Lecture 6: Modeling Complex Code

CSE 373: Data Structures and Algorithms

Warm Up!

Big-O

$f(n) \in O(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$

Big-Omega

$f(n) \in \Omega(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$, $f(n) \geq c \cdot g(n)$

Big-Theta

$f(n) \in \Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

$$f1(n) = 3^n$$

$$f2(n) = 2n^3 + 2$$

$$f3(n) = n^2 - 2n + 3$$

$$f4(n) = 7n^2 + 5n$$

$$f5(n) = 5n + 1$$

A. (answer this question A in the pollev) which of the above functions are in O(n^2)?

B. which of the above functions are in Theta(n^2)?

Take 2 Minutes

1. www.pollev.com/cse373activity for participating in our active learning questions.
2. https://www.pollev.com/cse373studentqs to ask your own questions

Warm Up!

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$$f5(n) = 5n + 1$$

For finding a tight Big-O of a function (process from Monday / in quiz section):
- drop the constant multipliers
- drop the non dominant terms
- the remaining term is your tight big O or Theta bound
- Remember Big-O is just an upperbound so we can say stuff like
  - f(n) = n is in O(n^2)

A. (answer this question A in the pollev)
which of the above functions are in O(n^2)?

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B. which of the above functions are in Theta(n^2)?

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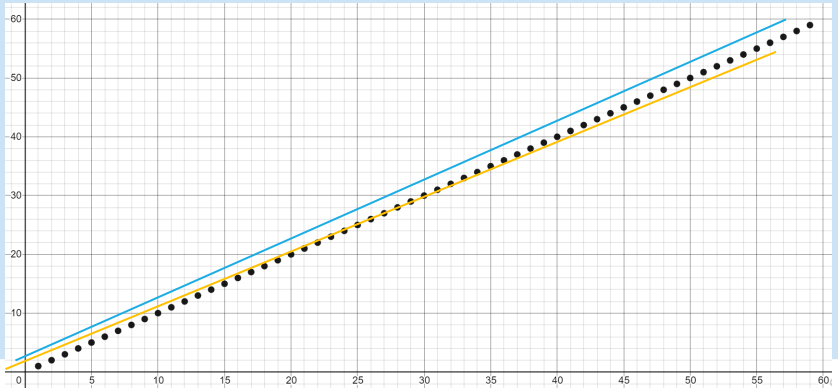
- drop the constant multipliers
- drop the non dominant terms
- the remaining term is your tight big O or Theta bound
- Remember Big-O is just an upperbound so we can say stuff like
  - f(n) = n is in O(n^2)

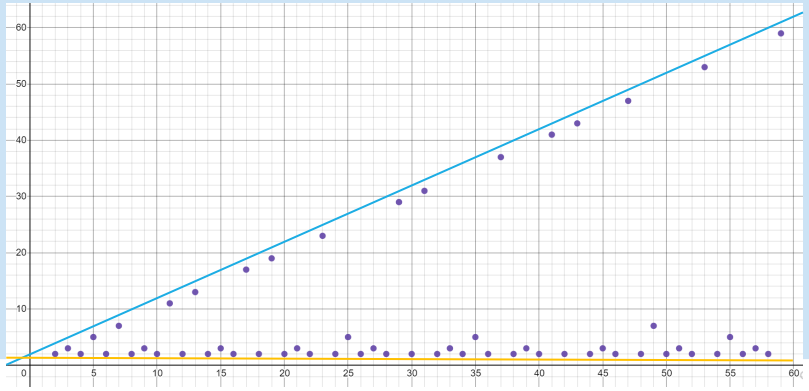
For theta it's exactly the same process here except we don't need to consider the loose upper bound part so it's actually simpler. We can just consider the biggest term directly. The only functions that are Theta(n^2) are those that have some n^2 term as the dominating term.

A “recap” about big-O vs big-Theta going forward

Mainly we'll use big-Theta for the bounds from now on since it's the most specific while still being generally direct process to go from f(n) = 2n^3 + 2 --> Theta(n^3) time

When to use Big-Theta (most of the time):	When you have to use Big-O/Big-Omega:
for any function that's just the sum of its terms like f(n) = 2^n + 3n^3 + 4n - 5 we can always just do the approach of dropping constant multipliers / removing the lower order terms to find the big Theta at a glance.	f(n) { n if n is prime, 1 otherwise} since in this case, the big O (n) and the big Omega(1) Omega don't overlap at the same complexity class, there is no reasonable big-Theta and we couldn't use it here.





Administrivia

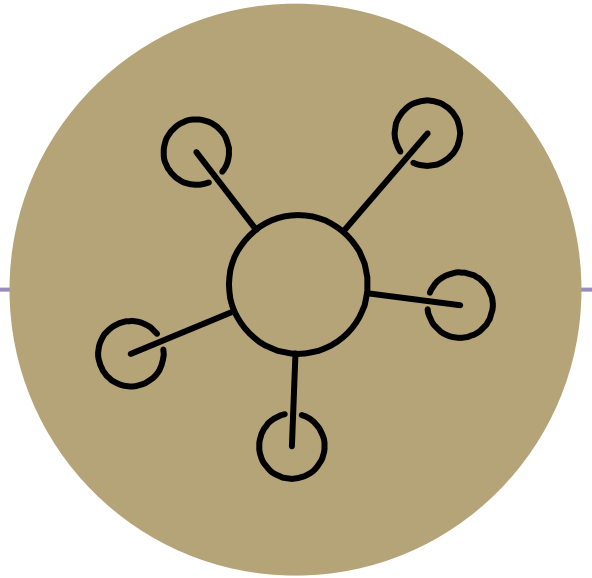
project1 released

- grader-only tests and rate limiting
- you get an additional submission every 10 minutes, can store up to 6 submissions (tokens)
- we don't want you spamming the auto-grader (extra computation on the server + guess and check is generally not that effective for your learning if you're doing it a lot) – instead you can take a breather and investigate the problems / errors. We want to encourage you all to do more testing / reviewing your code to grow to become more independent programmers throughout this course.
- Most students don't bump into the limit / run out of submissions and have to wait to recharge – the point of this isn't to block y'all. If you feel that it's too strict then please bring it up for discussion after trying this assignment.

exercise1 - out by midnight tonight, due next Friday 11:59pm

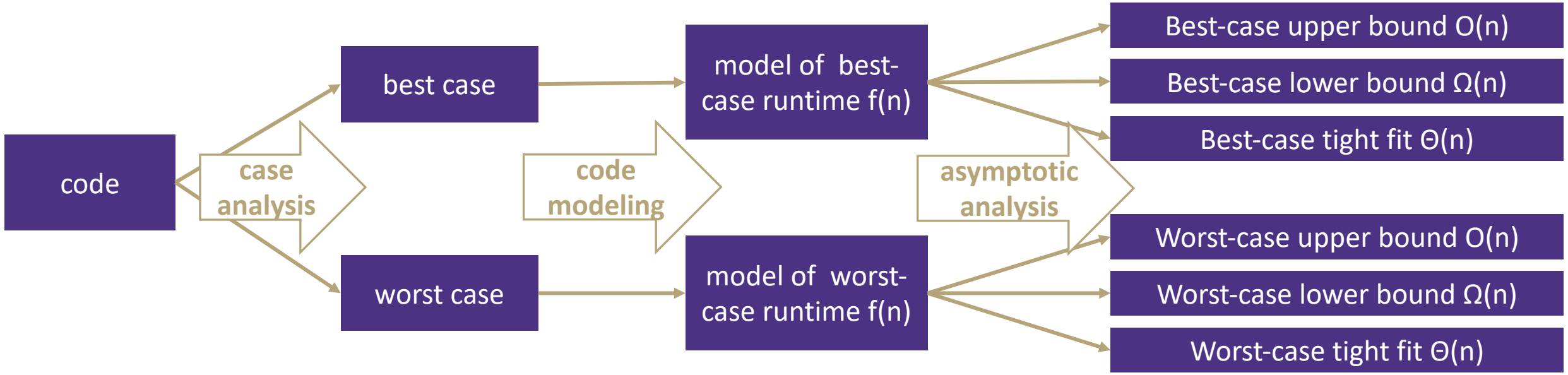
Piazza

- try to use the search bar before you post / use descriptive summary titles when possible - thank you



Questions

Code Analysis Process



Modeling Recursive Code

Recursive Patterns

Modeling and analyzing recursive code is all about finding patterns in how the input changes between calls and how much work is done within each call

Let's explore some of the more common recursive patterns

Pattern #1: Halving the Input

Pattern #2: Constant size input and doing work

Pattern #3: Doubling the Input

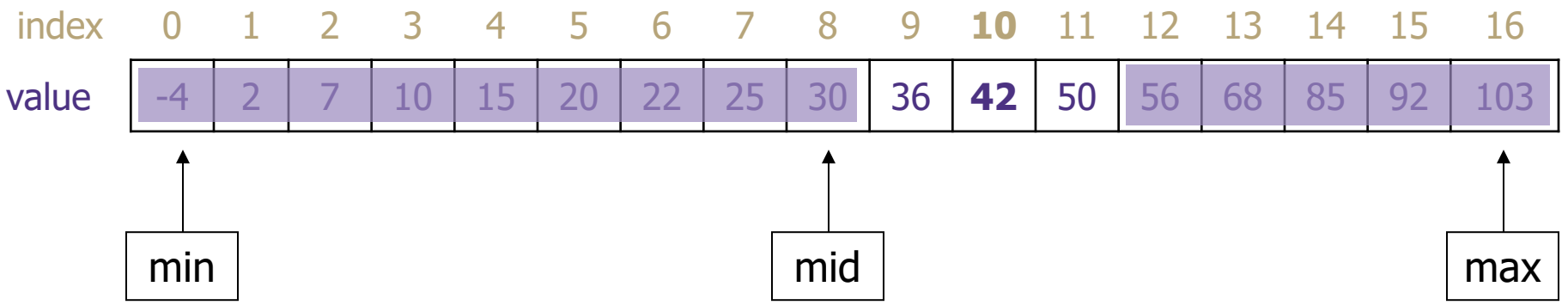
Binary Search

```
public int binarySearch(int[] arr, int toFind, int lo, int hi) {
    if( hi < lo ) {
        return -1;
    } if(hi == lo) {
        if(arr[hi] == toFind) {
            return hi;
        }
        return -1;
    }
    int mid = (lo+hi) / 2;
    if(arr[mid] == toFind) {
        return mid;
    } else if(arr[mid] < toFind) {
        return binarySearch(arr, toFind, mid+1, hi);
    } else {
        return binarySearch(arr, toFind, lo, mid-1);
    }
}
```

Binary Search Runtime

binary search: Locates a target value in a *sorted* array or list by successively eliminating half of the array from consideration.

- Example: Searching the array below for the value **42**:



How many elements will be examined?

- What is the best case?

  element found at index 8, 1 item examined, O(1)

- What is the worst case?

  element not found, ½ elements examined, then ½ of that…

Pattern #1 – Halving the input

Take 1 min to respond to activity

www.pollev.conm/cse373activity
Take a guess! What is the tight Big-O of worst case binary search?

Binary search runtime

For an array of size N, it eliminates ½ until 1 element remains.

N, N/2, N/4, N/8, ..., 4, 2, 1

- How many divisions does it take?

Think of it from the other direction:

- How many times do I have to multiply by 2 to reach N?

  1, 2, 4, 8, ..., N/4, N/2, N
- Call this number of multiplications "x".

$2^x = N$

$x = \log_2 N$

Binary search is in the **logarithmic** complexity class.

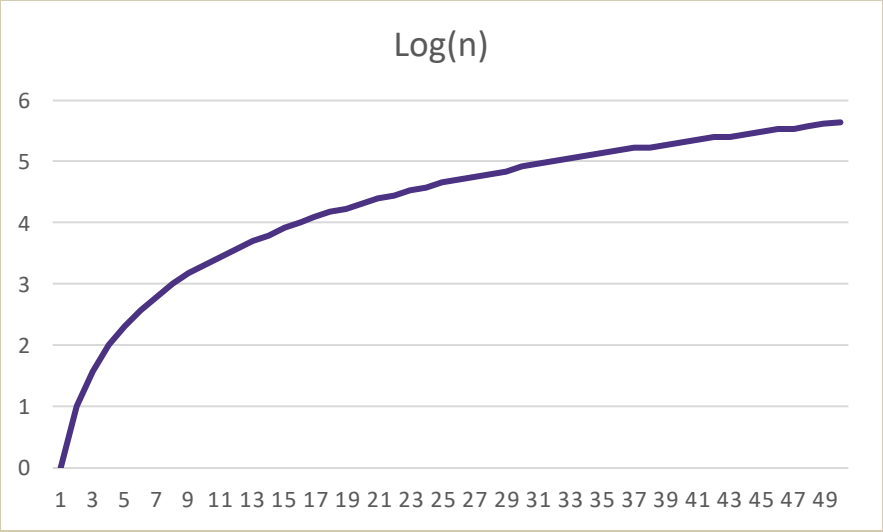
Logarithm – inverse of exponentials

$y = \log_b x \ \ is\ equal\ to\ b^y = x$

Examples:

$2^2 = 4 \Rightarrow 2 = \log_2 4$

$3^2 = 9 \Rightarrow 2 = \log_3 9$

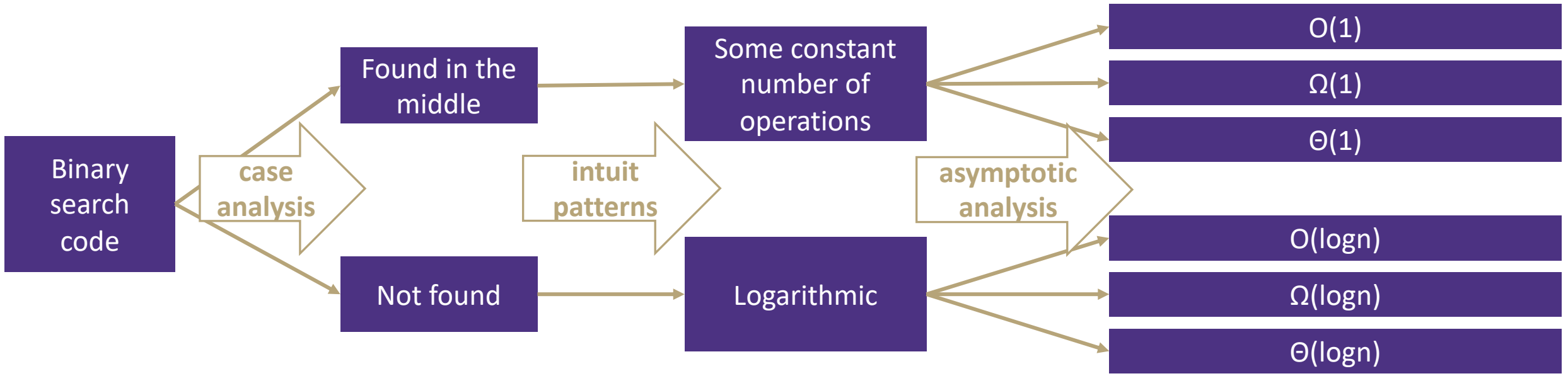


Moving Forward

While this analysis is correct it relied on our ability to think through the pattern intuitively

This works for binary search, but most recursive code is too complex to rely on our intuition.

We need more powerful tools to form a proper code model.



Model

Let's start by just getting a model. Let $F(n)$ be our model for the worst-case running time of binary search.

```
public int binarySearch(int[] arr, int toFind, int lo, int hi) {
    if( hi < lo ) {
        return -1;
    } if(hi == lo) {
        if(arr[hi] == toFind) {
            return hi;
        }
        return -1;
    }
    int mid = (lo+hi) / 2;
    if(arr[mid] == toFind) {
        return mid;
    } else if(arr[mid] < toFind) {
        return binarySearch(arr, toFind, mid+1, hi);
    } else {
        return binarySearch(arr, toFind, lo, mid-1);
    }
}
```

2

4

6

2 + ??

How do you model recursive calls?

With a recursive math function!

Meet the Recurrence

A **recurrence** relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term(s)

It's a lot like recursive code:

- At least one base case and at least one recursive case
- Each case should include the values for n to which it corresponds
- The recursive case should reduce the input size in a way that eventually triggers the base case
- The cases of your recurrence usually correspond exactly to the cases of the code

$$T(n) = \begin{cases} 5 & \text{if } n < 3 \\ 2T\left(\frac{n}{2}\right) + 10 & \text{otherwise} \end{cases}$$

Write a Recurrence

```
public int recursiveFunction(int n){
    if(n < 3) {
        return 3;
    }
    for(int int i=0; i < n; i++) { +n
        System.out.println(i);
    }
    int val1 = recursiveFunction(n/3);
    int val2 = recursvieFunction(n/3);
    return val1 * val2;
}
```

Base Case **2**

Recursive Case
Non-recursive work **n+2**
Recursive work **2*T(n/3)**

$$T(n) = \begin{cases} 2 & \text{if } n < 3 \\ 2T\left(\frac{n}{3}\right) + n & \text{otherwise} \end{cases}$$

Recurrence to Big-Θ

$$T(n) = \begin{cases} 2 & \text{if } n < 3 \\ 2T\left(\frac{n}{3}\right) + n & \text{otherwise} \end{cases}$$

It's still really hard to tell what the big-O is just by looking at it.

But fancy mathematicians have a formula for us to use!

Master Theorem

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Where $f(n)$ is $\Theta(n^c)$

If $\log_b a < c$ then $T(n) \in \Theta(n^c)$

If $\log_b a = c$ then $T(n) \in \Theta(n^c \log n)$

If $\log_b a > c$ then $T(n) \in \Theta\left(n^{\log_b a}\right)$

a=2 b=3 and c=1

$y = \log_b x$ *is equal to* $b^y = x$

$\log_3 2 = x \Rightarrow 3^x = 2 \Rightarrow x \cong 0.63$

$\log_3 2 < 1$

We're in case 1

$T(n) \in \Theta(n)$

Understanding Master Theorem

Master Theorem

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

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- A measures how many recursive calls are triggered by each method instance
- B measures the rate of change for input
- C measures the dominating term of the non recursive work within the recursive method
- D measures the work done in the base case

The $\log_b a < c$ case

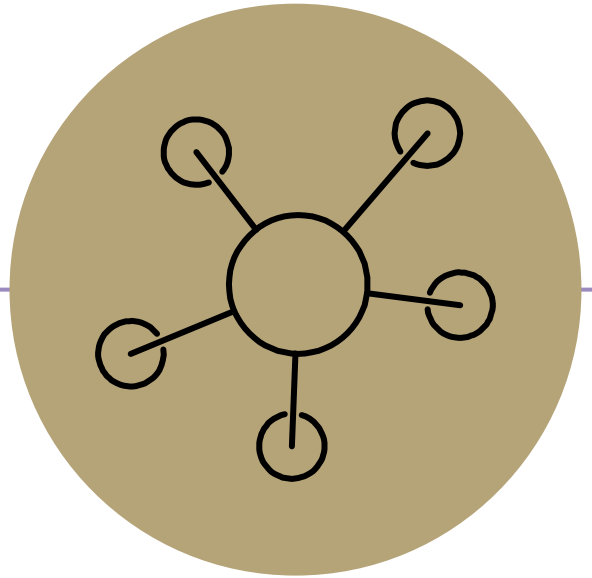
- Recursive case does a lot of non recursive work in comparison to how quickly it divides the input size
- Most work happens in beginning of call stack
- Non recursive work in recursive case dominates growth, n^c term

The $\log_b a = c$ case

- Recursive case evenly splits work between non recursive work and passing along inputs to subsequent recursive calls
- Work is distributed across call stack

The $\log_b a > c$ case

- Recursive case breaks inputs apart quickly and doesn't do much non recursive work
- Most work happens near bottom of call stack



Questions

Recursive Patterns

Pattern #1: Halving the Input

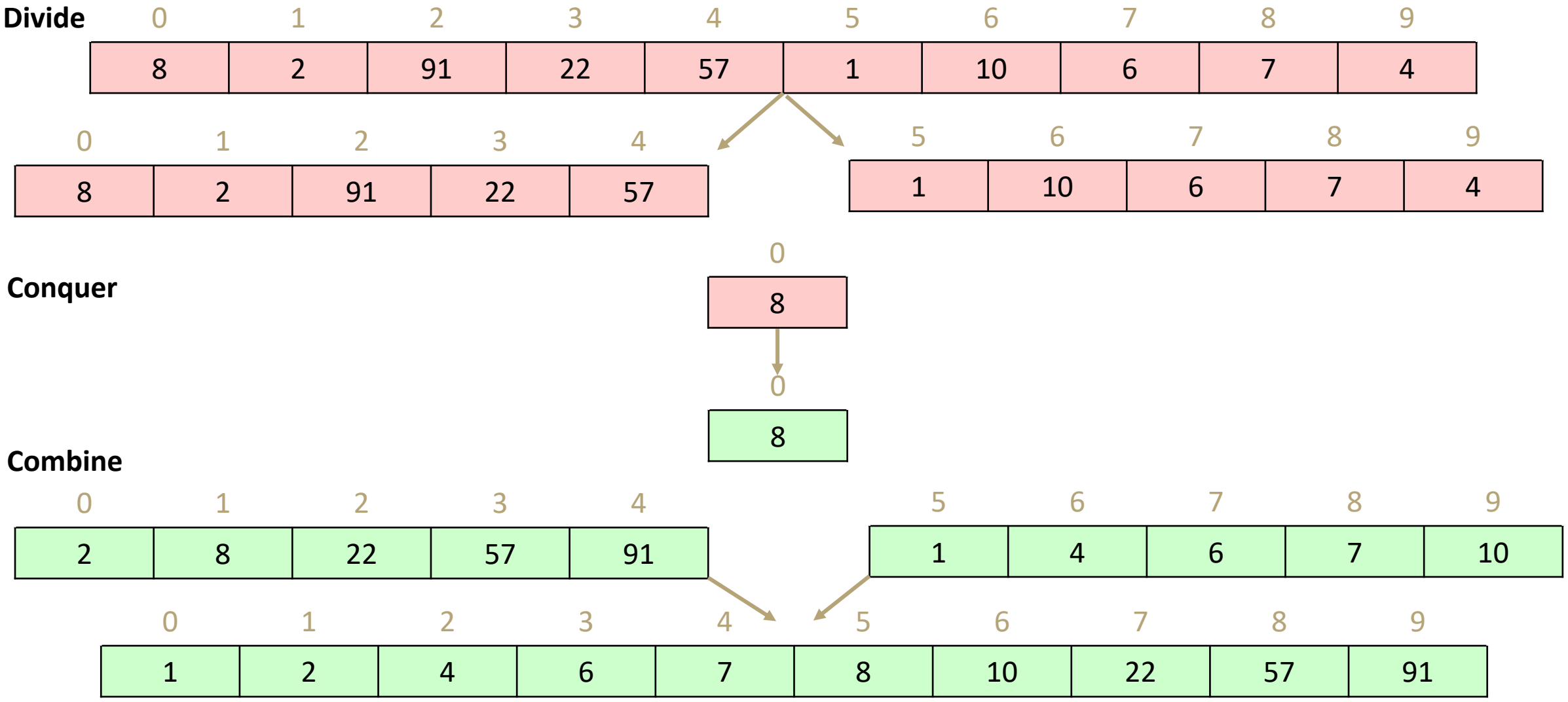
Binary Search $\Theta(\log n)$

Pattern #2: Constant size input and doing work

Merge Sort

Pattern #3: Doubling the Input

Merge Sort


Divide
0 1 2 3 4 5 6 7 8 9
8 2 91 22 57 1 10 6 7 4
0 1 2 3 4
8 2 91 22 57
5 6 7 8 9
1 10 6 7 4
Conquer
0
8
0
8
Combine
0 1 2 3 4
2 8 22 57 91
5 6 7 8 9
1 4 6 7 10
0 1 2 3 4 5 6 7 8 9
1 2 4 6 7 8 10 22 57 91

Merge Sort

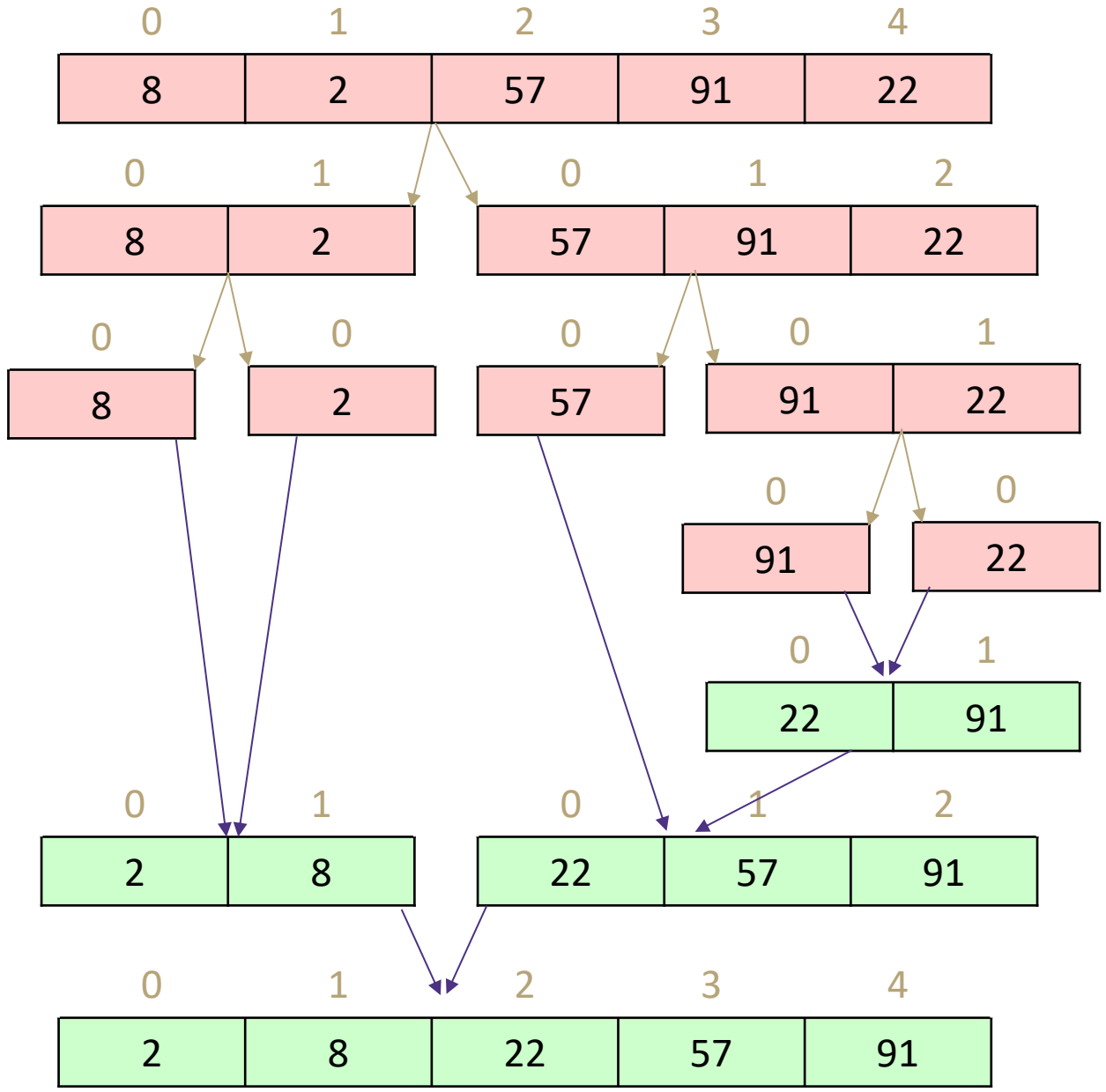
```
mergeSort(input) {
   if (input.length == 1)
      return
   else
      smallerHalf = mergeSort(new [0, ..., mid])
      largerHalf = mergeSort(new [mid + 1, ...])
      return merge(smallerHalf, largerHalf)
}
```

$$T(n) = \begin{cases} 1 \text{ if } n <= 1 \\ 2T(n/2) + n \text{ otherwise} \end{cases}$$

Pattern #2 – Constant size input and doing work

Take 1 min to respond to activity

www.pollev.conm/cse373activity
Take a guess! What is the Big-O of worst case merge sort?



Merge Sort Recurrence to Big-Θ

$$T(n) = \begin{cases} 1 \text{ if } n <= 1 \\ 2T(n/2) + n \text{ otherwise} \end{cases}$$

Master Theorem

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a=2 b=2 and c=1

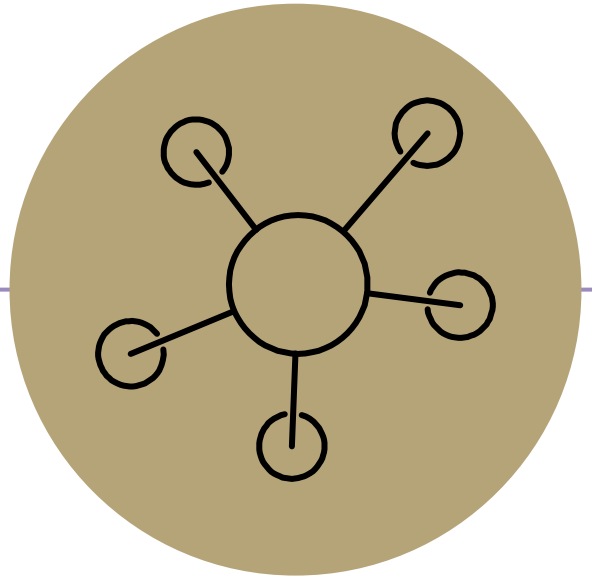
$y = \log_b x$ *is equal to* $b^y = x$

$\log_2 2 = x \Rightarrow 2^x = 2 \Rightarrow x = 1$

$\log_2 2 = 1$

We're in case 2

$T(n) \in \Theta(n \log n)$



Questions

Recursive Patterns

Pattern #1: Halving the Input

Binary Search Θ(logn)

Pattern #2: Constant size input and doing work

Merge Sort Θ(nlogn)

Pattern #3: Doubling the Input

Calculating Fibonacci

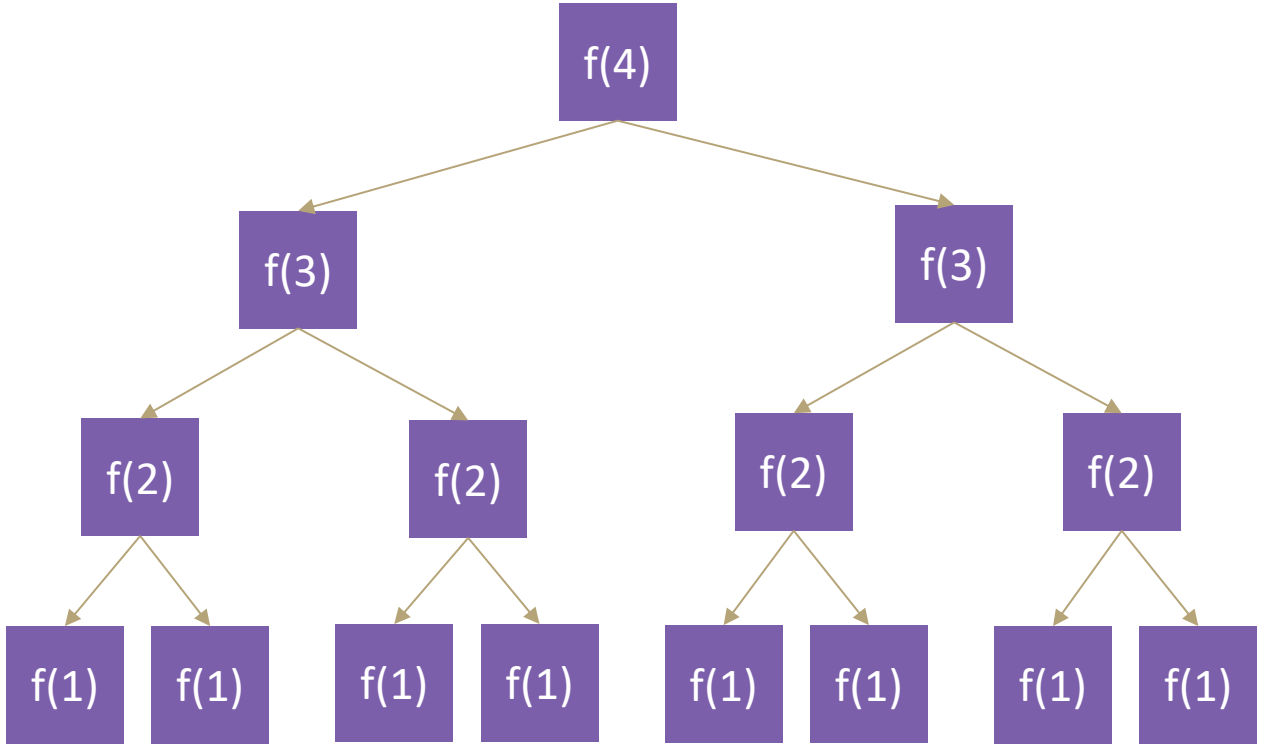
Calculating Fibonacci

```
public int fib(int n) {
    if (n <= 1) {
        return 1;
    }
    return fib(n-1) + fib(n-1);
}
```

- Each call creates 2 more calls
- Each new call has a copy of the input, almost
- Almost doubling the input at each call

Pattern #3 – Doubling the Input

Almost



Calculating Fibonacci Recurrence to Big-Θ

```
public int f(int n) {
    if (n <= 1) {
        return 1;
    }
    return f(n-1) + f(n-1);
}
```

d

2T(n-1) + c

$$T(n) = \begin{cases} d \text{ when } n \leq 1 \\ 2T(n-1) + c \text{ otherwise} \end{cases}$$

Take 1 min to respond to activity

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Finish the recurrence, what is the model for the recursive case?

Can we use master theorem?

Master Theorem

$$T(n) = \begin{cases} d & \text{if } n \text{ is at most some constant} \\ aT\left(\frac{n}{b}\right) + f(n) & \text{otherwise} \end{cases}$$

Uh oh, our model doesn't match that format…

Can we intuit a pattern?

T(1) = d
T(2) = 2T(2-1) + c = 2(d) + c
T(3) = 2T(3-1) + c = 2(2(d) + c) + c = 4d + 3c
T(4) = 2T(4-1) + c = 2(4d + 3c) + c = 8d + 7c
T(5) = 2T(5-1) + c = 2(8d + 7c) + c = 16d +25c

Looks like something's happening but it's tough

Maybe geometry can help!

Calculating Fibonacci Recurrence to Big-Θ

How many layers in the function call tree?

How many layers will it take to transform "n" to the base case of "1" by subtracting 1

For our example, 4 -> Height = n

$$T(n) = \begin{cases} d \text{ when } n \leq 1 \\ 2T(n-1) + c \text{ otherwise} \end{cases}$$

How many function calls per layer?

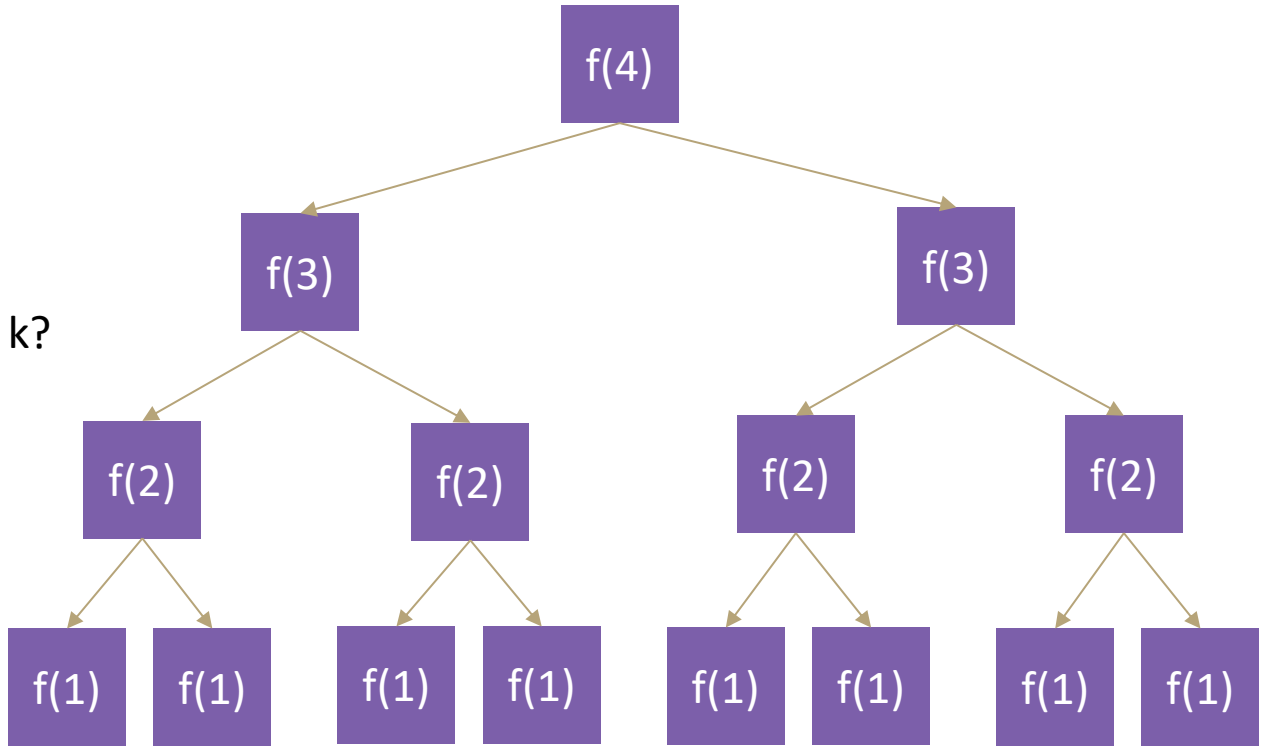
Layer	Function calls
1	1
2	2
3	4
4	8

How many function calls on layer k?

$$\mathbf{2^{k-1}}$$

How many function calls TOTAL for a tree of k layers?

$$\mathbf{1 + 2 + 3 + 4 + \ldots + 2^{k-1}}$$



Calculating Fibonacci Recurrence to Big-Θ

Patterns found:

How many layers in the function call tree? n

How many function calls on layer k? 2^{k-1}

How many function calls TOTAL for a tree of k layers?

$1 + 2 + 4 + 8 + \ldots + 2^{k-1}$

Total runtime = (total function calls) x (runtime of each function call)

Total runtime = $(1 + 2 + 4 + 8 + \ldots + 2^{k-1})$ x (constant work)

$$1 + 2 + 4 + 8 + \ldots + 2^{k-1} = \sum_{i=1}^{k-1} 2^i = \frac{2^k - 1}{2 - 1} = 2^k - 1$$

Summation Identity
Finite Geometric Series

$$\sum_{i=1}^{k-1} x^i = \frac{x^k - 1}{x - 1}$$

$$\boldsymbol{T(n) = 2^n - 1 \in \Theta(2^n)}$$

Recursive Patterns

Pattern #1: Halving the Input

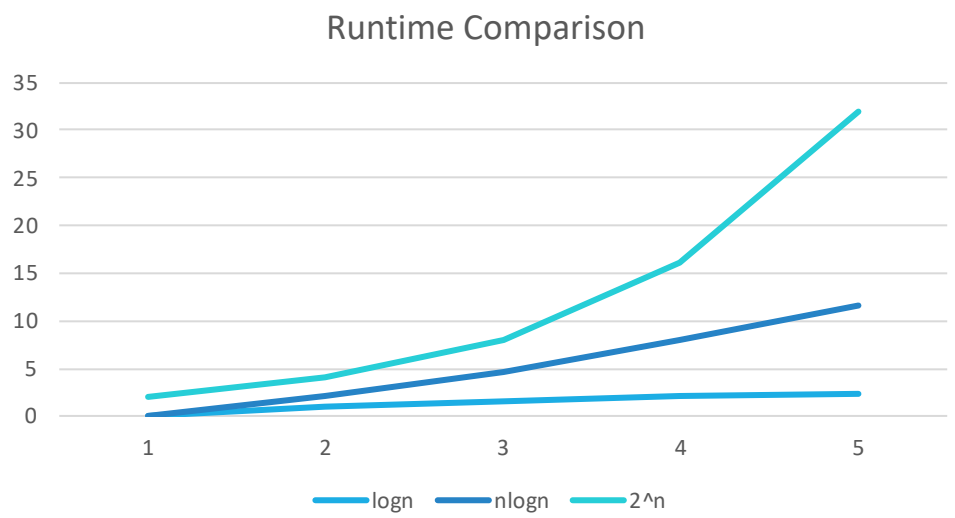
Binary Search Θ(logn)

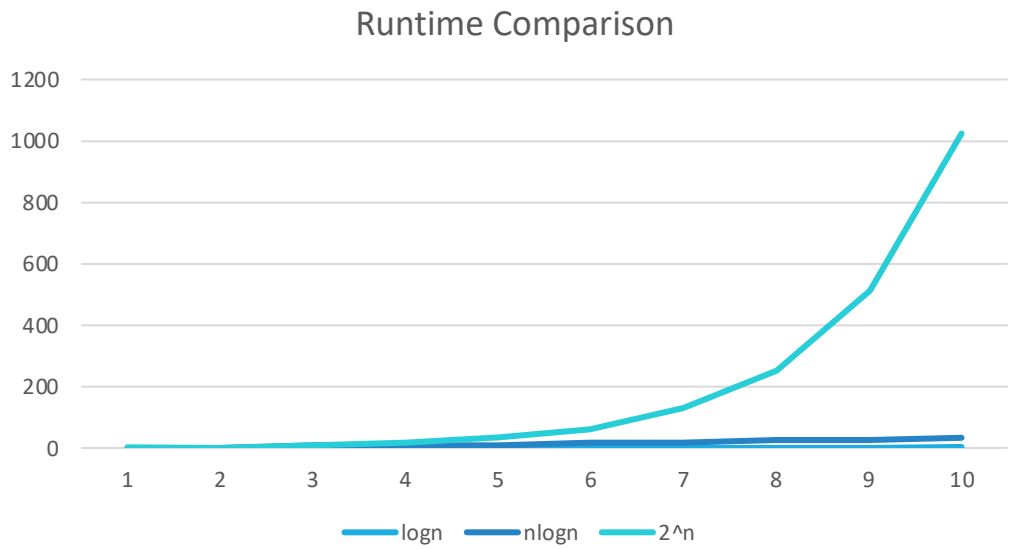
Pattern #2: Constant size input and doing work

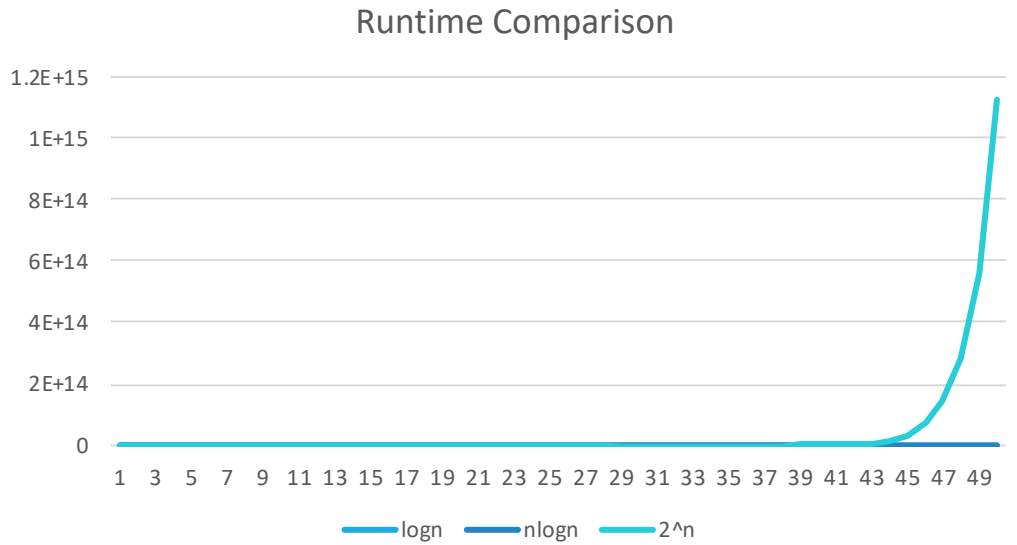
Merge Sort Θ(nlogn)

Pattern #3: Doubling the Input

Calculating Fibonacci $\Theta(2^n)$







Appendix

Another example

```
public int Mystery(int n){
    if(n == 1) {                          +1
        return 1;                         +1
    } else {
        for(int i = 0; i < n; i++){
            for(int j = 0; j < n; j++){
                System.out.println("hi!");  +1
            }
        }
        return Mystery(n/2)
    }
}
```

n

n

$$T(n) = \begin{cases} C \ \ when\, n = 1 \\ T(n/2) + n^2 \text{ if n} > 1 \end{cases}$$

There is no magic shortcut for these problems (except in a few well-behaved cases).

We'll expect you to know these two summations since they're common patterns.

$$1 + 2 + 3 + 4 + \cdots + Q = \frac{Q(Q+1)}{2} \quad \in \Theta(Q^2)$$

$$1 + 2 + 4 + 8 + \cdots + Q = 2Q - 1 \quad \in \Theta(Q)$$

Strategies.

Find the exact count of steps.

Write out examples.

Use a geometric argument–visualizations!