LEC 04

CSE 373

Asymptotic Analysis

BEFORE WE START

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Announcements

- Project 0 (CSE 143 Review) due Wednesday 10/7 11:59pm
- Project 1 (Deques) comes out the same day
  - Three options for projects:
    - **Choose a partner** – someone you know or meet in the class
    - **Join the partner pool** – we'll assign you a partner
    - **Opt to work alone** – not recommended, but available
- Exercise 1 (written, individual) released Friday
- Option to choose your breakout room for class sessions!
  - See Ed announcement for details on how to sign up! Can modify at any time.
  - Will still use random assignment for those in class who don't sign up.

Learning Objectives

After this lecture, you should be able to...

1. Describe the difference between Code Modeling and Asymptotic Analysis (both components of Algorithmic Analysis)
2. Model a (simple) piece of code with a function describing its runtime
3. Explain why we can throw away constants when we compute Big-Oh bounds.
   - From a practical perspective and from the “definition” perspective.

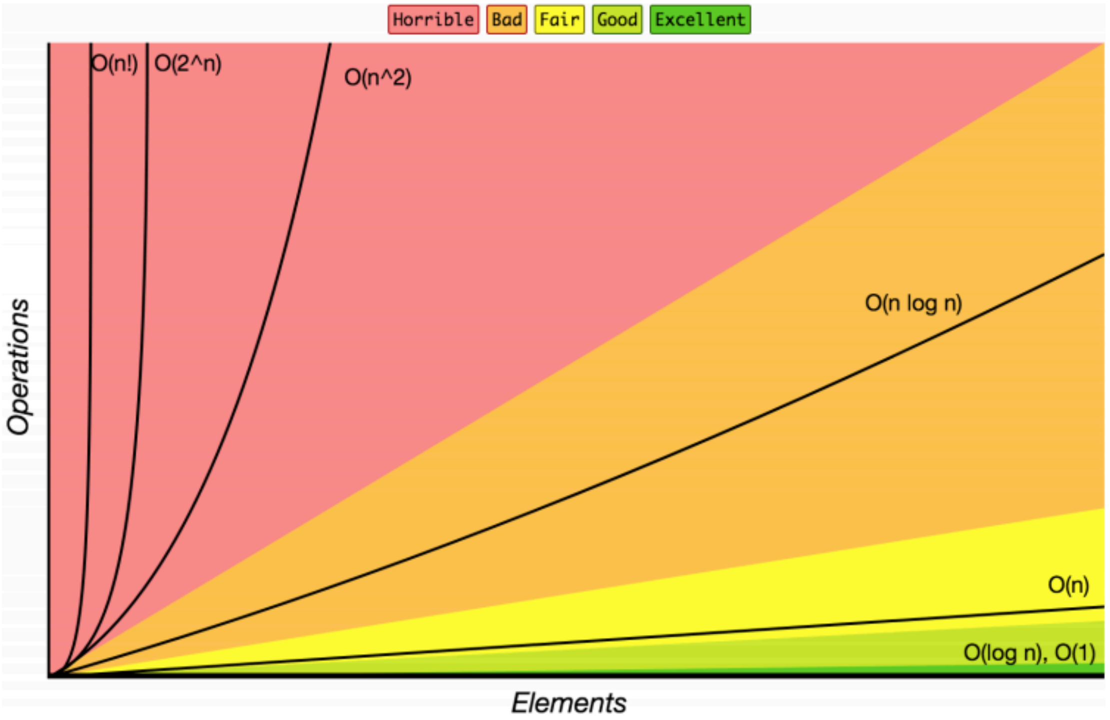
Lecture Outline

- **Overview: Algorithmic Analysis**
- Code Modeling
- Asymptotic Analysis
- Big-O Definition

143 Review Complexity Class

- **Complexity Class**: a category of algorithm efficiency based on the algorithm's relationship to the input size N

Complexity Class	Big-O	Runtime if you double N
constant	**O(1)**	**unchanged**
logarithmic	$O(\log_2 N)$	increases slightly
linear	**O(N)**	**doubles**
log-linear	$O(N \log_2 N)$	slightly more than doubles
quadratic	$\mathbf{O(N^2)}$	**quadruples**
…	…	…
exponential	$O(2^N)$	multiplies drastically



Review Big-Oh Analysis: Why?

	ArrayList	LinkedList
`add (front)`	O(n) linear	O(1) constant
`remove (front)`	O(n) linear	O(1) constant
`add (back)`	O(1) constant usually	O(n) linear
`remove (back)`	O(1) constant	O(n) linear
`get`	O(1) constant	O(n) linear
`insert (anywhere)`	O(n) linear	O(n) linear

- Complexity classes help us differentiate between data structures
  - "Just change first node" vs. "Change every element" is clearly different
  - To *evaluate* data structures, need to understand impact of design decisions

Review Big-Oh Analysis: Why?

- We need a tool to analyze code, and we want it to be:

A **Simple**
We don't care about tiny differences in implementation, want the big picture result

B **Mathematically Rigorous**
Use mathematical functions as a precise, flexible basis

C **Decisive**
Produce a clear comparison indicating which code takes "longer"

Review Big-Oh Analysis: ... How?!

CODE

BIG-OH

```
for (i = 0; i < n; i++) {
  a[i] = 1;
  b[i] = 2;
}
```

O(n)

- 143 general patterns: “O(1) constant is no loops, O(n) is one loop, O(n^2) is nested loops”
  - This is still useful!
  - But in 373 we’ll go much more in depth: we can explain more about *why*, and how to handle more complex cases when they arise (which they will!)

Overview: Algorithmic Analysis


```
for (i = 0; i < n; i++) {
  a[i] = 1;
  b[i] = 2;
}
```

$f(n) = 2n$

$O(n)$

- **Algorithmic Analysis**: The overall process of characterizing code with a complexity class, consisting of:
  - **Code Modeling**: Code → Function describing code's runtime
  - **Asymptotic Analysis**: Function → Complexity class describing asymptotic behavior

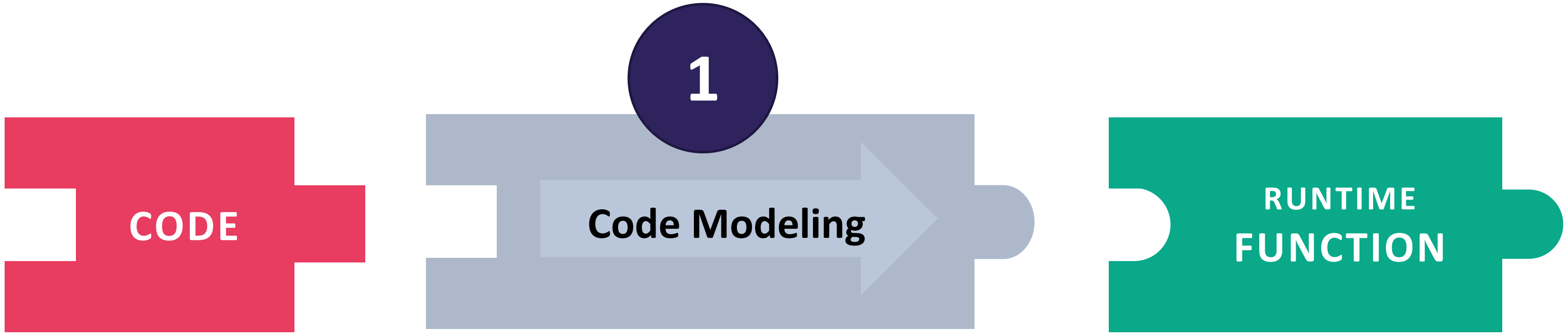
Lecture Outline

- Overview: Algorithmic Analysis
- **Code Modeling**
- Asymptotic Analysis
- Big-O Definition

Talking About Code

- **Cost Model**: An analysis mindset to express the resource whose growth rate is being measured
- For simplicity, we'll discuss everything in terms of runtime today
  - But other cost models exist! For example, storage space is common

Code Modeling



- **Code Modeling** – the process of mathematically representing how many operations a piece of code will run in relation to the input size n.
  - Convert from code to a function representing its runtime

What is an operation?

- We don’t know exact runtime of every operation, but for now let’s try simplifying assumption: all basic operations take the same time

- Basics:
  - +, -, /, *, %, ==
  - Assignment
  - Returning
  - Variable/array access
- Function Calls
  - Total runtime in body
  - Remember: new calls a function (constructor)
- Conditionals
  - Test + time for the followed branch
    - Learn how to reason about branch later
- Loops
  - Number of iterations * total runtime in condition and body

Code Modeling Example I

```
public void method1(int n) {
    int sum = 0;            +1
    int i = 0;              +1
    while (i < n) {         +1
        sum = sum + (i * 3); +3
        i = i + 1;          +2
    }
    return sum;             +1
}
```

Loop runs n times

+6 *n

$f(n) = 6n + 3$

Code Modeling Example II

```
public void method2(int n) {
    int sum = 0;              +1
    int i = 0;                +1
    while (i < n) {           +1
        int j = 0;            +1
        while (j < n) {       +1
            if (j % 2 == 0) { +2
                // do nothing
            }
            sum = sum + (i * 3) + j;  +4
            j = j + 1;        +2
        }
        i = i + 1;            +2
    } return sum;             +1
}
```

This inner loop runs n times

+9 *n

This outer loop runs n times

9n + 4 *n

f(n) = (9n+4)n + 3

Lecture Outline

- Overview: Algorithmic Analysis
- Code Modeling
- **Asymptotic Analysis**
- Big-O Definition

Where are we?

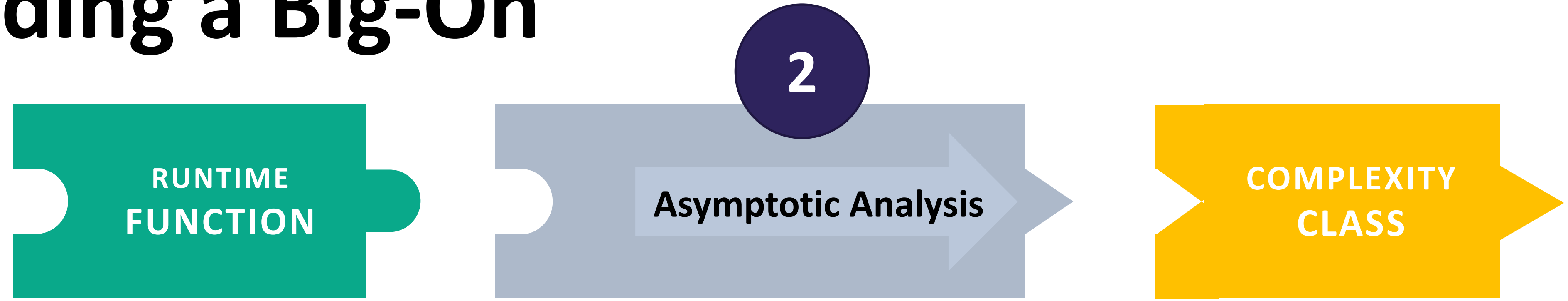

```
for (i = 0; i < n; i++) {
  a[i] = 1;
  b[i] = 2;
}
```

$f(n) = 2n$

$O(n)$

- We just turned a piece of code into a function!
  - We'll look at better alternatives for code modeling later
- Now to focus on step 2, asymptotic analysis

Finding a Big-Oh



- We have an expression for $f(n)$. How do we get the $O()$ that we've been talking about?

1. Find the "dominating term" and delete all others.
    - The "dominating" term is the one that is largest as n gets bigger. In this class, often the largest power of n.
2. Remove any constant factors.

f(n) = (9n+3)n + 3

$= 9n^2 + 3n + 3$

$\approx 9n^2$

$\approx n^2$

f(n) is $O(n^2)$

Asymptotic Analysis

What is the complexity class for the following function?

$$f(n) = 1{,}0000 + 400n + 0.00001n^3 + 20n^2$$

Is it okay to throw away all that info?

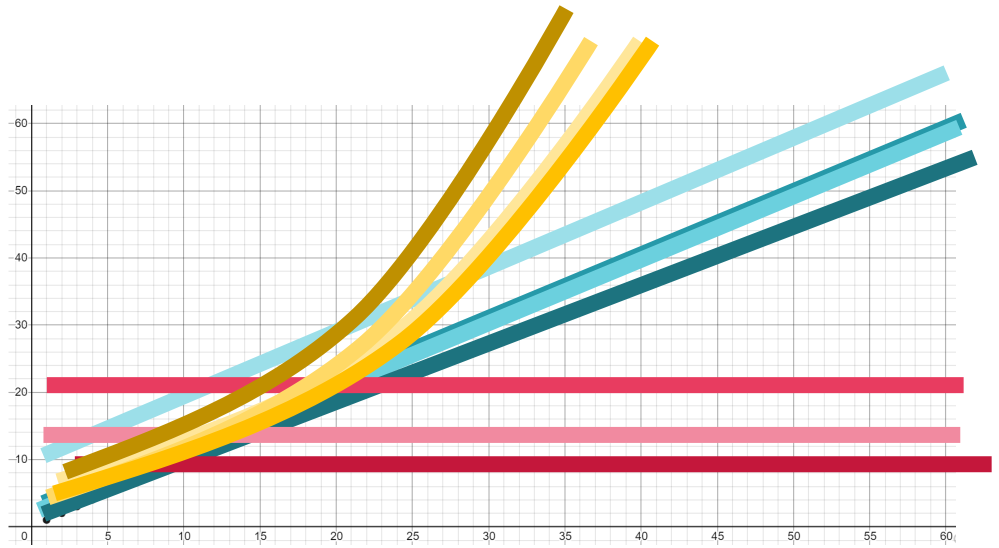
- Big-Oh is like the "significant digits" of computer science
- **Asymptotic Analysis**: Analysis of function behavior as its input approaches infinity
  - We only care about what happens when n approaches infinity
  - For small inputs, doesn't really matter: all code is "fast enough"
  - Since we're dealing with infinity, constants and lower-order terms don't meaningfully add to the final result. The highest-order term is what drives growth!

Remember our goals:

A **Simple**
We don't care about tiny differences in implementation, want the big picture result

C **Decisive**
Produce a clear comparison indicating which code takes "longer"

No seriously, this is really okay?



- There are tiny variations in these functions (2n vs. 3n vs. 3n+1)
  - But at infinity, will be clearly grouped together
  - We care about which *group* a function belongs in

- Let's convince ourselves this is the right thing to do:
  - https://www.desmos.com/calculator/t9qvn56yyb

What *is* an operation, again?

```
public void method1(int n) {
    int sum = 0;
    int i = 0;
    while (i < n) {
        sum = sum + (i * 3);
        i = i + 1;
    }
    return sum;
}
```

Operation	Count
Assignment	2 + 2n
<	n
+	2n
*	n
Return	1

- We could try being more precise, and count up individual operations
  - Then, sum the time each operation takes
  - But how long *do* they take? Some architectures are really fast at +, others faster at assignment
  - And when we compile it, our code gets expressed as lower-level operations anyway! **It's almost impossible to stare at code and know the "true" constants.**

```
public static void method1(int[]); Code:
0: iconst_0          10: iload_2       18: istore_2
1: istore_1          11: iconst_3      19: goto 4
2: iconst_0          12: imul          22: iload_1
3: istore_2          13: iadd          23: ireturn
4: iload_2           14: istore_1
5: iload_0           15: iload_2
6: if_icmpge 22      16: iconst_1
9: iload_1           17: iadd
```

Code Modeling Anticipating Asymptotic Analysis

- We can't accurately model the constant factors just by staring at the code.
  - And the lower-order terms matter even less than the constant factors.

- Since they're going to be thrown away anyway, you can anticipate which constants are unnecessary to count precisely during Code Modeling
  - e.g. a loop body containing a constant 2 vs. 10 operations is unimportant here

- This does not mean you shouldn't care about constant factors ever – they *are* important in real code!
  - Asymptotic analysis is just one tool, but other perspectives that do consider constants are also valid and useful!

Big-Oh Analysis: Why?

- We need a tool to analyze code, and we want it to be:

A **Simple**
We don't care about tiny differences in implementation, want the big picture result

B **Mathematically Rigorous**
Use mathematical functions as a precise, flexible basis

C **Decisive**
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- **Big-O Definition**

Using Formal Definitions

- If analyzing simple or familiar functions, don't bother with the formal definition. You *can* be comfortable using your intuition!
- We're going to be making more subtle big-O statements in this class.
  - We need a mathematical definition to be sure we know exactly where we are.
- We're going to teach you how to use the formal definition, so if you come across a weird edge case, you know how to get your bearings.

B

Mathematically Rigorous

Use mathematical functions as a precise, flexible basis

Big-Oh Definition

- We wanted to find an upper bound on our algorithm's running time, but
  - We only care about what happens as n gets large.
  - We don't want to care about constant factors.

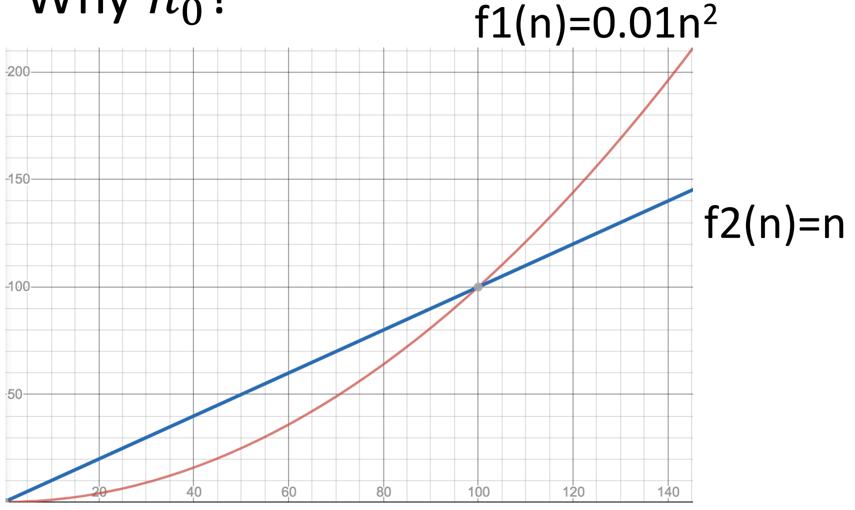
Big-Oh

$f(n)$ is $O(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

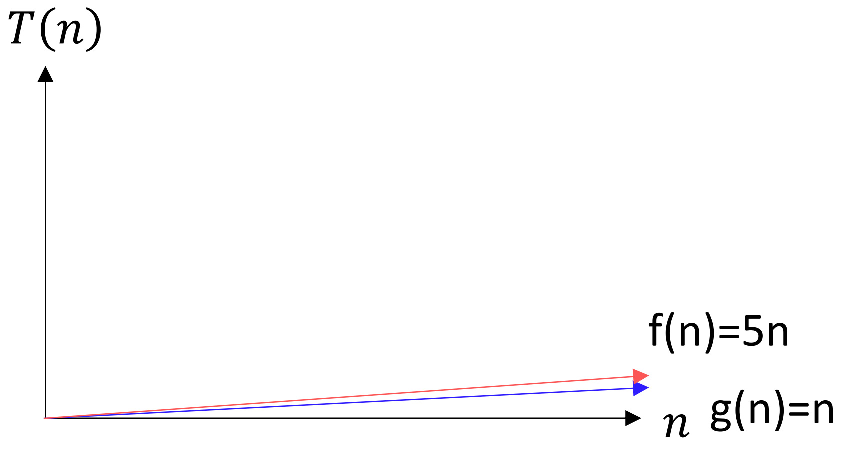
$$f(n) \leq c \cdot g(n)$$

Intuition: $g(n)$ "eventually dominates" $f(n)$

Why n_0?



Why c?



Big-Oh Proofs

Big-Oh

$f(n)$ is $O(g(n))$ if there exist positive constants c, n_0 such that for all $n \geq n_0$,

$$f(n) \leq c \cdot g(n)$$

Show that $f(n) = 10n + 15$ is $O(n)$.

Apply definition term by term

$10n \leq c \cdot n$ when $c = 10$ for all values of n. $So\ 10n \leq 10n$, for all n

$15 \leq c \cdot n$ when $c = 15\ for\ n \geq 1$. $So\ 15 \leq 15n$, when $n \geq 1$

Add up all your truths

$10n + 15 \leq 10n + 15n = 25n$ for $n \geq 1$

10n + 15 <= 25n for n >= 1.

which is in the form of the definition

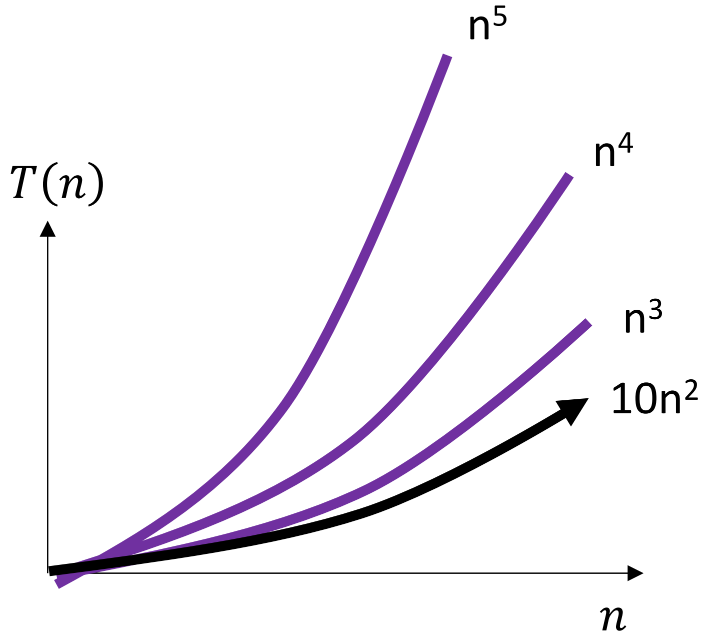
f(n) <= c * g(n)

where c = 25 and n_0 = 1.

Big-Oh Doesn't Have to be Tight

- True or False: $10n^2$ is $O(n^3)$
- It's true – it fits the definition

  $10n^2 \leq c \cdot n^3$ $when$ $c = 10$ for $n \geq 1$



- Big-O is just an upper bound that may be loose and not describe the function fully. For example, all of the following are true:

$10n^2$ is $O(n^3)$
$10n^2$ is $O(n^4)$
$10n^2$ is $O(n^5)$
$10n^2$ is $O(n^n)$
$10n^2$ is $O(n!)$ … and so on