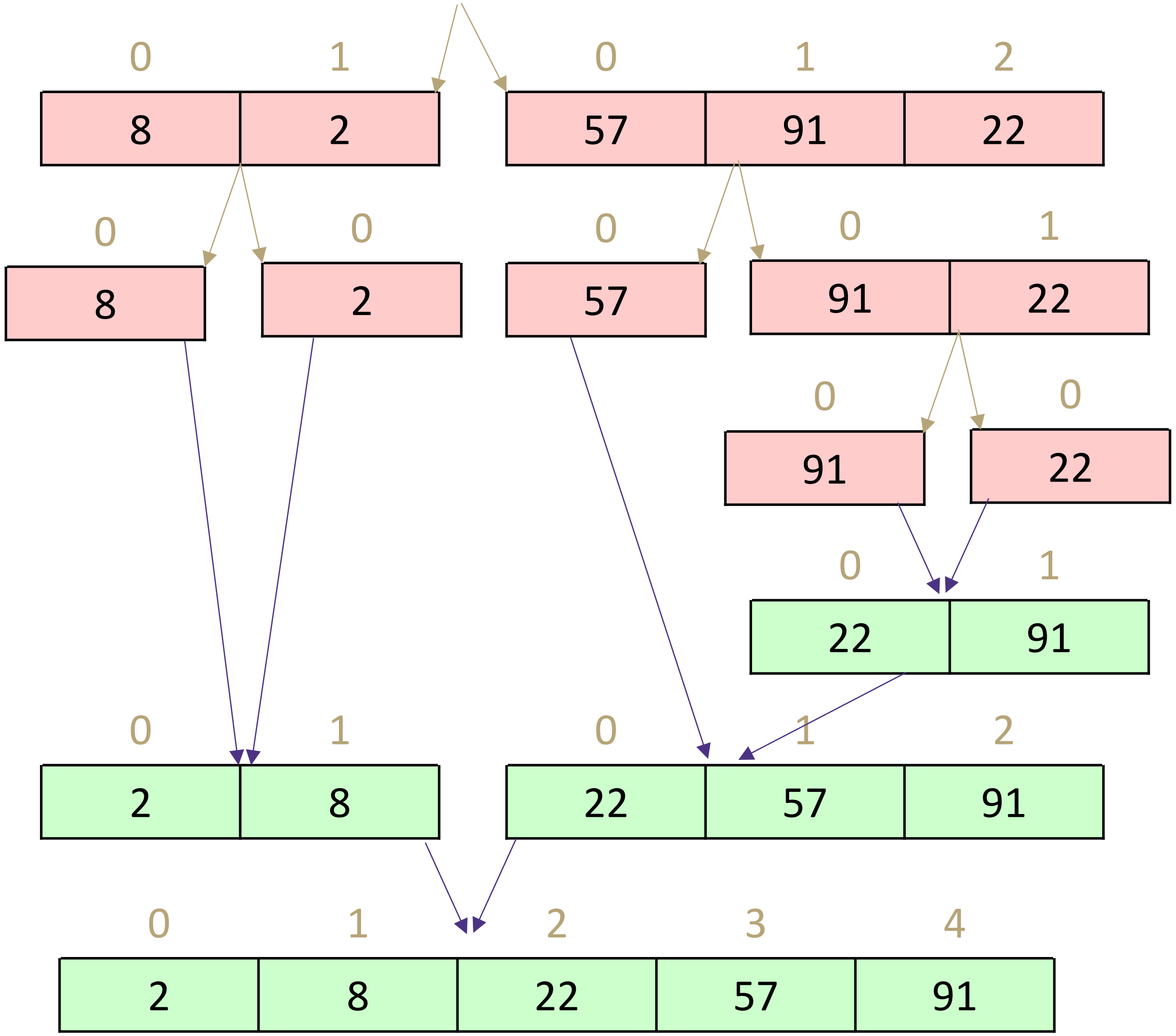
$$T(n) = \begin{cases} 1 & \text{if } n \leq 3 \\ 2T\left(\frac{n}{2}\right) + n & \text{otherwise} \end{cases}$$



Take a guess: What is the Big-Theta of worst-case Merge Sort? Why?

Top

Start the presentation to see live content. For screen share software, share the entire screen. Get help at pollev.com/app



Merge Sort Recurrence to Big-Θ

$$T(n) = \begin{cases} 1 & \text{if } n \leq 3 \\ 2T\left(\frac{n}{2}\right) + n & \text{otherwise} \end{cases}$$

MASTER THEOREM

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a < c$ then $T(n) \in \Theta(n^c)$

If $\log_b a = c$ then $T(n) \in \Theta(n^c \log n)$

If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

a=2 b=2 and c=1

$y = \log_b x \ \text{is equal to}\ b^y = x$

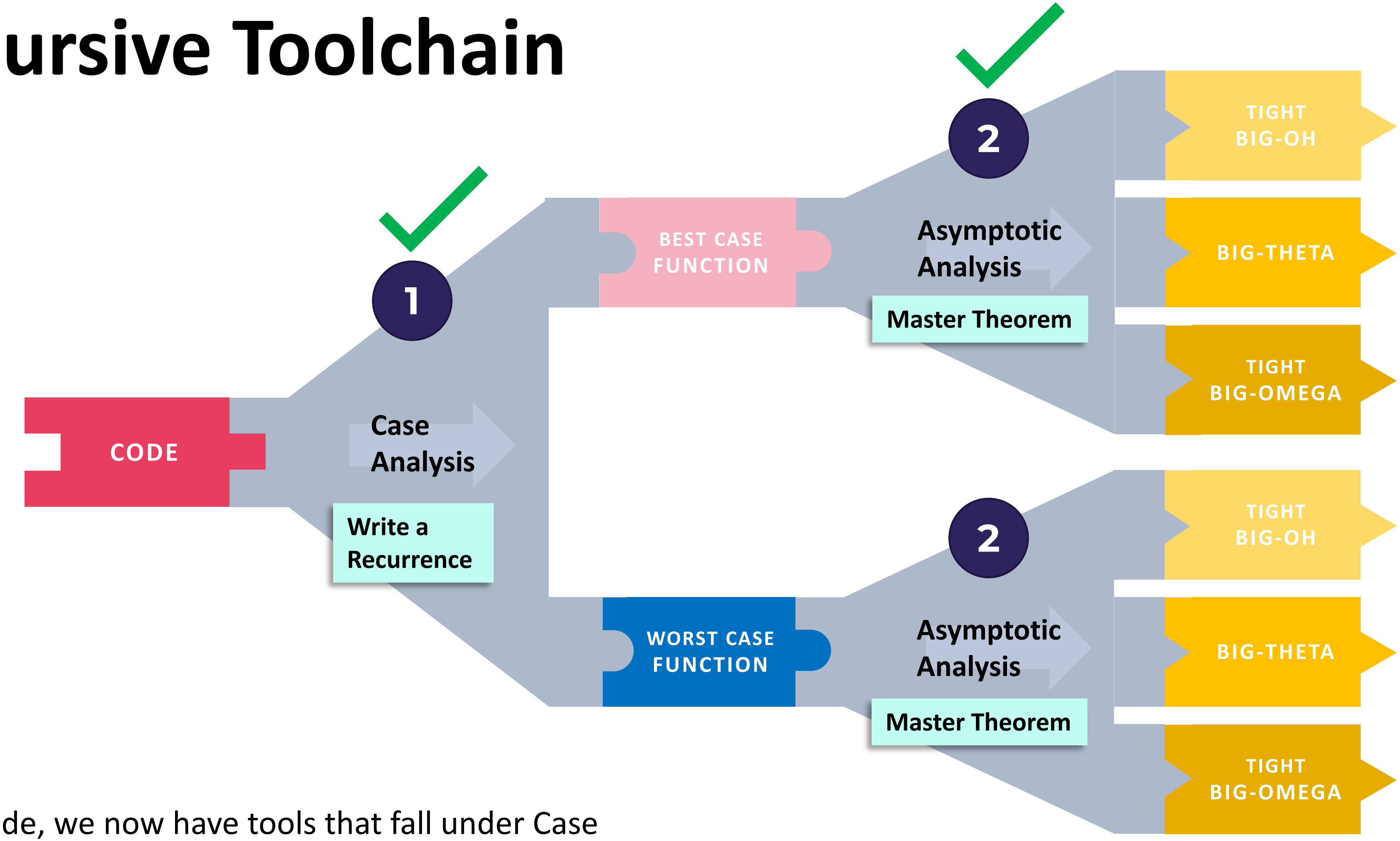
$\log_2 2 = x \Rightarrow 2^x = 2 \Rightarrow x = 1$

$\log_2 2 = 1$

We're in case 2

$T(n) \in \Theta(n \log n)$

Recursive Toolchain



For recursive code, we now have tools that fall under Case Analysis (Writing Recurrences) and Asymptotic Analysis (The Master Theorem).

Lecture Outline

- *Review* Asymptotic Analysis & Case Analysis
- **Analyzing Recursive Code**

Recursive code usually falls into one of 3 common patterns:

1

Halving the Input

Binary Search
Θ (log n)

2

Constant size Input

Merge Sort
Θ (n log n)

3

Doubling the Input

Fibonacci

Lecture Outline

- *Review* Asymptotic Analysis & Case Analysis
- **Analyzing Recursive Code**

Recursive code usually falls into one of 3 common patterns:

1

Halving the Input

Binary Search
Θ (log n)

2

Constant size Input

Merge Sort
Θ (n log n)

3

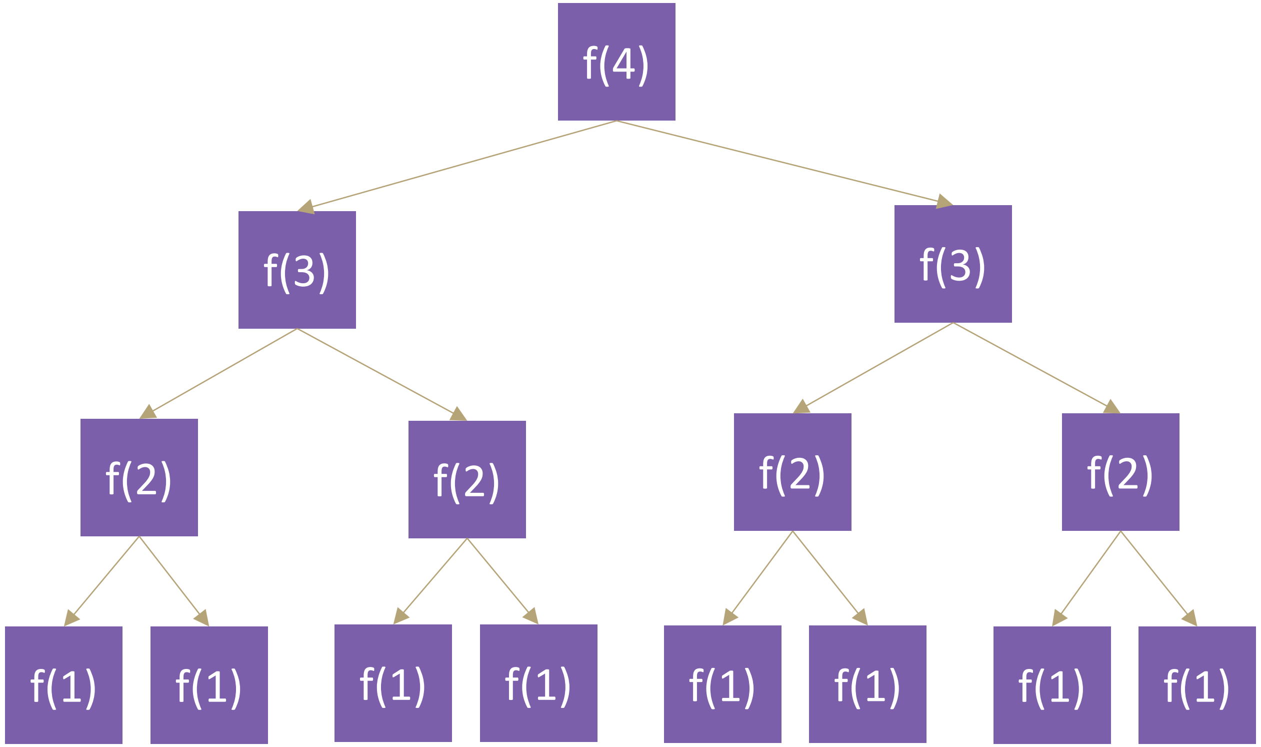
at

ibonacci

NEXT LECTURE

Calculating Fibonacci

```
public int fib(int n) {
    if (n <= 1) {
        return 1;
    }
    return fib(n-1) + fib(n-1);
}
```



- Each call creates 2 more calls
- Each new call has a copy of the input, almost
- Almost doubling the input at each call

3 **Doubling the Input**

Almost

Fibonacci Recurrence to Big-Θ

```
public int fib(int n) {
    if (n <= 1) {
        return 1;
    }
    return fib(n-1) + fib(n-1);
}
```

d

2T(n-1) + c

$$T(n) = \begin{cases} d & \text{if } n \leq 1 \\ 2T(n-1) + c & \text{otherwise} \end{cases}$$

Can we use the Master Theorem?

MASTER THEOREM

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Uh oh, our model doesn’t match that format…

Can we intuit a pattern?

T(1) = d

T(2) = 2T(2-1) + c = 2(d) + c

T(3) = 2T(3-1) + c = 2(2(d) + c) + c = 4d + 3c

T(4) = 2T(4-1) + c = 2(4d + 3c) + c = 8d + 7c

T(5) = 2T(5-1) + c = 2(8d + 7c) + c = 16d +25c

Looks like something’s happening but it’s tough

Maybe geometry can help!

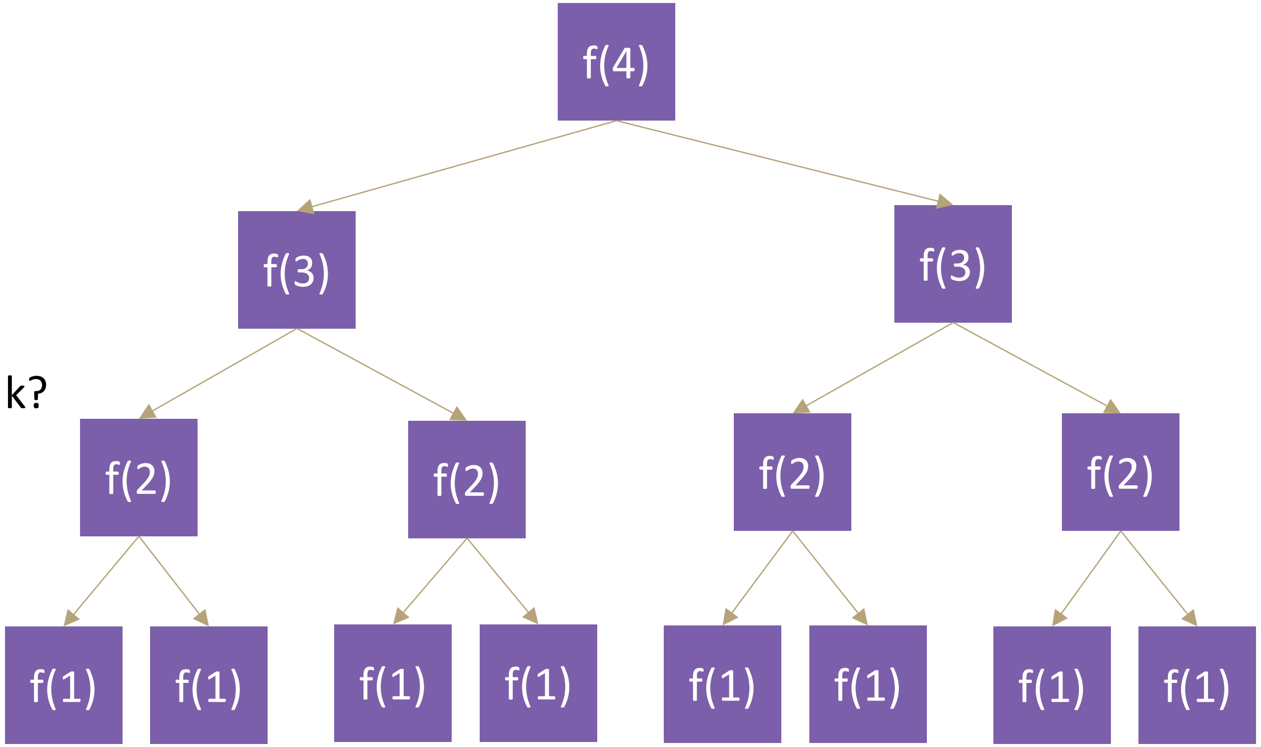
Fibonacci Recurrence to Big-Θ

How many layers in the function call tree?

How many layers will it take to transform "n" to the base case of "1" by subtracting 1

For our example, 4 -> Height = n

$$T(n) = \begin{cases} d \text{ when } n \leq 1 \\ 2T(n-1) + c \text{ otherwise} \end{cases}$$



How many function calls per layer?

LAYER	FUNCTION CALLS
1	1
2	2
3	4
4	8

How many function calls on layer k?

2^{k-1}

How many function calls TOTAL for a tree of k layers?

$1 + 2 + 3 + 4 + \ldots + 2^{k-1}$

Fibonacci Recurrence to Big-Θ

- Patterns found:

How many layers in the function call tree? n

How many function calls on layer k? 2^{k-1}

How many function calls TOTAL for a tree of k layers?

$1 + 2 + 4 + 8 + \ldots + 2^{k-1}$

Total runtime = (total function calls) x (runtime of each function call)

Total runtime = ($1 + 2 + 4 + 8 + \ldots + 2^{k-1}$) x (constant work)

$$1 + 2 + 4 + 8 + \ldots + 2^{k-1} = \sum_{i=1}^{k-1} 2^i = \frac{2^k - 1}{2 - 1} = 2^k - 1$$

Summation Identity
Finite Geometric Series

$$\sum_{i=1}^{k-1} x^i = \frac{x^k - 1}{x - 1}$$

$$\boldsymbol{T(n) = 2^n - 1 \in \Theta(2^n)}$$

Fibonacci Recurrence to Big-Θ

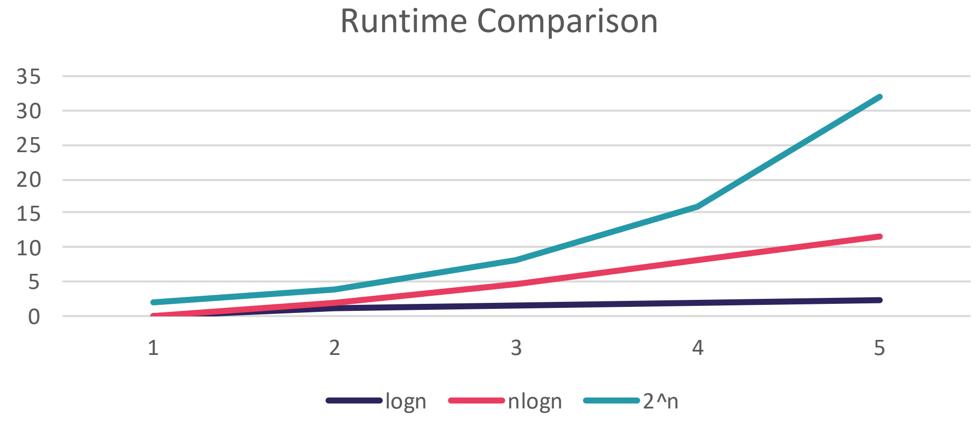
How many layers in the function call tree?	n
How many function calls on layer k?	2^{k-1}
How many function calls TOTAL for a tree of k layers?	$1 + 2 + 4 + 8 + \ldots + 2^{k-1}$
Total runtime = (total function calls) * (runtime of each function call)	$(1 + 2 + 4 + 8 + \ldots + 2^{k-1})$ x (constant work) $1 + 2 + 4 + 8 + \ldots + 2^{k-1} = \sum_{i=1}^{k-1} 2^i = \frac{2^k - 1}{2 - 1} = 2^k - 1$ $\boldsymbol{T(n) = 2^n - 1 \in \Theta(2^n)}$

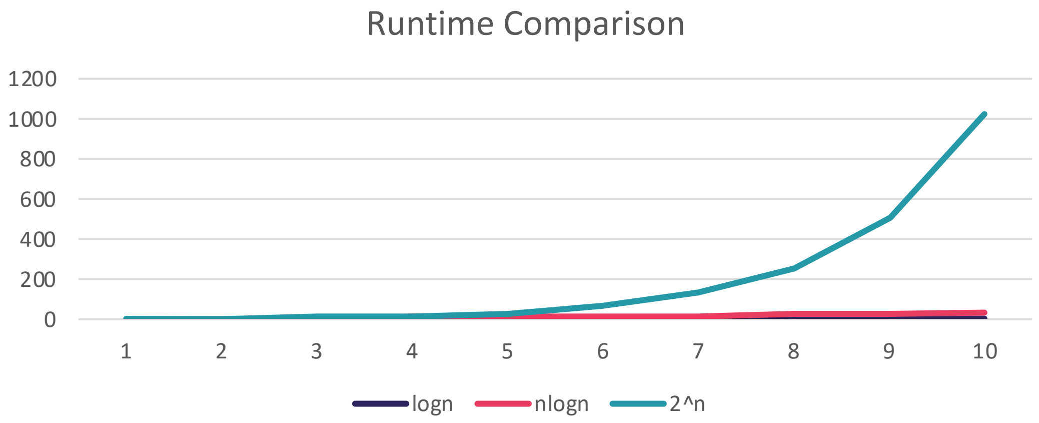
Summation Identity

Finite Geometric Series

$$\sum_{i=1}^{k-1} x^i = \frac{x^k - 1}{x - 1}$$

3 Patterns for Recursive Code





1

Halving the Input

Binary Search

Θ (log n)

2

Constant size Input

Merge Sort

Θ (n log n)

3

Doubling the Input

Fibonacci

Θ (2^n)