Before we start

Which of the following is true about minimum spanning trees?

a) Every graph has at least one MST
b) Every graph has at most one MST
c) The shortest path from any vertex \( u \) to any vertex \( v \) is the one that follows the MST
d) Every MST has a source vertex
e) Dijkstra’s algorithm computes an MST
f) None of the above
Announcements

• P3 due **TONIGHT** at 11:59pm PDT
  - Don’t forget to fill out the [P3 Feedback Survey](#)!

• EX3 due Friday, 8/07 11:59pm PDT

• P4, *the last project of the quarter*, released tonight!
  - If you haven’t already, please fill out the [P4 Partner Form](#) so we can send out partner assignments
Learning Objectives

After this lecture, you should be able to...

1. Describe Kruskal’s Algorithm, evaluate why it works, and describe why it needs a new ADT

2. Compare and contrast QuickUnion with QuickFind and describe how the two structure optimize for different operations

3. Implement WeightedQuickUnion and describe why making the change protects against the worst case find runtime

4. Describe path compression at a high level and argue for why it improves runtimes despite not following an invariant
Review Minimum Spanning Trees (MSTs)

- A Minimum Spanning Tree for a graph is a set of that graph’s edges that connect all of that graph’s vertices (spanning) while minimizing the total weight of the set (minimum)
  - Note: does NOT necessarily minimize the path from each vertex to every other vertex
  - Any tree with V vertices will have V-1 edges
  - A separate entity from the graph itself! More of an “annotation” applied to the graph, just like a Shortest Paths Tree (SPT)
Review Why do MST Algorithms Work?

• Two useful properties for MST edges. We can think about them from either perspective:
  - **Cycle Property:** The heaviest edge along a cycle is NEVER part of an MST.
  - **Cut Property:** Split the vertices of the graph into any two sets A and B. The lightest edge between A and B is ALWAYS part of an MST. *(Prim’s thinks this way)*

• Whenever you add an edge to a tree you create exactly one cycle. Removing any edge from that cycle gives another tree!

• This observation, combined with the cycle and cut properties form the basis of all of the **greedy algorithms** for MSTs.
  - **greedy algorithm:** chooses best known option at each point *and commits*, rather than waiting for a global view of the graph before deciding
Review Adapting Dijkstra’s: Prim’s Algorithm

• Normally, Dijkstra’s checks for a shorter path from the start.
• But MSTs don’t care about individual paths, only the overall weight!
• New condition: “would this be a smaller edge to connect the current known set to the rest of the graph?”

```
prims (G graph, V start)
Map edgeTo, distTo;
initialize distTo with all nodes mapped to ∞, except start to 0
PriorityQueue<V> perimeter; perimeter.add(start);

while (!perimeter.isEmpty()):
    u = perimeter.removeMin()
    known.add(u)
    for each edge (u,v) to unknown v with weight w:
        oldDist = distTo.get(v) // previous smallest edge to v
        newDist = distTo.get(u) + w // is this a smaller edge to v?
        if (newDist < oldDist):
            distTo.put(u, newDist)
            edgeTo.put(u, v)
            if (perimeter.contains(v)):
                perimeter.changePriority(v, newDist)
            else:
                perimeter.add(v, newDist)
```

Normally, Dijkstra’s checks for a shorter path from the start. But MSTs don’t care about individual paths, only the overall weight! New condition: “would this be a smaller edge to connect the current known set to the rest of the graph?”
A Different Approach

• Suppose the MST on the right was produced by Prim’s
• **Observation**: We basically chose all the smallest edges in the entire graph (1, 2, 3, 4, 6)
  - The only exception was 5. Why shouldn’t we add edge 5?
  - Because adding 5 would create a cycle, and to connect A, C, & D we’d rather choose 1 & 4 than 1 & 5 or 4 & 5.

• Prim’s thinks “vertex by vertex”, but what if you think “edge by edge” instead?
  - Start with the smallest edge in the entire graph and work your way up
  - Add the edge to the MST as long as it connects two new groups (meaning don’t add any edges that would create a cycle)

Building an MST “edge by edge” in this graph:

• Add edge 1
• Add edge 2
• Add edge 3
• Add edge 4
• Skip edge 5 (would create a cycle)
• Add edge 6
• Finished: all vertices in the MST!
Kruskal’s Algorithm

• This “edge by edge” approach is how Kruskal’s Algorithm works!

• **Key Intuition:** Kruskal’s keeps track of isolated “islands” of vertices (each is a sub-MST)
  - Start with each vertex as its own “island”
  - If an edge connects two vertices within the same “island”, it forms a cycle! Discard it.
  - If an edge connects two vertices in different “islands”, add it to the MST! Now those “islands” need to be combined.

```
kruskalMST(G graph)
Set(?) msts; Set finalMST;
initialize msts with each vertex as single-element MST
sort all edges by weight (smallest to largest)
for each edge (u,v) in ascending order:
  uMST = msts.find(u)
  vMST = msts.find(v)
  if (uMST != vMST):
    finalMST.add(edge (u, v))
    msts.union(uMST, vMST)
```

“A
B
C
D
E
F

"islands"
Kruskal’s Algorithm

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```

A  B  D  E
C
4  2  11  1
3
6  2  8  9  10  7  5
F
"islands"
Kruskal’s Algorithm

• This “edge by edge” approach is how Kruskal’s Algorithm works!

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```
Prim’s Demos and Visualizations

• Dijkstra’s Visualization
  - https://www.youtube.com/watch?v=1oiQ0hrVwJk
  - Dijkstra’s proceeds radially from its source, because it chooses nearby edges by path length from source

• Prim’s Visualization
  - https://www.youtube.com/watch?v=6uq0cQZOyoY
  - Prim’s jumps around the perimeter, because it chooses nearby edges by edge weight (there’s no source)

• Kruskal’s Visualization
  - https://www.youtube.com/watch?v=ggLyKfBTABo
  - Kruskal’s jumps around the entire graph, because it chooses from all edges purely by edge weight (while preventing cycles)
Selecting an ADT

- Kruskal’s needs to **find** what MST a vertex belongs to, and **union** those MSTs together
  - Our existing ADTs don’t lend themselves well to “unioning” two sets...
  - Let’s define a new one!

```
kruskalMST(G graph)
  Set(?) msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)

  for each edge (u,v) in ascending order:
    uMST = msts.find(u)
    vMST = msts.find(v)
    if (uMST != vMST):
      finalMST.add(edge (u, v))
      msts.union(uMST, vMST)
```
Disjoint Sets ADT (aka “Union-Find”)

- Kruskal’s will use a Disjoint Sets ADT under the hood
  - Conceptually, a single instance of this ADT contains a “family” of sets that are disjoint (no element belongs to multiple sets)

```java
CSE 373 Summer 2020
LEC 18: Disjoint Sets I

DisjointSetsADT

State
Family of Sets
• disjoint: no shared elements
• each set has a representative (either a member or a unique ID)

Behavior
makeSet(value) - new set with value as only member (and representative)
find(value) - return representative of the set containing value
union(x, y) - combine sets containing x and y into one set with all elements, choose single new representative

kruskalMST(G graph)

DisjointSets<V> msts; Set finalMST;
initialize msts with each vertex as single-element MST
sort all edges by weight (smallest to largest)

for each edge (u,v) in ascending order:
  uMST = msts.find(u)
  vMST = msts.find(v)
  if (uMST != vMST):
    finalMST.add(edge (u, v))
    msts.union(uMST, vMST);
```
Project 4: Mazes!

• You find yourself trapped in the Labyrinth of Greek legend – bummer!

• How do you solve a maze?
  - If we could model a maze as a graph, we’d just need an algorithm to find a path from s to t... Maybe even the shortest path?

• How do you *generate* a maze?
  - We’d love an algorithm that is guaranteed to connect s to t (spanning), but only produces one path from s to t (tree)...

```
s
  
  
  
  
  
  
  
  
  
  
  
  
  
  
  
  t
```
Project 4: Mazes

• It turns out that *randomizing* the weights of a graph and then computing the MST is a **fantastic way to generate mazes**!

• In P4, you’ll do both: Implement Dijkstra’s to solve an arbitrary maze, then implement Kruskal’s (and a Disjoint Set) to generate those mazes

• This project is really application-heavy!
  - Graphical User Interface (GUI) for viewing mazes and solving them
  - Significantly more starter code than past projects, to give you practice integrating with an existing codebase
  - A major part of the challenge in P4 is reading through the starter code to understand what you need to interface with! Don’t underestimate the time that takes.

• 2 week project, and **2 weeks worth of work**. It’s never been more important to start early!
Lecture Outline

1. QuickFind
   - Optimizes for the Union operation

2. QuickUnion
   - Avoids the worst case runtime for Find

3. Weighted QuickUnion
   - Makes future Find operations faster

4. Weighted QuickUnion + Path Compression
Case Study: Disjoint Sets

• Today’s lecture on the data structures which implement the Disjoint Sets ADT is an interesting case study in data structure design and iterative design improvements
  - Good chance to dust off your metacognitive skills!
  - In particular, try to identify what observations we make in each data structure that inspire improvements in the next data structure. How could you apply a similar skill to your own data structures?
Can we use an existing data structure? aka “Can we just throw maps at the problem?”

Maps to Sets (baseline): map from representative ID to set of values

<table>
<thead>
<tr>
<th>Maps to Sets</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Keanu, Farrell</td>
</tr>
<tr>
<td>2</td>
<td>Joyce, Brian, Eric</td>
</tr>
</tbody>
</table>

**DISJOINT SETS ADT**

State
- Family of Sets
  - disjoint: no shared elements
  - each set has a representative (either a member or a unique ID)

Behavior
- **makeSet(value)** – new set with value as only member (and representative)
- **find(value)** – return representative of the set containing value
- **union(x, y)** – combine sets containing x and y into one set with all elements, choose single new representative

<table>
<thead>
<tr>
<th>Operation</th>
<th>Time Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>makeSet(value)</td>
<td>Θ(1)</td>
</tr>
<tr>
<td>find(value)</td>
<td>Θ(n)</td>
</tr>
<tr>
<td>union(x, y)</td>
<td>Θ(n)</td>
</tr>
</tbody>
</table>

find(value): scan through every set under every representative
union(x, y): copy all elements from set pointed to by x into set pointed to by y
QuickFind Implementation

**QuickFind**: map from value to representative ID

| Keanu | 1 |
| Joyce | 2 |
| Farrell | 1 |
| Brian | 2 |
| Eric | 2 |

- **find(Farrell) → 1**
- **find(Eric) → 2**
- **find(Farrell) != find(Eric)**
- **find(Farrell) == find(Keanu)**

- If we store values as the keys, we can take advantage of fast lookup to make find fast!
- But what about union?

**DISJOINT SETS ADT**

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</thead>
<tbody>
<tr>
<td>makeSet(value)</td>
<td>$\Theta(1)$</td>
<td>$\Theta(1)$</td>
</tr>
<tr>
<td>find(value)</td>
<td>$\Theta(n)$</td>
<td>$\Theta(1)$</td>
</tr>
<tr>
<td>union(x, y)</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
</tr>
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</table>

- **find(value)**: lookup element and return corresponding rep.
- **union(x, y)**: scan through all elements to find those in same set, update to new rep.
Lecture Outline

**DISJOINT SETS ADT**

1. **QuickFind**
   - Optimizes for the Union operation

2. **QuickUnion**
   - Avoids the worst case runtime for Find

3. **Weighted QuickUnion**
   - Makes future Find operations faster

4. **Weighted QuickUnion + Path Compression**
QuickUnion Data Structure

• Fundamental idea:
  - QuickFind tracks each element’s ID
  - QuickUnion tracks each element’s parent. Only the root has an ID!
    - Each set becomes tree-like, but something slightly different called an up-tree: store pointers from children to parents!

Abstract Idea of “Disjoint Sets”

Implementation using QuickUnion
QuickUnion: Find

find(Eric):
  jump to Eric node
  travel upward until root
  return ID

• Key idea: can travel upward from any node to find its representative ID

• How do we jump to a node quickly?
  - Also store a map from value to its node (Omitted in future slides)

find(Farrell) \to 1
find(Eric) \to 2
find(Farrell) \not= find(Eric)
find(Farrell) \equiv find(Keanu)
QuickUnion: Union

- Key idea: easy to simply rearrange pointers to union entire trees together!
- Which of these implementations would you prefer?
  - Vote in the Participants Panel!

```
union(Eric, Farrell):
    rootF = find(Farrell)
    set Eric to point to rootF
```

RESULT:
QuickUnion: Union

union(Eric, Farrell):
   rootF = find(Farrell)
   set Eric to point to rootF

RESULT:

• We prefer the right implementation because by changing just the root, we effectively pull the entire tree into the new set!
  • If we change the first node instead, we have to do more work for the rest of the old tree
  • A rare example of constant time work manipulating a factor of n elements
QuickUnion: Why bother with the second root?

- Key idea: will help minimize runtime for future find() calls if we keep the height of the tree short!
  - Pointing directly to the second element would make the tree taller

\[
\text{union(Eric, Farrell):} \\
\quad \text{rootE = find(Eric)} \\
\quad \text{rootF = find(Farrell)} \\
\quad \text{set rootE to point to rootF}
\]

Why not just use:

\[
\text{union(Eric, Farrell):} \\
\quad \text{rootE = find(Eric)} \\
\quad \text{set rootE to point to Farrell}
\]
QuickUnion: Checking in on those runtimes

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</table>

- **QuickUnion**

  • Only if we discount the runtime from union’s calls to find! Otherwise, $\Theta(n)$.
  - However, for Kruskal’s, not a bad assumption: we only ever call union with roots anyway!

```java
union(A, B):  
  rootA = find(A)  
  rootB = find(B)  
  set rootA to point to rootB
```

```java
kruskalMST(G graph)
  DisjointSets<V> msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)
  for each edge (u, v) in ascending order:
    uMST = msts.find(u)  
    vMST = msts.find(v)  
    if (uMST != vMST):
      finalMST.add(edge (u, v))  
      msts.union(uMST, vMST);
```
QuickUnion: Let’s Build a Worst Case

Even with the "use-the-roots" implementation of union, try to come up with a series of calls to union that would create a worst-case runtime for find on these Disjoint Sets:

QuickUnion: Let's Build a Worst Case

\[
\begin{align*}
\text{find}(A): & \quad \text{jump to A node} \\
& \quad \text{travel upward until root} \\
& \quad \text{return ID}
\end{align*}
\]

\[
\begin{align*}
\text{union}(A, B): & \quad \text{rootA} = \text{find}(A) \\
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\end{align*}
\]
QuickUnion: Let’s Build a Worst Case

Even with the “use-the-roots” implementation of union, try to come up with a series of calls to union that would create a worst-case runtime for find on these Disjoint Sets:

union(A, B)
union(B, C)
union(C, D)
find(A)

```
union(A, B):
    rootA = find(A)
    rootB = find(B)
    set rootA to point to rootB

find(A):
    jump to A node
    travel upward until root
    return ID
```
Analyzing the QuickUnion Worst Case

• How did we get a degenerate tree?
  - Even though pointing a root to a root usually helps with this, we can still get a degenerate tree if we put the root of a large tree under the root of a small tree.
  - In QuickUnion, rootA always goes under rootB
    - But what if we could ensure the smaller tree went under the larger tree?

union(C, D)

What currently happens

What would help avoid degenerate tree
Lecture Outline

DISJOINT SETS ADT

1. QuickFind
   - Optimizes for the Union operation

2. QuickUnion
   - Avoids the worst case runtime for Find

3. Weighted QuickUnion
   - Makes future Find operations faster

4. Weighted QuickUnion + Path Compression
**WeightedQuickUnion**

- **Goal:** Always pick the smaller tree to go under the larger tree

- **Implementation:** Store the number of nodes (or “weight”) of each tree in the root
  - Constant-time lookup instead of having to traverse the entire tree to count

```plaintext
union(A, B):
    rootA = find(A)
    rootB = find(B)
    put lighter root under heavier root
```

```
union(A, B)
union(B, C)
union(C, D)
find(A)
```

Now what happens?

```
A
B
C
D
```

Perfect! Best runtime we can get.
WeightedQuickUnion: Performance

• union()’s runtime is still dependent on find()’s runtime, which is a function of the tree’s height

• What’s the worst-case height for WeightedQuickUnion?

```
union(A, B):
    rootA = find(A)
    rootB = find(B)
    put lighter root under heavier root
```
WeightedQuickUnion: Performance

• Consider the worst case where the tree height grows as fast as possible

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WeightedQuickUnion: Performance

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WeightedQuickUnion: Performance

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<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
</tr>
</tbody>
</table>
WeightedQuickUnion: Performance

• Consider the worst case where the tree height grows as fast as possible
• Worst case tree height is $\Theta(\log N)$
Why Weights Instead of Heights?

• We used the number of items in a tree to decide upon the root

• Why not use the height of the tree?
  - HeightedQuickUnion’s runtime is asymptotically the same: $\Theta(\log(n))$
  - It’s easier to track weights than heights, even though WeightedQuickUnion can lead to some suboptimal structures like this one:
WeightedQuickUnion Runtime

<table>
<thead>
<tr>
<th></th>
<th>Maps to Sets</th>
<th>QuickFind</th>
<th>QuickUnion</th>
<th>WeightedQuickUnion</th>
</tr>
</thead>
<tbody>
<tr>
<td>makeSet(value)</td>
<td>$\Theta(1)$</td>
<td>$\Theta(1)$</td>
<td>$\Theta(1)$</td>
<td>$\Theta(1)$</td>
</tr>
<tr>
<td>find(value)</td>
<td>$\Theta(n)$</td>
<td>$\Theta(1)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(\log n)$</td>
</tr>
<tr>
<td>union(x, y) assuming root args</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(1)$</td>
<td>$\Theta(1)$</td>
</tr>
<tr>
<td>union(x, y)</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(n)$</td>
<td>$\Theta(\log n)$</td>
</tr>
</tbody>
</table>

• This is pretty good! But there’s one final optimization we can make: path compression
Lecture Outline

DISJOINT SETS ADT

1. QuickFind
   - Optimizes for the Union operation

2. QuickUnion

3. Weighted QuickUnion
   - Avoids the worst case runtime for Find

4. Weighted QuickUnion + Path Compression
   - Makes future Find operations faster
Modifying Data Structures for Future Gains

• Thus far, the modifications we’ve studied are designed to *preserve invariants*
  - E.g. Performing rotations to preserve the AVL invariant
  - We rely on those invariants always being true so every call is fast

• Path compression is entirely different: we are modifying the tree structure to *improve future performance*
  - Not adhering to a specific invariant
  - The first call may be slow, but will optimize so future calls can be fast
Path Compression: Idea

• This is the worst-case topology if we use WeightedQuickUnion

• Idea: When we do find(15), move all visited nodes under the root
  - Additional cost is insignificant (we already have to visit those nodes, just constant time work to point to root too)
Path Compression: Idea

- This is the worst-case topology if we use WeightedQuickUnion

  ![Diagram showing path compression](image)

  - Idea: When we do find(15), move all *visited nodes* under the root
    - Additional cost is insignificant (we already have to visit those nodes, just constant time work to point to root too)
  - Perform Path Compression on every find(), so future calls to find() are faster!