Lecture 6: Modeling Complex Code

CSE 373: Data Structures and Algorithms

Warm Up!

What's the theta bound for the runtime function for this piece of code?

```
public void method1(int n) {
    if (n <= 100) {
        System.out.println(":3");
    } else {
        System.out.println(":D");
        for (int i = 0; i<16; i++) {
            method1(n / 4);
        }
    }
}
```

Master Theorem

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a < c$ then $T(n) \in \Theta(n^c)$

If $\log_b a = c$ then $T(n) \in \Theta(n^c \log n)$

If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

Take 2 Minutes

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2. https://www.pollev.com/cse373studentqs to ask your own questions

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$$T(n) = \begin{cases} constant\ work & \text{if } n \leq 100 \\ 16T\left(\frac{n}{4}\right) + constant\ work & \text{otherwise} \end{cases}$$

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a = 16, b = 4, c = 0

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$\log_4 16 \;\; ? \;\; 0$

a = 16, b = 4, c = 0

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$\log_4 16 \;\; ? \;\; 0$

$\log_4 16 > 0 \qquad T(n) \in \Theta\left(n^{\log_b a}\right)$

a = 16, b = 4, c = 0

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$\log_4 16 \ ? \ 0$

$\log_4 16 > 0$ $\quad T(n) \in \Theta\left(n^{\log_b a}\right)$

a = 16, b = 4, c = 0

$\Theta\left(n^{\log_4 16}\right) = \Theta(n^2)$

Master Theorem

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Administrivia

project1 due Wednesday 11:59pm

exercise1 due Friday 11:59pm

Piazza
- try to use the search bar before you post / use descriptive summary titles when possible - thank you

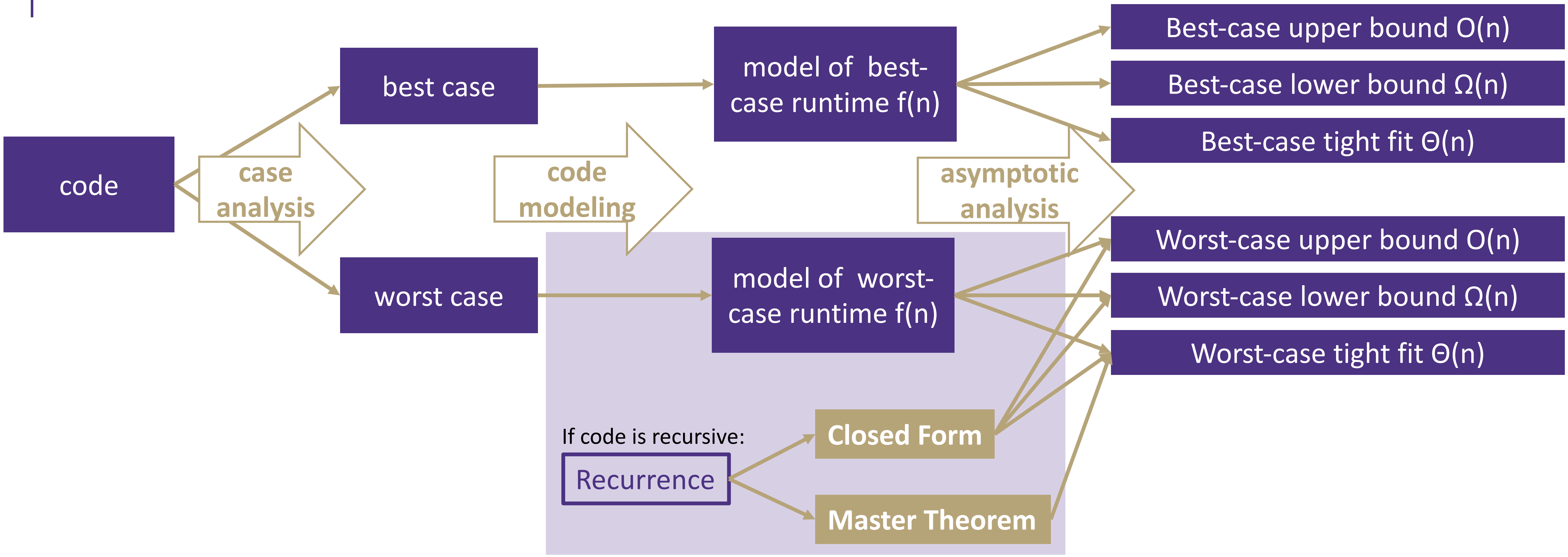
Office Hours

- please come to them! Piazza is super busy and office hours are less busy. We promise we won't bite ☺

anonymous feedback
- lectures + projects are so different?
  - it'll be better soon - we didn't have time to cram in Deques (it's probably the least relevant ADT) or make a new assignment this quarter. We're aware, should be better for the future since the next few projects are all implementing new data structures and algorithms (HashMap, Binary Heaps, Graphs, ArrayDisjointSets, Kruskal's and Dijkstra's algorithms) we talk about directly in lecture.
  - exercises + experiments should help connect lecture topics (while also not being _too_ much time)
- is there time to give an intro to the projects in lecture?
  - maybe barely – we can try to see if it fits on a week-to-week

Questions

Code Analysis Process



Recursive Patterns

Pattern #1: Halving the Input

Binary Search $\Theta(\log n)$

Pattern #2: Constant size input and doing work

Merge Sort $\Theta(n \log n)$

Pattern #3: Doubling the Input

Calculating Fibonacci

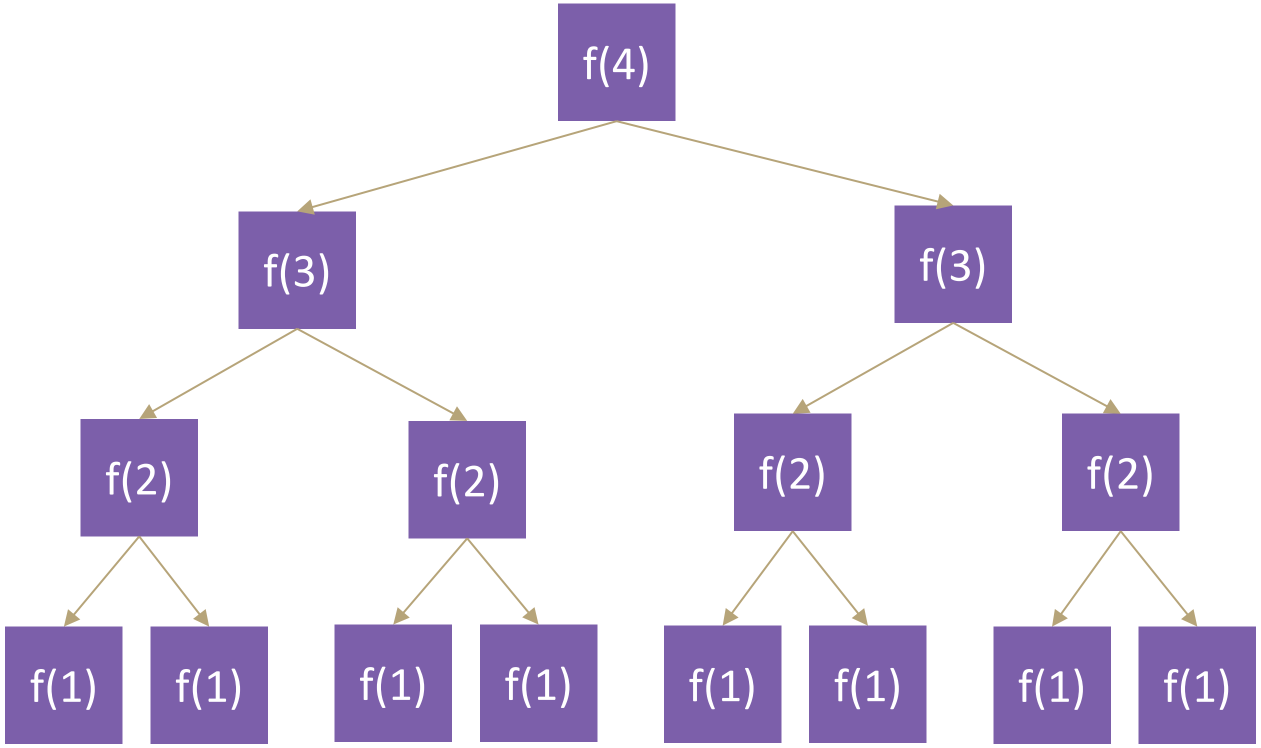
Calculating Fibonacci

```
public int fib(int n) {
     if (n <= 1) {
          return 1;
     }
     return fib(n-1) + fib(n-1);
}
```

- Each call creates 2 more calls
- Each new call has a copy of the input, almost
- Almost doubling the input at each call

Pattern #3 – Doubling the Input

Almost



Calculating Fibonacci Recurrence to Big-Θ

```
public int f(int n) {
    if (n <= 1) {
        return 1;
    }
    return f(n-1) + f(n-1);
}
```

d

2T(n-1) + c

$$T(n) = \begin{cases} d \text{ when } n \le 1 \\ 2T(n-1) + c \text{ otherwise} \end{cases}$$

Take 1 min to respond to activity

www.pollev.conm/cse373activity
Finish the recurrence, what is the model for the recursive case?

Can we use master theorem?

Master Theorem

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Uh oh, our model doesn't match that format…

- Can we intuit a pattern?
- T(1) = d
- T(2) = 2T(2-1) + c = 2(d) + c
- T(3) = 2T(3-1) + c = 2(2(d) + c) + c = 4d + 3c
- T(4) = 2T(4-1) + c = 2(4d + 3c) + c = 8d + 7c
- T(5) = 2T(5-1) + c = 2(8d + 7c) + c = 16d +25c
- Looks like something's happening but it's tough
  - Maybe geometry can help!

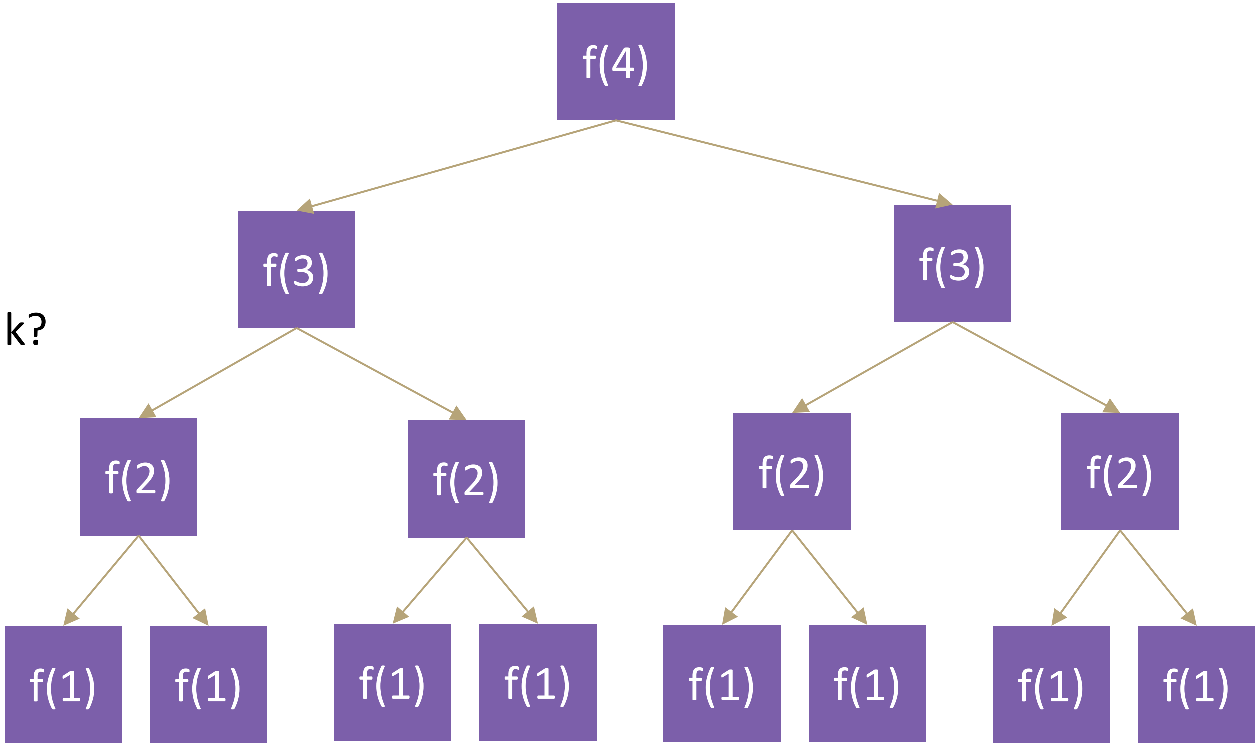
Calculating Fibonacci Recurrence to Big-Θ

How many layers in the function call tree?

$$T(n) = \begin{cases} d \text{ when } n \leq 1 \\ 2T(n-1) + c \text{ otherwise} \end{cases}$$

How many layers will it take to transform “n” to the base case of “1” by subtracting 1

For our example, 4 -> Height = n



How many function calls per layer?

Layer	Function calls
0	1
1	2
2	4
3	8

How many function calls on layer k?

$$\mathbf{2^{k-1}}$$

How many function calls TOTAL for a tree of k layers?

$$\mathbf{1 + 2 + 3 + 4 + \ldots + 2^{k-1}}$$

Calculating Fibonacci Recurrence to Big-Θ

Patterns found:

How many layers in the function call tree? n

How many function calls on layer k? 2^{k-1}

How many function calls TOTAL for a tree of k layers?

$1 + 2 + 4 + 8 + \ldots + 2^{k-1}$

Total runtime = (total function calls) x (runtime of each function call)

Total runtime = $(1 + 2 + 4 + 8 + \ldots + 2^{k-1})$ x (constant work)

$$1 + 2 + 4 + 8 + \ldots + 2^{k-1} = \sum_{i=1}^{k-1} 2^i = \frac{2^k - 1}{2 - 1} = 2^k - 1$$

Summation Identity
Finite Geometric Series

$$\sum_{i=1}^{k-1} x^i = \frac{x^k - 1}{x - 1}$$

$$\boldsymbol{T(n) = 2^n - 1 \in \Theta(2^n)}$$

Recursive Patterns

Pattern #1: Halving the Input

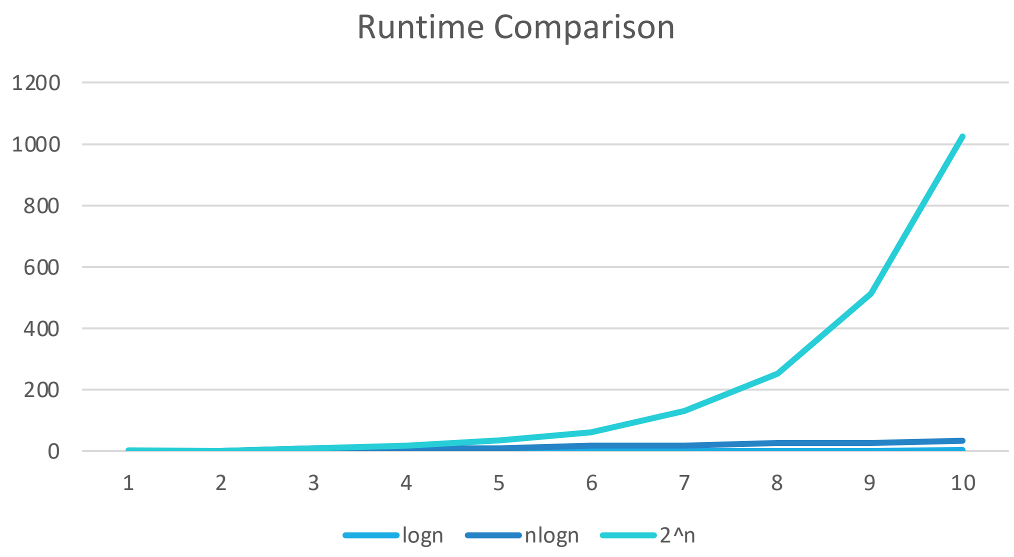
Binary Search Θ(logn)

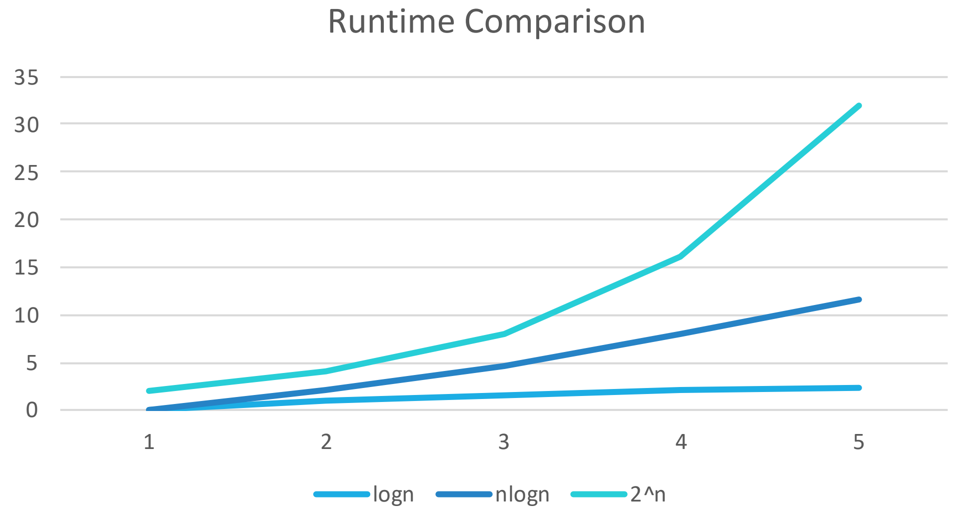
Pattern #2: Constant size input and doing work

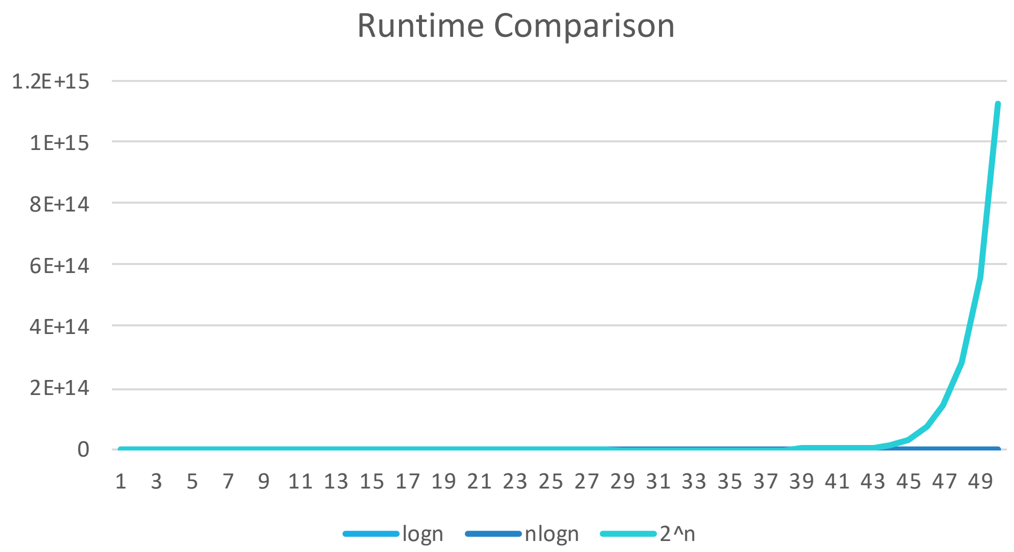
Merge Sort Θ(nlogn)

Pattern #3: Doubling the Input

Calculating Fibonacci Θ(2^n)



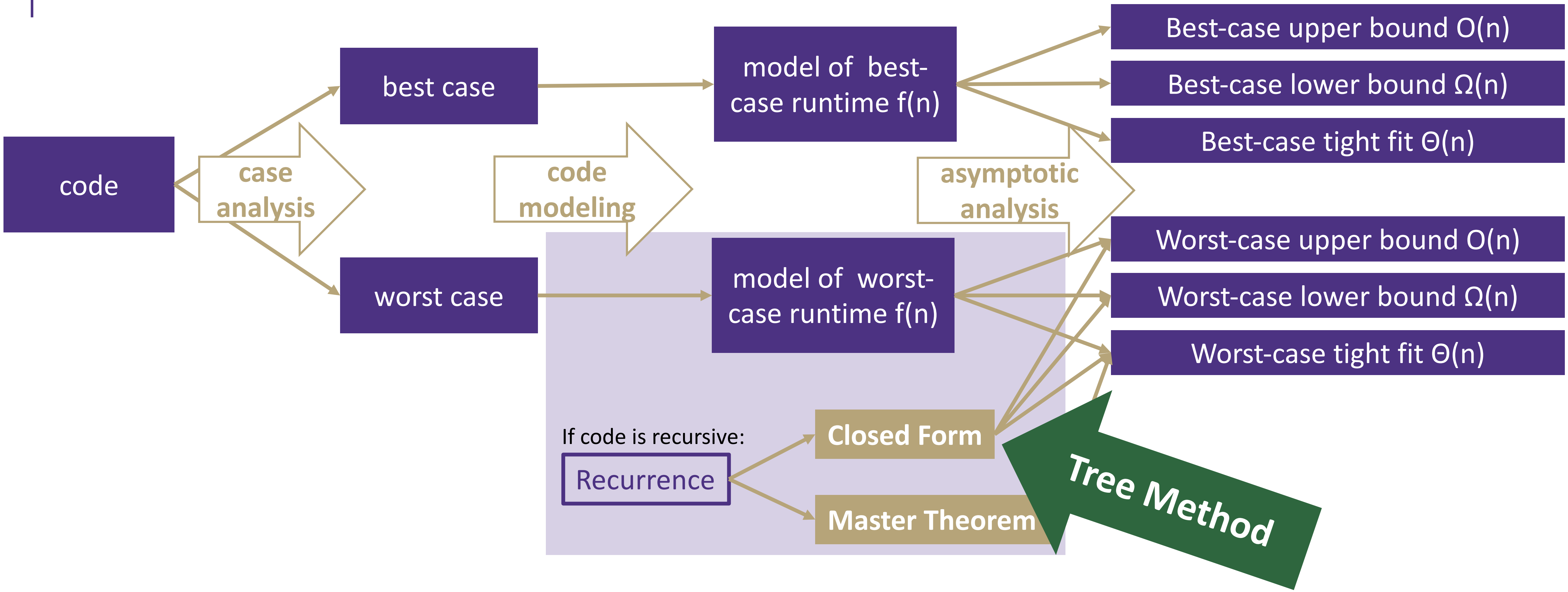




Questions

Tree Method

Code Analysis Process



Recurrence to Big Θ Techniques

A recurrence is a mathematical function that includes itself in its definition

This makes it very difficult to find the dominating term that will dictate the asymptotic growth

Solving the recurrence or "finding the closed form" is the process of eliminating the recursive definition. So far, we've seen three methods to do so:

1. Apply Master Theorem
   - Pro: Plug and chug convenience
   - Con: only works for recurrences of a certain format
2. Unrolling
   - Pro: Least complicated setup
   - Con: requires intuitive pattern matching
3. Tree Method
   - Pro: Plug and chug
   - Con: Complex setup

$$T(n) = \begin{cases} d \text{ when } n \leq 1 \\ 2T(n-1) + c \text{ otherwise} \end{cases}$$

Master Theorem

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

T(1) = d
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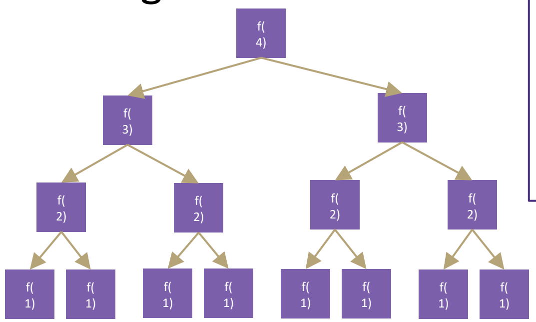


Tree Method

Draw out call stack, what is the input to each call? How much work is done by each call?

How much work is done at each layer?

64 for this example -> n work at each layer

Work is variable per layer, but across the entire layer work is constant - always n

How many layers are in our function call tree?

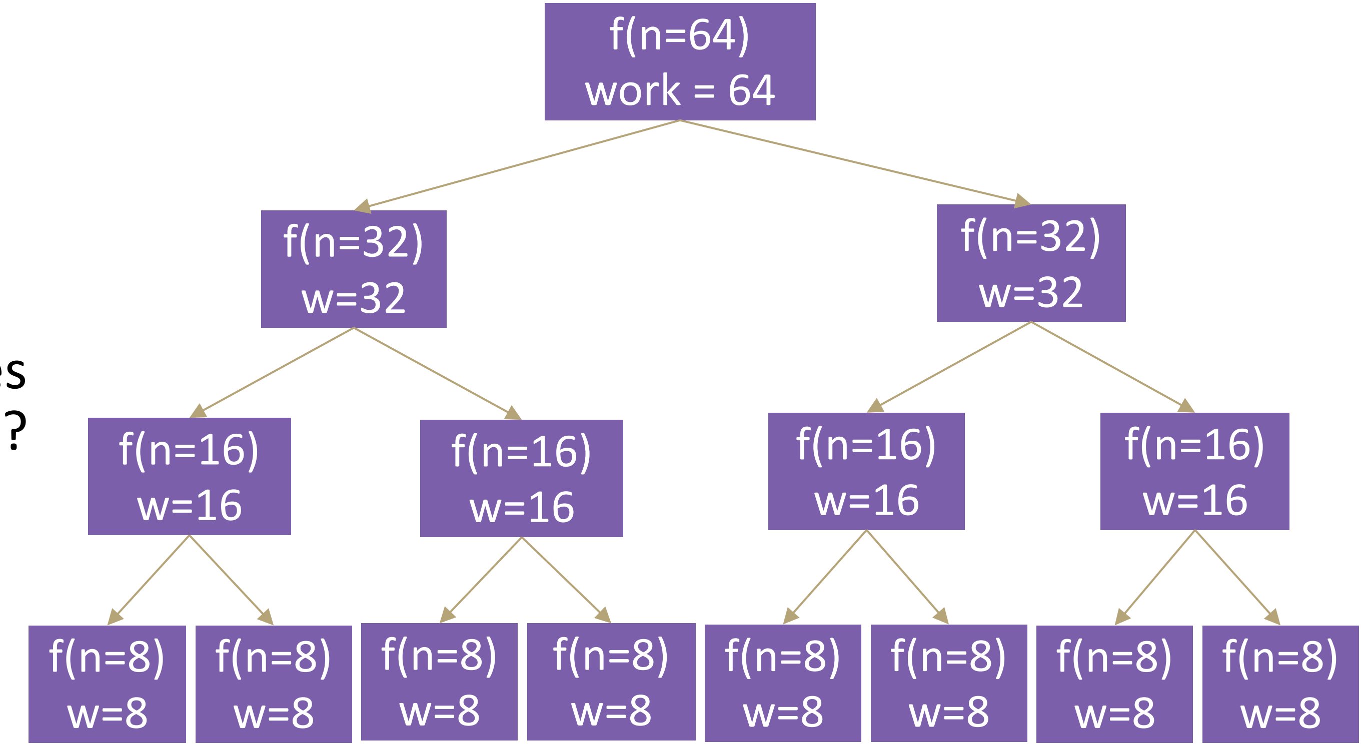
Hint: how many levels of recursive calls does it take *binary search* to get to the base case?

Height = $\log_2 n$

It takes $\log_2 n$ divisions by 2 for n to be reduced to the base case 1

$\log_2 64 = 6$ -> 6 levels of this tree

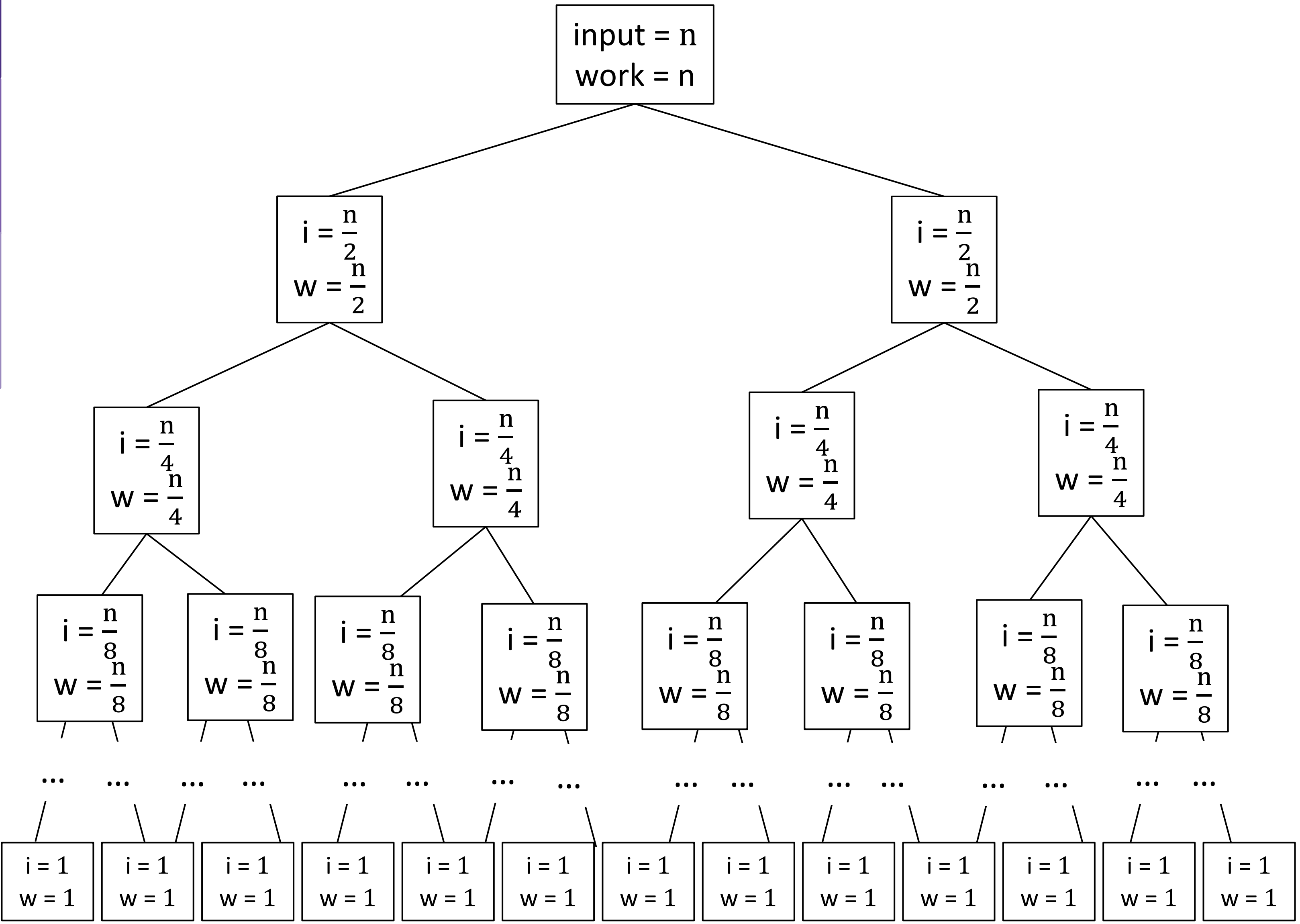
Merge Sort $T(n) = \begin{cases} 1 & \text{if } n \le 1 \\ 2T(n/2) + n & \text{otherwise} \end{cases}$



… and so on…

Tree Method

$$T(n) = \begin{cases} 1 \text{ when } n \le 1 \\ 2T\left(\frac{n}{2}\right) + n \text{ otherwise} \end{cases}$$



Recursive level	How many nodes at each level?	How much work done by each node?	How much work across each level?
0	1	n	n
1	2	$\frac{n}{2}$	n
2	4	$\frac{n}{4}$	n
3	8	$\frac{n}{8}$	n
logn	n	1	n

Tree Method Practice

$$T(n) = \begin{cases} 1 \text{ when } n \le 1 \\ 2T\left(\frac{n}{2}\right) + n \text{ otherwise} \end{cases}$$

1. What is the size of the input on level i? $\frac{n}{2^i}$

2. What is the work done by each node on the i^{th} recursive level? $\left(\frac{n}{2^i}\right)$

3. What is the number of nodes at level i? 2^i

4. What is the total work done at the i^{th} recursive level?

   $numNodes * workPerNode = 2^i\left(\frac{n}{2^i}\right) = n$

5. What value of i does the last level occur?

   $\frac{n}{2^i} = 1 \rightarrow n = 2^i \rightarrow i = \log_2 n$

6. What is the total work across the base case level?

   $numNodes * workPerNode = 2^{\log_2 n}(1) = n$

Level (i)	Number of Nodes	Work per Node	Work per Level
0	1	n	n
1	2	$\frac{n}{2}$	n
2	4	$\frac{n}{4}$	n
3	8	$\frac{n}{8}$	n
$\log_2 n$	n	1	

Combining it all together…

$$T(n) = \sum_{i=0}^{\log_2 n - 1} n + n = n\log_2 n + n = \Theta(\text{nlogn})$$

power of a log

$$x^{\log_b y} = y^{\log_b x}$$

Summation of a constant

$$\sum_{i=0}^{n-1} c = cn$$

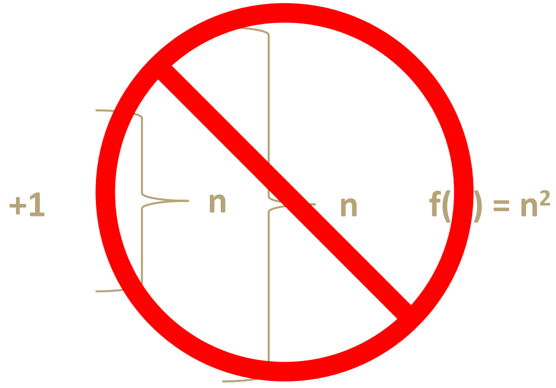
Questions

Summations

Modeling Complex Loops

Write a mathematical model of the following code

```
for (int i = 0; i < n; i++) {
    for (int j = 0; j < i; j++) {
        System.out.println("Hello!");
    }
}
```



Keep an eye on loop bounds!

Modeling Complex Loops

```
for (int i = 0; i < n; i++) {
    for (int j = 0; j < i; j++) {
        System.out.print(“Hello! ”);
    }
    Sysem.out.println();
}
```

+1

0 + 1 + 2 + 3 +…+ i-1

n

T(n) = (0 + 1 + 2 + 3 +…+ i-1)

How do we model this part?

Summations!

1 + 2 + 3 + 4 +… + n = $\sum_{i=1}^{n} i$

Definition: Summation

$\sum_{i=a}^{b} f(i)$ = f(a) + f(a + 1) + f(a + 2) + … + f(b-2) + f(b-1) + f(b)

T(n) = $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1$

What is the Big O?

Simplifying Summations

Find closed form using summation identities (given on exams)

```
for (int i = 0; i < n; i++) {
   for (int j = 0; j < i; j++) {
      System.out.println("Hello!");
   }
}
```

→ $T(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1$ → closed form → simplified tight big O

$$T(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1 = \sum_{i=0}^{n-1} 1 \cdot i = 1 \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n = \boldsymbol{O(n^2)}$$

Summation of a constant

$$\sum_{i=0}^{n-1} c = cn$$

Factoring out a constant

$$\sum_{i=a}^{b} cf(i) = c \sum_{i=a}^{b} f(i)$$

Gauss's Identity

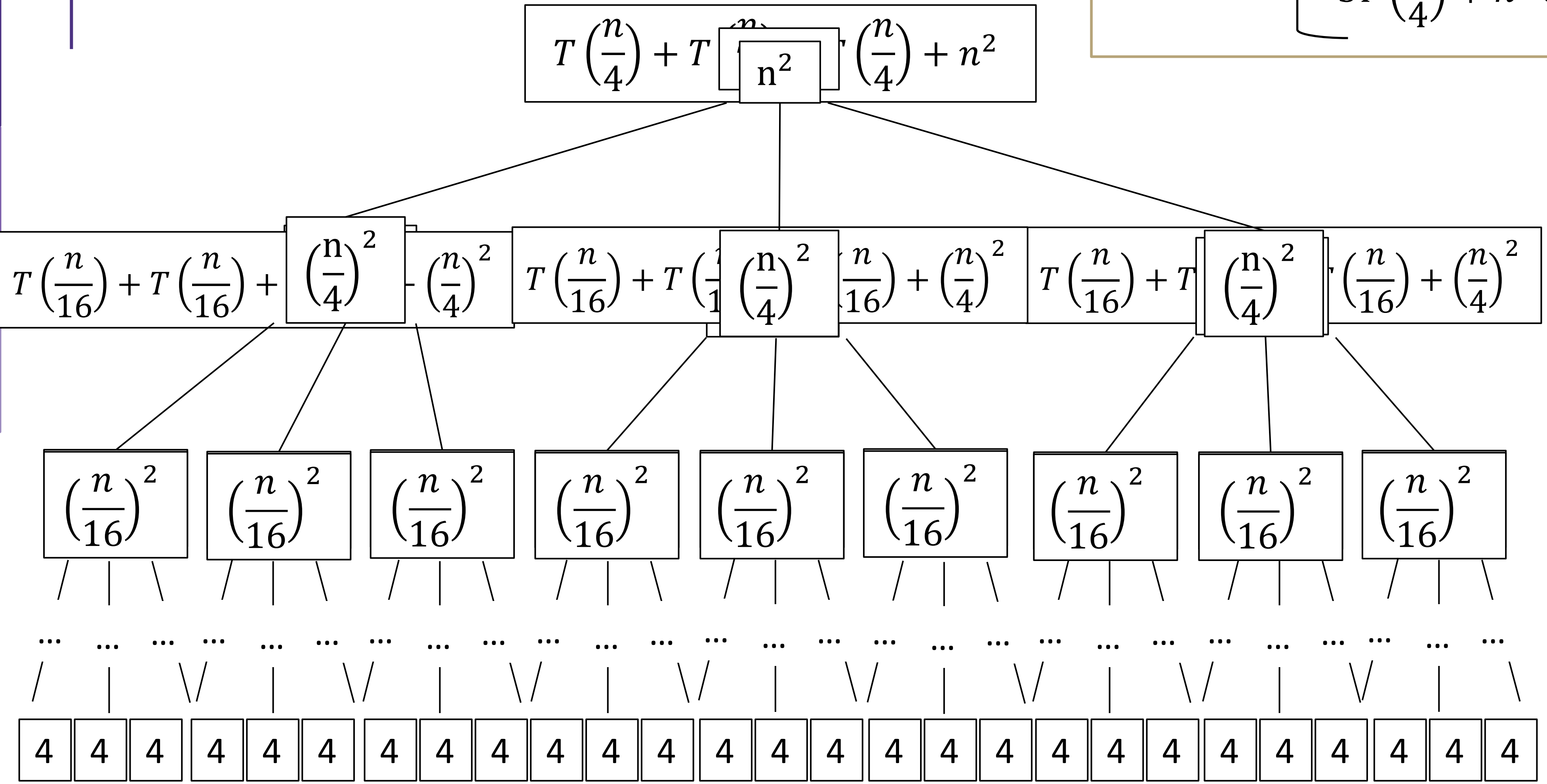
$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$

Questions

Appendix

Tree Method Practice

$$T(n) = \begin{cases} 4 \text{ when } n \leq 1 \\ 3T\left(\frac{n}{4}\right) + n^2 \text{ otherwise} \end{cases}$$



Answer the following questions:

1. What is the size of the input on level i?
2. What is the work done by each node on the i^{th} recursive level
3. What is the number of nodes at level i?
4. What is the total work done at the i^th recursive level?
5. What value of i does the last level occur?
6. What is the total work across the base case level?

EXAMPLE MODIFIED FROM PROVIDED BY CS 161 – JESSICA SU
HTTPS://WEB.STANFORD.EDU/CLASS/ARCHIVE/CS/CS161/CS161.1168/LECTURE3.PDF

Tree Method Practice

5 Minutes

$$T(n) = \begin{cases} 4 \text{ when } n \le 1 \\ 3T\left(\frac{n}{4}\right) + n^2 \text{ otherwise} \end{cases}$$

1. What is the size of the input on level i? $\frac{n}{4^i}$
2. What is the work done by each node on the i^{th} recursive level? $\left(\frac{n}{4^i}\right)^2$
3. What is the number of nodes at level i? 3^i
4. What is the total work done at the i^{th} recursive level? $3^i\left[\left(\frac{n}{4^i}\right)\right]^2 = \left(\frac{3}{16}\right)^i n^2$
5. What value of i does the last level occur? $\frac{n}{4^i} = 1 \rightarrow n = 4^i \rightarrow i = \log_4 n$
6. What is the total work across the base case level? $3^{\log_4 n} \cdot 4$

power of a log
$x^{\log_b y} = y^{\log_b x}$

$4 \cdot n^{\log_4 3}$

Level (i)	Number of Nodes	Work per Node	Work per Level
0	1	n^2	n^2
1	3	$\left(\frac{n}{4}\right)^2$	$\frac{3}{4^2}n^2$
2	9	$\left(\frac{n}{4^2}\right)^2$	$\frac{3^2}{4^4}n^2$
base	$3^{\log_4 n}$	4	$4 * 3^{\log_4 n}$

Combining it all together…

$$T(n) = \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i n^2 + 4n^{\log_4 3}$$

Tree Method Practice

$$T(n) = \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i n^2 + 4n^{\log_4 3}$$

factoring out a constant

$$\sum_{i=a}^{b} cf(i) = c\sum_{i=a}^{b} f(i)$$

$$T(n) = n^2 \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i + 4n^{\log_4 3}$$

Identities are on the webpage.
You don't need to memorize them.

finite geometric series

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}$$

Closed form:

$$T(n) = n^2 \left(\frac{\left(\frac{3}{16}\right)^{\log_4 n} - 1}{\frac{3}{16} - 1} \right) + 4n^{\log_4 3}$$

So what's the big-Θ…

$$T(n) = n^2 \left(-\frac{16}{13}\right)\left(\frac{3}{16}\right)^{\log_4 n} + \left(\frac{16}{13}\right) n^2 + 4n^{\log_4 3}$$

$$T(n) = n^2 \left(-\frac{16}{13}\right)(n)^{\log_4 \frac{3}{16}} + \left(\frac{16}{13}\right) n^2 + 4n^{\log_4 3}$$

$$T(n) \in \Theta(n^2)$$

More Tree Method

$$T(n) = \begin{cases} 6T\left(\frac{n}{2}\right) + 2n \text{ if } n > 8 \\ 3 \qquad\qquad\quad \text{otherwise} \end{cases}$$

Tree Method Practice

$$T(n) = \begin{cases} 6T\left(\frac{n}{2}\right) + 2n \text{ if } n > 8 \\ 3 \qquad\qquad \text{otherwise} \end{cases}$$

Answer the following questions:

1. What is the size of the input on level i?
2. What is the work done by each node on the i^{th} recursive level
3. What is the number of nodes at level i?
4. What is the total work done at the i^th recursive level?
5. What value of i does the last level occur?
6. What is the total work across the base case level?

5 Minutes

Tree Method Practice

$$T(n) = \begin{cases} 6T\left(\frac{n}{2}\right) + 2n \text{ if } n > 2 \\ 3 \quad \text{otherwise} \end{cases}$$

1. What is the size of the input on level i? $\frac{n}{2^i}$

2. What is the work done by each node on the i^{th} recursive level? $2\frac{n}{2^i}$

3. What is the number of nodes at level i? 6^i

4. What is the total work done at the i^{th} recursive level? $6^i\left[2\frac{n}{2^i}\right] = 2 \cdot 3^i \cdot n$

5. What value of i does the last level occur?
$\frac{n}{2^i} = 2 \rightarrow n = 2^{i+1} \rightarrow i = \log_2(n) - 1$

6. What is the total work across the base case level? $6^{\log_2(n)-1} \cdot 3$

power of a log
$x^{\log_b y} = y^{\log_b x}$

$$\frac{3 \cdot 6^{\log_2 n}}{6} = \frac{1}{2} \cdot n^{\log_2 6} = \frac{1}{2} \cdot n^{\log_2 6}$$

Level (i)	Number of Nodes	Work per Node	Work per Level
0	1	$2n$	$2n$
1	2	$\frac{2n}{8}$	$\frac{n}{2}$
2	4	$2\left(\frac{n}{8^2}\right)$	$\frac{n}{8}$
base	$2^{\log_8 n - 1}$	3	$\frac{3}{2}n^{1/3}$

Combining it all together…

$$T(n) = \sum_{i=0}^{\log_2(n)-2} 2 \cdot 3^i n + \frac{1}{2} n^{\log_2 6}$$

$$T(n) = \sum_{i=0}^{\log_2(n)-2} 2 \cdot 3^i n + \frac{1}{2} n^{\log_2 6}$$

$$= 2n \sum_{i=0}^{\log_2(n)-2} 3^i + \frac{1}{2} n^{\log_2 6}$$

$$= 2n \frac{3^{\log_2(n)-1}}{3-1} + \frac{1}{2} n^{\log_2 6}$$

$$= n \cdot \frac{n^{\log_2(3)}}{3} + \frac{1}{2} n^{\log_2 6}$$

$$= \frac{n^{\log_2(3)+1}}{3} + \frac{1}{2} n^{\log_2 6}$$

$$= \frac{n^{\log_2(6)}}{3} + \frac{1}{2} n^{\log_2 6} = \frac{5}{6} n^{\log_2 6}$$

finite geometric series

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}$$

power of a log

$$x^{\log_b y} = y^{\log_b x}$$

$$1 = \log_2 2$$

$$\log_a b + \log_a c = \log_a(bc)$$

Summation Practice

```
public static void primesUpToN(int n) {
    System.out.print("1 2 ");           +1
    for (int i = 3; i <= n; i++) {
        for (int j = 2; j < i; j++){
            if (j != i && j % i == 0) {  +4
                System.out.print(i + " ");  +1
                break;
            }
        }
    }
    System.out.println();  +1
}
```

$\sum_{j=2}^{i-1} 5$ $\quad \sum_{i=3}^{n} \sum_{j=2}^{i-1} 5$

$T(n) = 1 + \sum_{i=3}^{n} \sum_{j=2}^{i-1} 5 = 1 + \sum_{i=0}^{n-3} \sum_{j=0}^{i-3} 5 = 1 + \sum_{i=0}^{n-3} 5(i-2) = 1 + 5(\sum_{i=0}^{n-3} i - \sum_{i=0}^{n-3} 2) - = 1 + 5(\frac{(n-2)(n-3)}{2}$ - (n-2)(2))

Adjusting summation bounds	Summation of a constant	Factoring out a constant	Gauss's identity