LEC 19

CSE 373

Disjoint Sets I

BEFORE WE START

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Announcements

- P3 due **Wednesday 11/18** at 11:59pm PST
- P4, *the last project of the quarter*, released tonight!
  - If you haven’t already, please fill out the P4 Partner Form so we can send out partner assignments

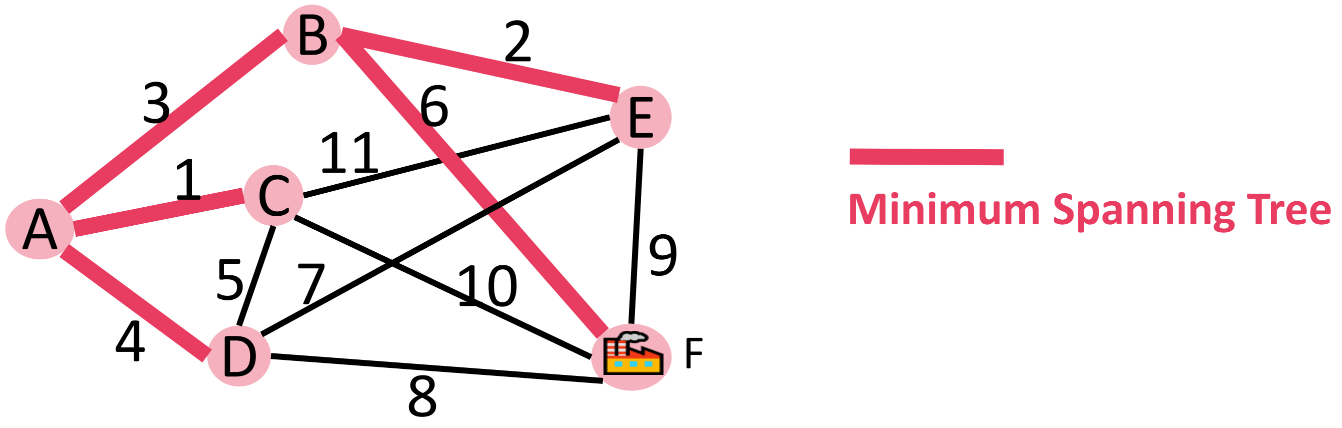
Learning Objectives

After this lecture, you should be able to...

1. Describe Kruskal's Algorithm, evaluate why it works, and describe why it needs a new ADT
2. Compare and contrast QuickUnion with QuickFind and describe how the two structure optimize for different operations
3. Implement WeightedQuickUnion and describe why making the change protects against the worst case find runtime

Review Minimum Spanning Trees (MSTs)

- A Minimum Spanning Tree for a graph is a set of that graph's edges that connect all of that graph's vertices (**spanning**) while minimizing the total weight of the set (**minimum**)
  - Note: does NOT necessarily minimize the path from each vertex to every other vertex
  - Any tree with V vertices will have V-1 edges
  - A separate entity from the graph itself! More of an "annotation" applied to the graph, just like a Shortest Paths Tree (SPT)

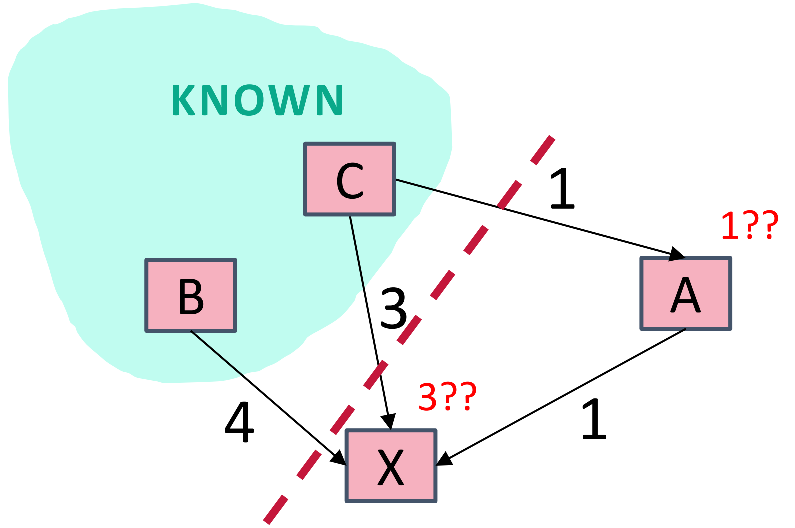


Review Why do MST Algorithms Work?

- Two useful properties for MST edges. We can think about them from either perspective:
  - **Cycle Property**: The heaviest edge along a cycle is NEVER part of an MST.
  - **Cut Property**: Split the vertices of the graph into any two sets A and B. The lightest edge between A and B is ALWAYS part of an MST. *(Prim's thinks this way)*
- Whenever you add an edge to a tree you create exactly one cycle. Removing any edge from that cycle gives another tree!
- This observation, combined with the cycle and cut properties form the basis of all of the **greedy algorithms** for MSTs.
  - **greedy algorithm**: chooses best known option at each point and *commits*, rather than waiting for a global view of the graph before deciding

Review Adapting Dijkstra's: Prim's Algorithm

- Normally, Dijkstra's checks for a shorter path from the start.
- But MSTs don't care about individual paths, only the overall weight!
- New condition: "would this be a smaller edge to connect the current known set to the rest of the graph?"



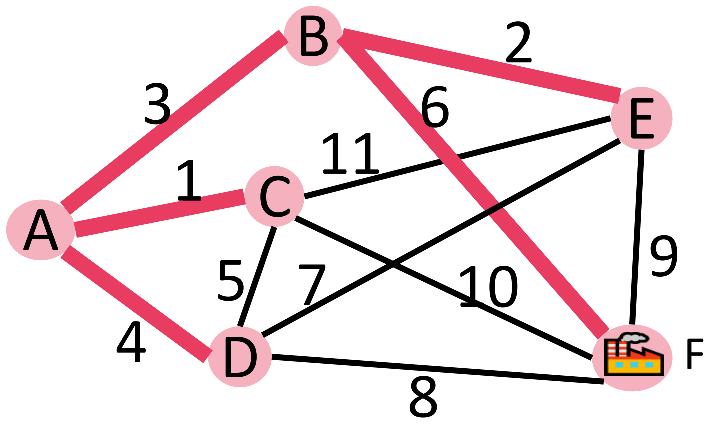

```
prims (G graph, V start)
Map edgeTo, distTo;
initialize distTo with all nodes mapped to ∞, except start to 0
PriorityQueue<V> perimeter; perimeter.add(start);

while (!perimeter.isEmpty()):
  u = perimeter.removeMin()
  known.add(u)
  for each edge (u,v) to unknown v with weight w:
    oldDist = distTo.get(v)      // previous smallest edge to v
    newDist = ~~distTo.get(u) +~~ w  // is this a smaller edge to v?
    if (newDist < oldDist):
      distTo.put(u, newDist)
      edgeTo.put(u, v)
      if (perimeter.contains(v)):
        perimeter.changePriority(v, newDist)
      else:
        perimeter.add(v, newDist)
```

A Different Approach

- Suppose the MST on the right was produced by Prim's
- **Observation**: We basically chose all the smallest edges in the entire graph (1, 2, 3, 4, 6)
  - The only exception was 5. Why shouldn't we add edge 5?
  - Because adding 5 would create a cycle, and to connect A, C, & D we'd rather choose 1 & 4 than 1 & 5 or 4 & 5.

- Prim's thinks "vertex by vertex", but what if you think "edge by edge" instead?
  - Start with the smallest edge in the entire graph and work your way up
  - Add the edge to the MST as long as it connects two new groups (meaning don't add any edges that would create a cycle)



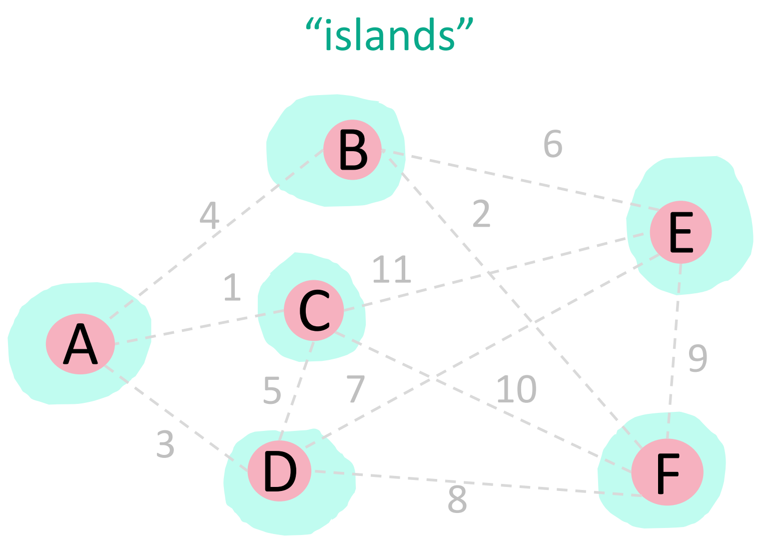
Building an MST "edge by edge" in this graph:

- Add edge 1
- Add edge 2
- Add edge 3
- Add edge 4
- Skip edge 5 (would create a cycle)
- Add edge 6
- Finished: all vertices in the MST!

Kruskal's Algorithm

- This "edge by edge" approach is how **Kruskal's Algorithm** works!

- **Key Intuition**: Kruskal's keeps track of isolated "islands" of vertices (each is a sub-MST)
  - Start with each vertex as its own "island"
  - If an edge connects two vertices within the same "island", it forms a cycle! Discard it.
  - If an edge connects two vertices in different "islands", add it to the MST! Now those "islands" need to be combined.




```
kruskalMST(G graph)
  Set(?) msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)

  for each edge (u,v) in ascending order:
    uMST = msts.find(u)
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    if (uMST != vMST):
      finalMST.add(edge (u, v))
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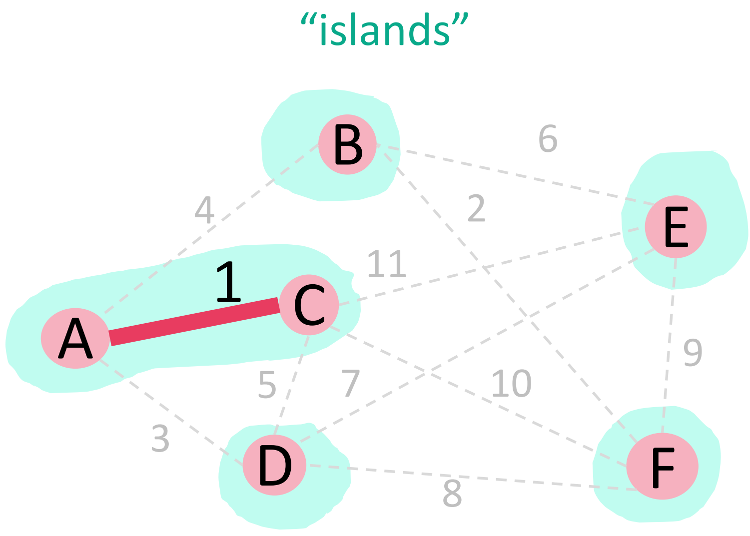


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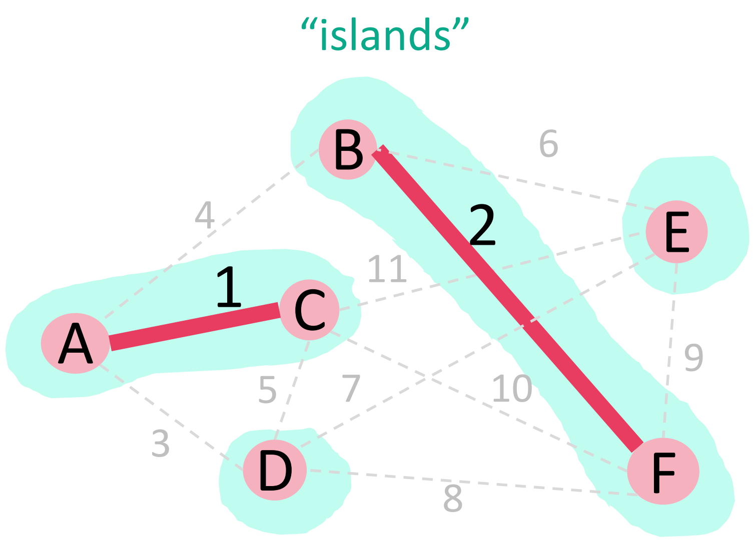

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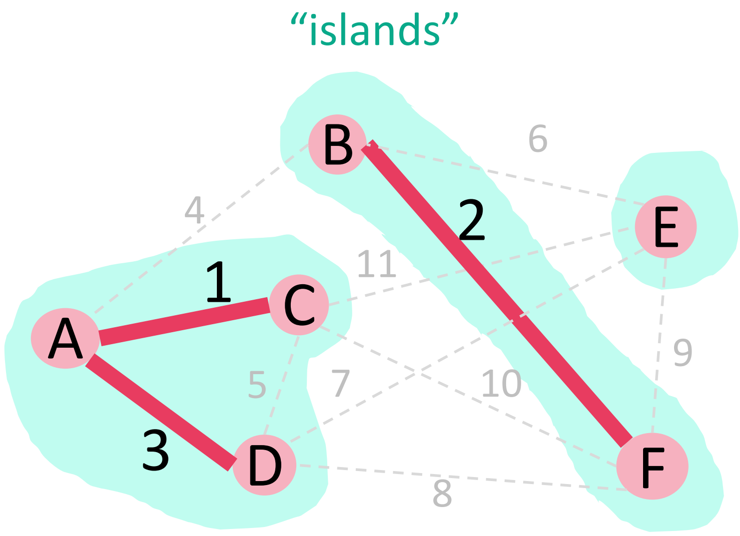

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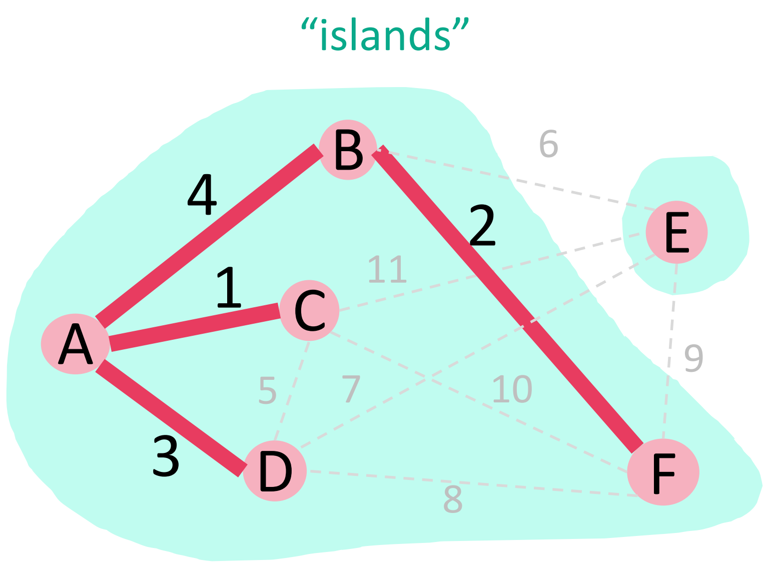


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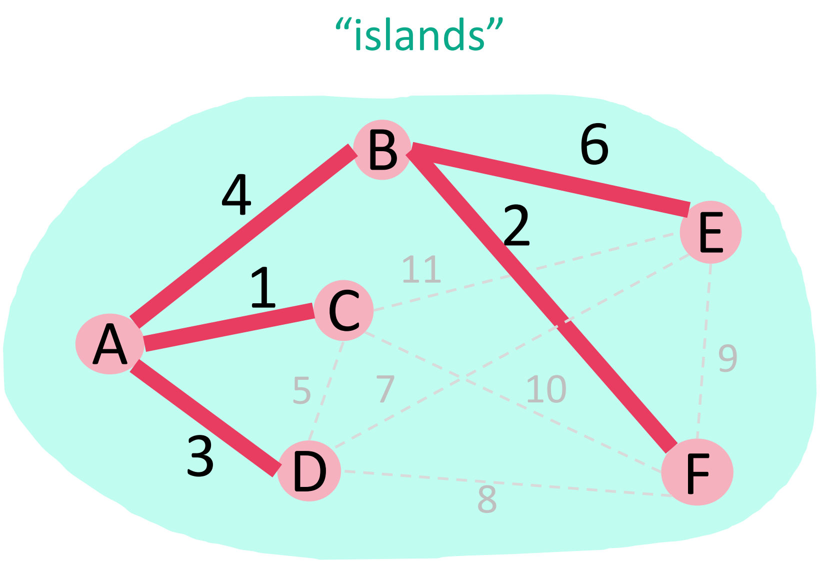

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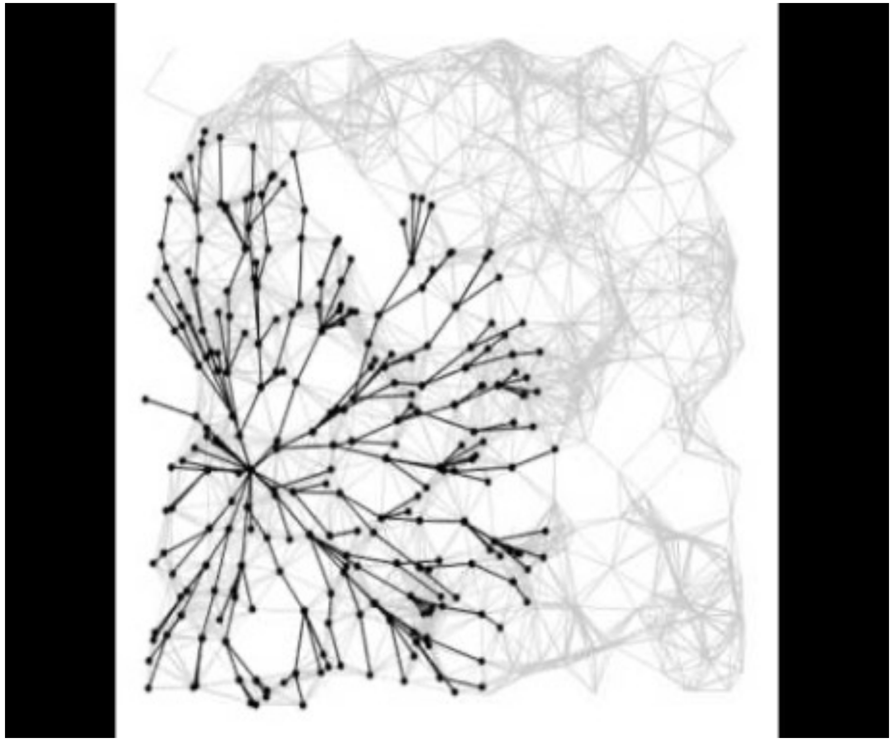
Prim's Demos and Visualizations

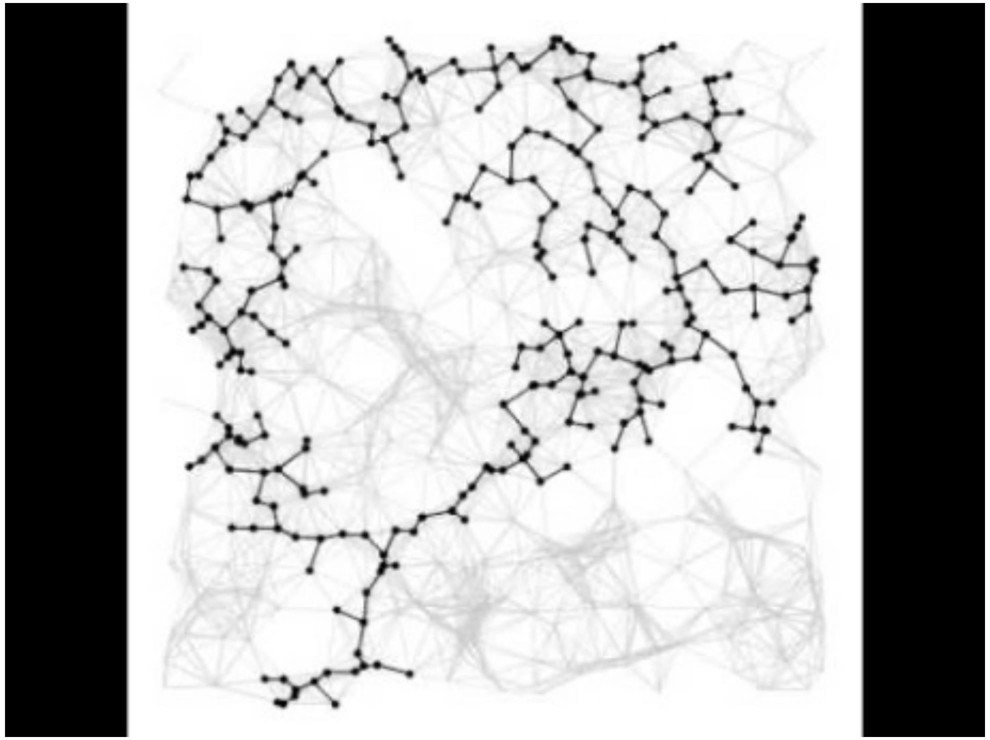
Dijkstra's Algorithm

Dijkstra's proceeds radially from its source, because it chooses edges by *path length from source*



Prim's Algorithm

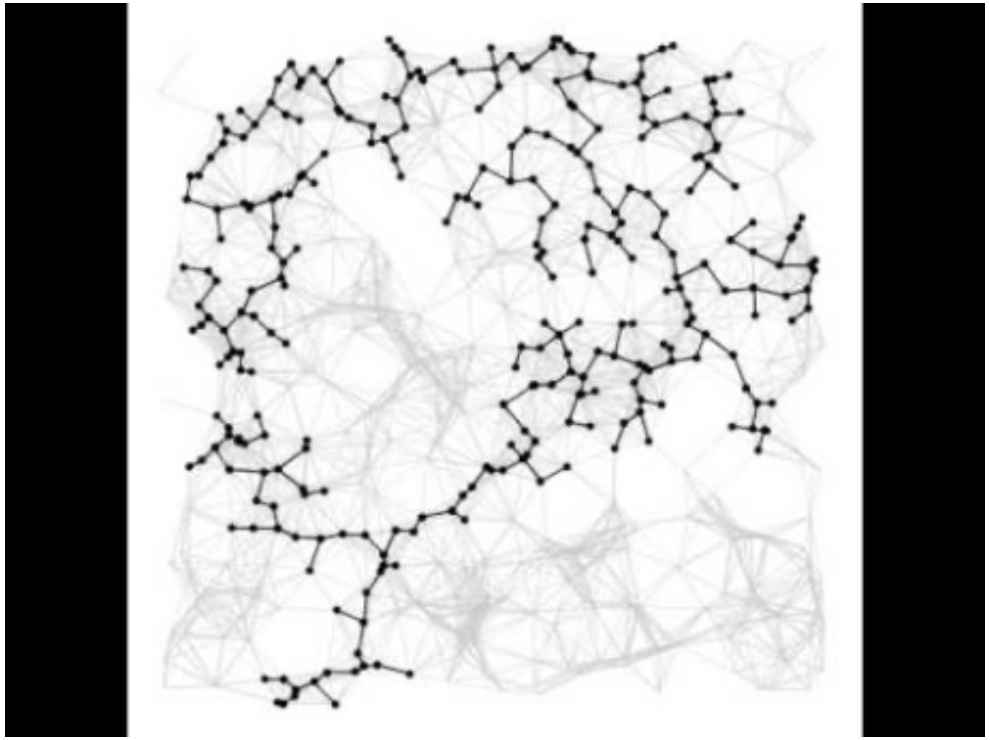
Prim's jumps around the graph (the perimeter), because it chooses edges by *edge weight* (there's no source)



Kruskal' Demos and Visualizations

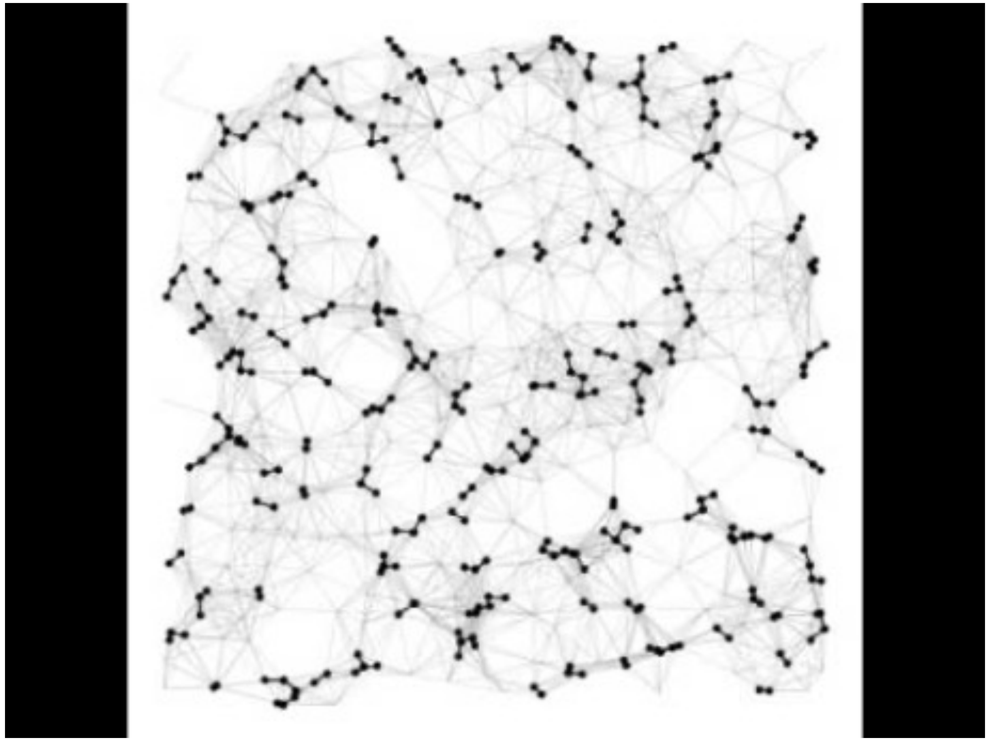
Prim's Algorithm

Prim's jumps around the graph (the perimeter), because it chooses edges by *edge weight* (there's no source)

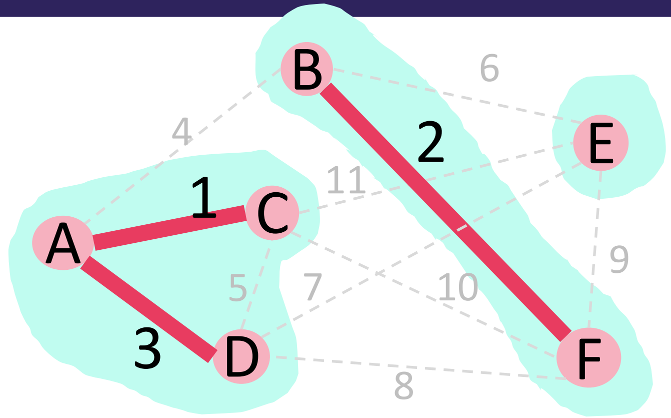


Kruskal's Algorithm

Kruskal's jumps around the entire graph, because it chooses from all edges purely by edge weight (while preventing cycles)



Selecting an ADT



- Kruskal's needs to **find** what MST a vertex belongs to, and **union** those MSTs together
  - Our existing ADTs don't lend themselves well to "unioning" two sets…
  - Let's define a new one!

```
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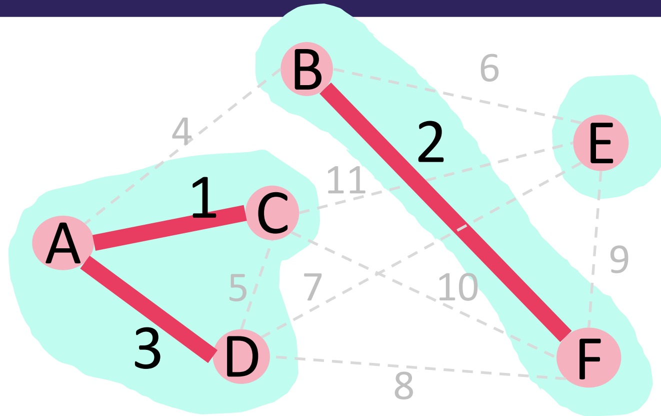
      msts.union(uMST, vMST)
```

Disjoint Sets ADT (aka "Union-Find")



- Kruskal's will use a Disjoint Sets ADT under the hood
  - Conceptually, a single instance of this ADT contains a "family" of sets that are disjoint (no element belongs to multiple sets)

```
kruskalMST(G graph)
  DisjointSets<V> msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)

  for each edge (u,v) in ascending order:
    uMST = msts.find(u)
    vMST = msts.find(v)
    if (uMST != vMST):
      finalMST.add(edge (u, v))
      msts.union(uMST, vMST);
```

DISJOINT SETS ADT

State

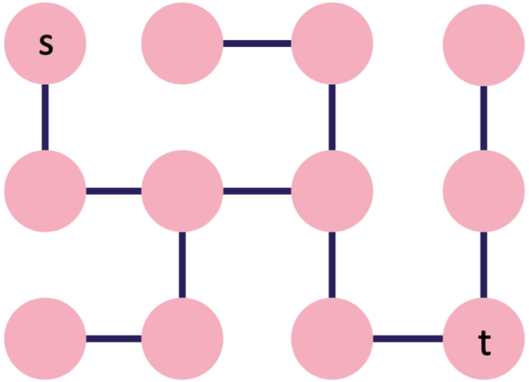
Family of Sets
- disjoint: no shared elements
- each set has a representative (either a member or a unique ID)

Behavior

<u>makeSet(value)</u> – new set with value as only member (and representative)
<u>find(value)</u> – return representative of the set containing value
<u>union(x, y)</u> – combine sets containing x and y into one set with all elements, choose single new representative

Project 4: Mazes!

- You find yourself trapped in the Labyrinth of Greek legend – bummer!
- How do you *solve* a maze?
    - If we could model a maze as a graph, we'd just need an algorithm to find a path from s to t… Maybe even the shortest path?



- How do you *generate* a maze?
    - We'd love an algorithm that is guaranteed to connect s to t (spanning), but only produces one path from s to t (tree)…

Project 4: Mazes

- It turns out that randomizing the weights of a graph and then computing the MST is a **fantastic way to generate mazes**!
- In P4, you'll do both: Implement Dijkstra's to solve an arbitrary maze, then implement Kruskal's (and a Disjoint Set) to generate those mazes
- This project is really application-heavy!
  - Graphical User Interface (GUI) for viewing mazes and solving them
  - Significantly more starter code than past projects, to give you practice integrating with an existing codebase
  - A major part of the challenge in P4 is reading through the starter code to understand what you need to interface with! Don't underestimate the time that takes.
- 2* week project, and 2* weeks worth of work. It's never been more important to start early!
  - You really have 3 weeks because of Thanksgiving in the middle, but don't let that fool you!

Lecture Outline

DISJOINT SETS ADT

1 **QuickFind**

2 **QuickUnion**

Optimizes for the Union operation

3 **Weighted QuickUnion**

Avoids the worst case runtime for Find

4 **Weighted QuickUnion + Path Compression**

Makes future Find operations faster

Case Study: Disjoint Sets

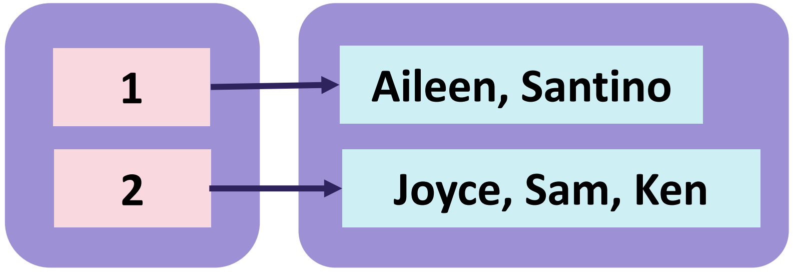
- Today's lecture on the data structures which implement the Disjoint Sets ADT is an interesting case study in data structure design and iterative design improvements
  - Good chance to dust off your metacognitive skills!
  - In particular, try to identify what observations we make in each data structure that inspire improvements in the next data structure. How could you apply a similar skill to your own data structures?

Metacognitive Opportunities Ahead!

Can we use an existing data structure?

aka "Can we just throw maps at the problem?"

Maps to Sets (baseline): map from representative ID to set of values



	Maps to Sets
makeSet(value)	$\Theta(1)$
find(value)	$\Theta(n)$
union(x, y)	$\Theta(n)$

find(value): scan through every set under every representative

union(x, y): copy all elements from set pointed to by x into set pointed to by y

DISJOINT SETS ADT

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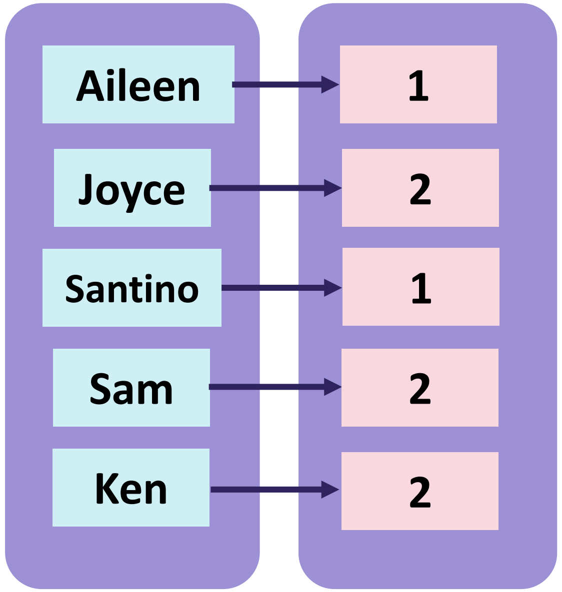
QuickFind Implementation

QuickFind: map from value to representative ID




```
find(Santino) → 1
find(Ken) → 2
find(Santino) != find(Ken)
find(Santino) == find(Aileen)
```

- If we store values as the keys, we can take advantage of fast lookup to make find fast!
- But what about union?

	Maps to Sets	QuickFind
makeSet(value)	$\Theta(1)$	$\Theta(1)$
find(value)	$\Theta(n)$	$\Theta(1)$
union(x, y)	$\Theta(n)$	$\Theta(n)$

find(value): lookup element and return corresponding rep.

union(x, y): scan through all elements to find those in same set, update to new rep.

DISJOINT SETS ADT

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Lecture Outline

DISJOINT SETS ADT

1. **QuickFind**
2. **QuickUnion**
   Optimizes for the Union operation
3. **Weighted QuickUnion**
   Avoids the worst case runtime for Find
4. **Weighted QuickUnion + Path Compression**
   Makes future Find operations faster

QuickUnion Data Structure

- Fundamental idea:
  - QuickFind tracks each element's ID
  - QuickUnion tracks each element's *parent*. Only the root has an ID!
    - Each set becomes tree-like, but something slightly different called an **up-tree**: store pointers from children to parents!

Abstract Idea of "Disjoint Sets"

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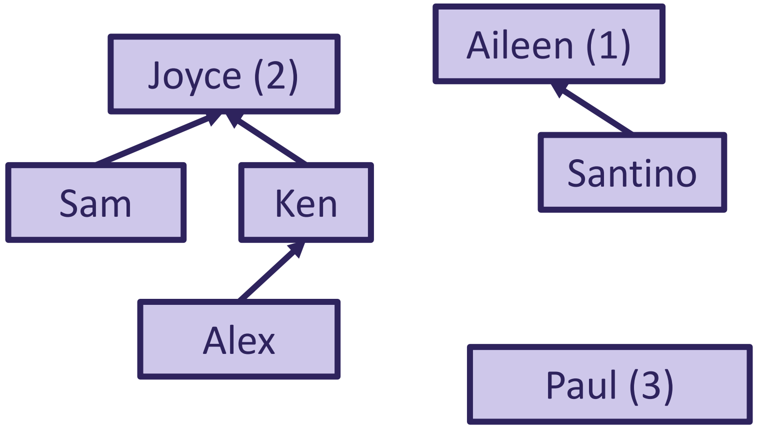

Implementation using QuickUnion

QuickUnion: Find

```
find(Ken):
  jump to Ken node
  travel upward until root
  return ID
```

- Key idea: can travel upward from any node to find its representative ID
- How do we jump to a node quickly?
  - *Also* store a map from value to its node (Omitted in future slides)

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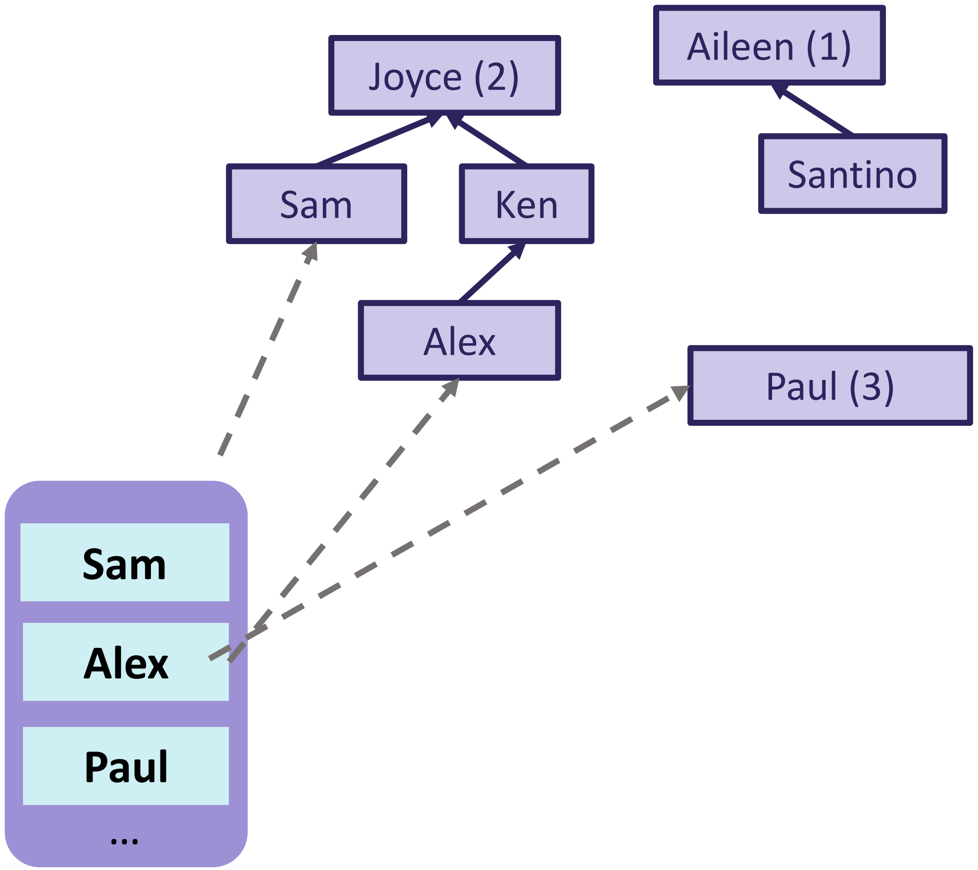
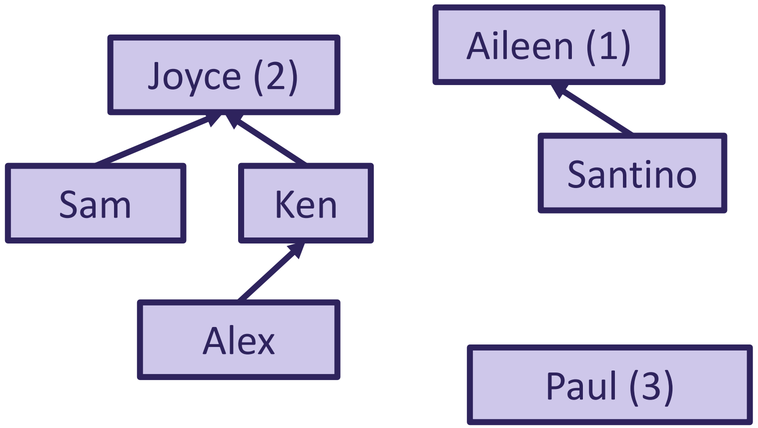


QuickUnion: Union

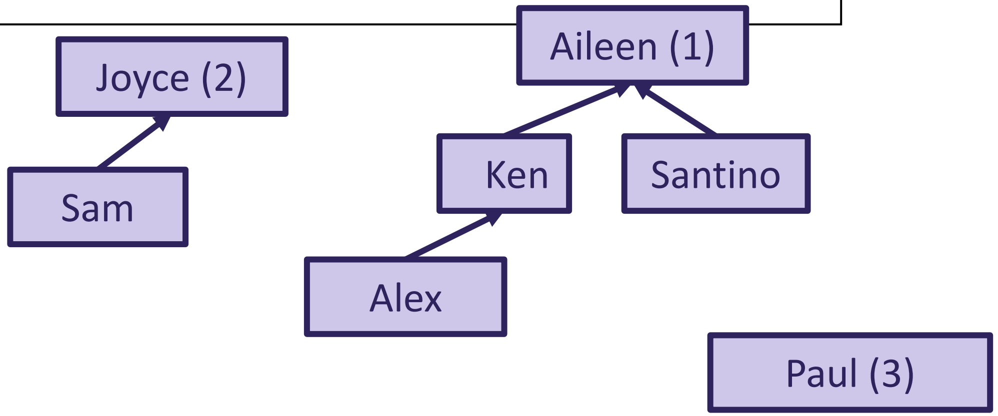


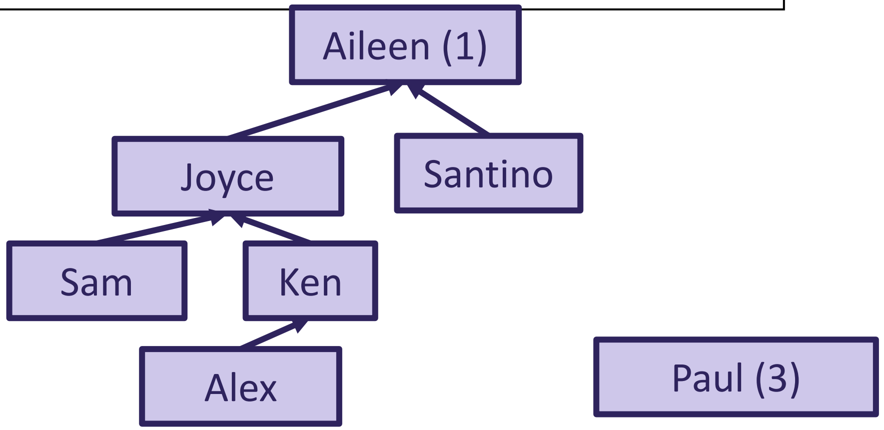
- Key idea: easy to simply rearrange pointers to union entire trees together!
- Which of these implementations would you prefer?

```
union(Ken, Santino):
  rootS = find(Santino)
  set Ken to point to rootS
```

```
union(Ken, Santino):
  rootK = find(Ken)
  rootS = find(Santino)
  set rootK to point to rootS
```

RESULT:





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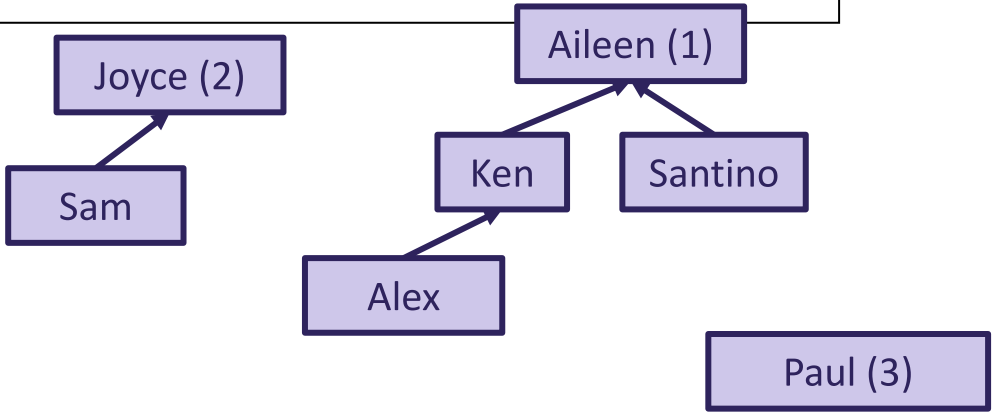
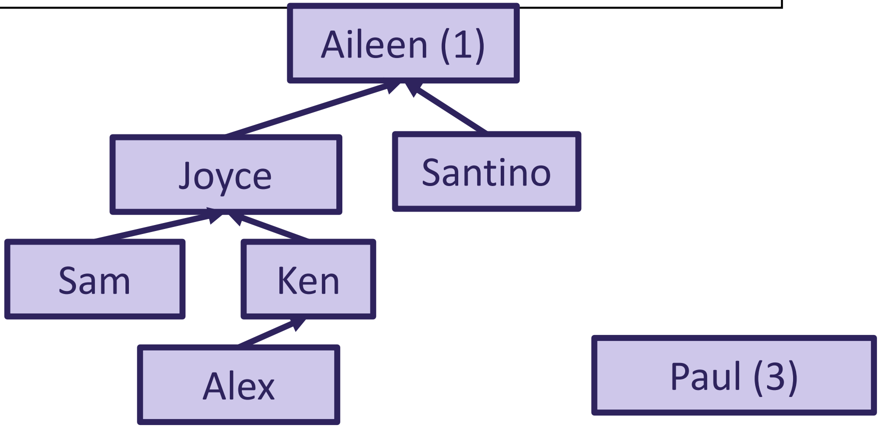
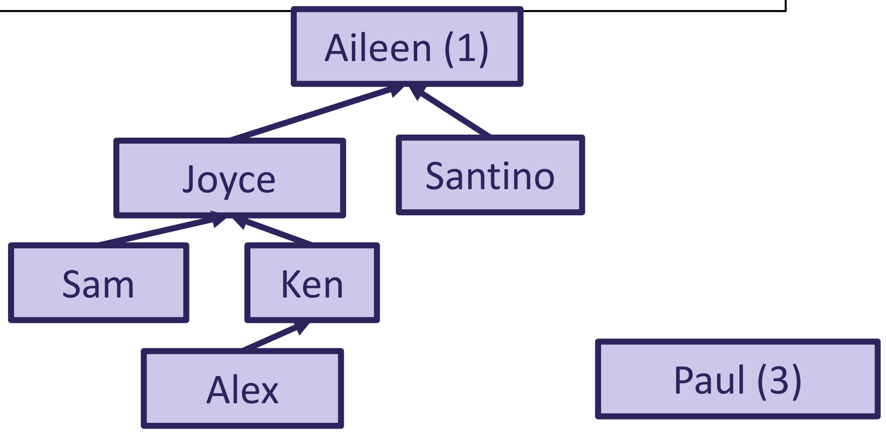




- We prefer the right implementation because by changing just the root, we effectively pull the entire tree into the new set!
  - If we change the first node instead, we have to do more work for the rest of the old tree
  - A rare example of constant time work manipulating a factor of n elements

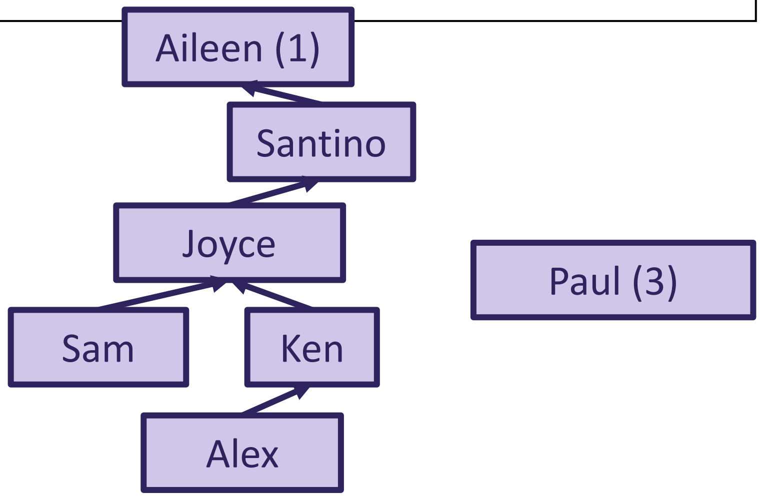
QuickUnion: Why bother with the second root?

```
union(Ken, Santino):
  rootK = find(Ken)
  rootS = find(Santino)
  set rootK to point to rootS
```



Why not just use:

```
union(Ken, Santino):
  rootK = find(Ken)
  set rootK to point to Santino
```



- Key idea: will help minimize runtime for future find() calls if we keep the height of the tree short!
  - Pointing directly to the second element would make the tree taller

QuickUnion: Checking in on those runtimes

	Maps to Sets	QuickFind	QuickUnion
makeSet(value)	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
findSet(value)	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$
union(x, y)	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$ *

* Only if we discount the runtime from union's calls to find! Otherwise, $\Theta(n)$.

- However, for Kruskal's, not a bad assumption: we only ever call union with roots anyway!

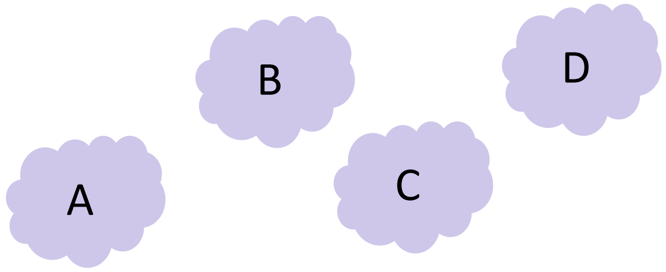
```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  set rootA to point to rootB
```

```
kruskalMST(G graph)
  DisjointSets<V> msts; Set finalMST;
  initialize msts with each vertex as single-element MST
  sort all edges by weight (smallest to largest)

  for each edge (u,v) in ascending order:
    uMST = msts.find(u)
    vMST = msts.find(v)
    if (uMST != vMST):
      finalMST.add(edge (u, v))
      msts.union(uMST, vMST);
```

QuickUnion: Let's Build a Worst Case

Even with the "use-the-roots" implementation of union, try to come up with a series of calls to union that would create a worst-case runtime for find on these Disjoint Sets:




```
find(A):
  jump to A node
  travel upward until root
  return ID
```

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  set rootA to point to rootB
```

QuickUnion: Let's Build a Worst Case

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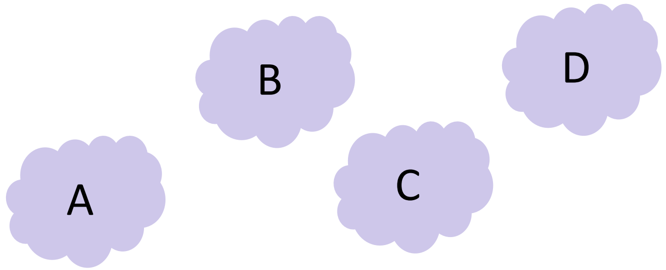
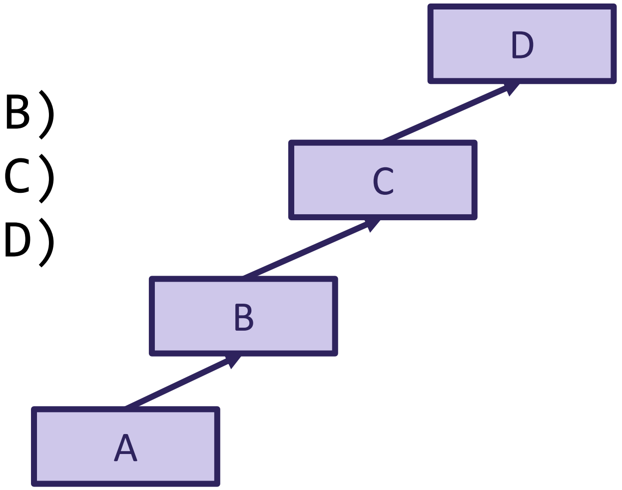



```
union(A, B)
union(B, C)
union(C, D)
find(A)
```



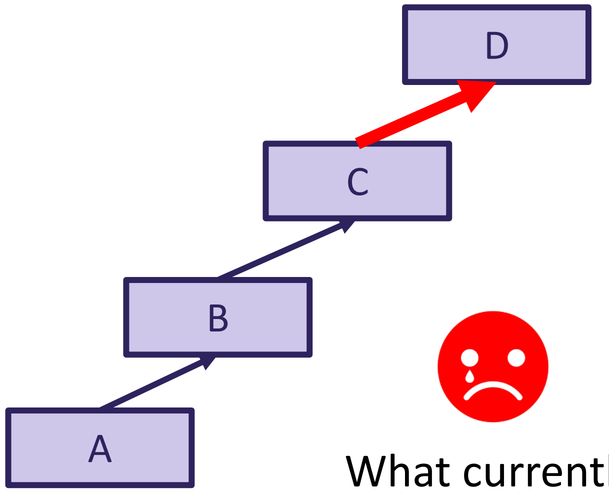

```
find(A):
  jump to A node
  travel upward until root
  return ID
```

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  set rootA to point to rootB
```

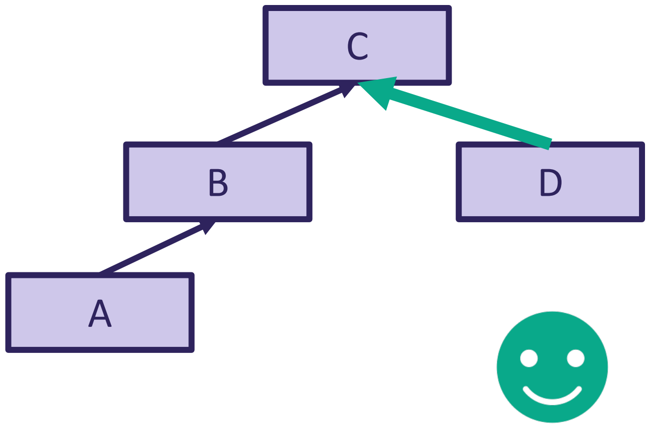
Analyzing the QuickUnion Worst Case

- How did we get a degenerate tree?
  - Even though pointing a root to a root usually helps with this, we can still get a degenerate tree **if we put the root of a large tree under the root of a small tree.**
  - In QuickUnion, rootA always goes under rootB
    - But what if we could ensure the smaller tree went under the larger tree?

`union(C, D)`



What currently happens



What would help avoid degenerate tree

Lecture Outline

DISJOINT SETS ADT

1. **QuickFind**

2. **QuickUnion**

   Optimizes for the Union operation

3. **Weighted QuickUnion**

   Avoids the worst case runtime for Find

4. **Weighted QuickUnion + Path Compression**

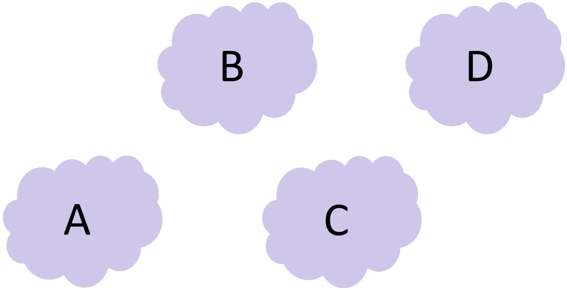
   Makes future Find operations faster

WeightedQuickUnion

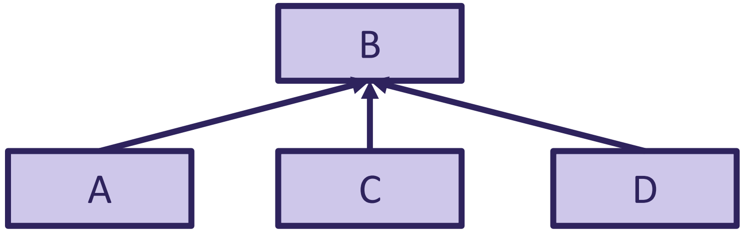
- Goal: Always pick the smaller tree to go under the larger tree
- Implementation: Store the number of nodes (or “weight”) of each tree in the root
  - Constant-time lookup instead of having to traverse the entire tree to count

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  put lighter root under heavier root
```

```
union(A, B)
union(B, C)
union(C, D)
find(A)
```



Now what happens?



Perfect! Best runtime we can get.

WeightedQuickUnion: Performance

- union()'s runtime is still dependent on find()'s runtime, which is a function of the tree's height
- What's the worst-case height for WeightedQuickUnion?

```
union(A, B):
  rootA = find(A)
  rootB = find(B)
  put lighter root under heavier root
```

WeightedQuickUnion: Performance

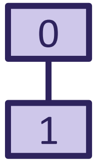
- Consider the worst case where the tree height grows as fast as possible

N	H
1	0

WeightedQuickUnion: Performance

- Consider the worst case where the tree height grows as fast as possible

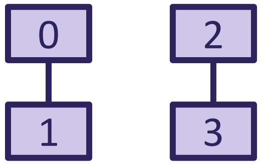
N	H
1	0
2	1



WeightedQuickUnion: Performance

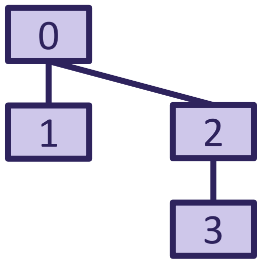
- Consider the worst case where the tree height grows as fast as possible

N	H
1	0
2	1
4	?



WeightedQuickUnion: Performance

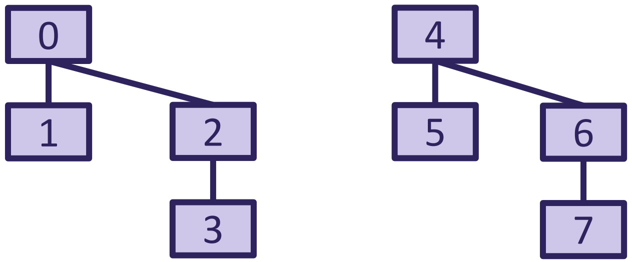
- Consider the worst case where the tree height grows as fast as possible



N	H
1	0
2	1
4	2

WeightedQuickUnion: Performance

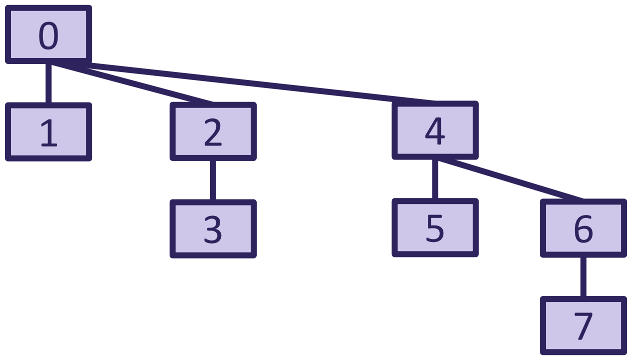
- Consider the worst case where the tree height grows as fast as possible



N	H
1	0
2	1
4	2
8	?

WeightedQuickUnion: Performance

- Consider the worst case where the tree height grows as fast as possible

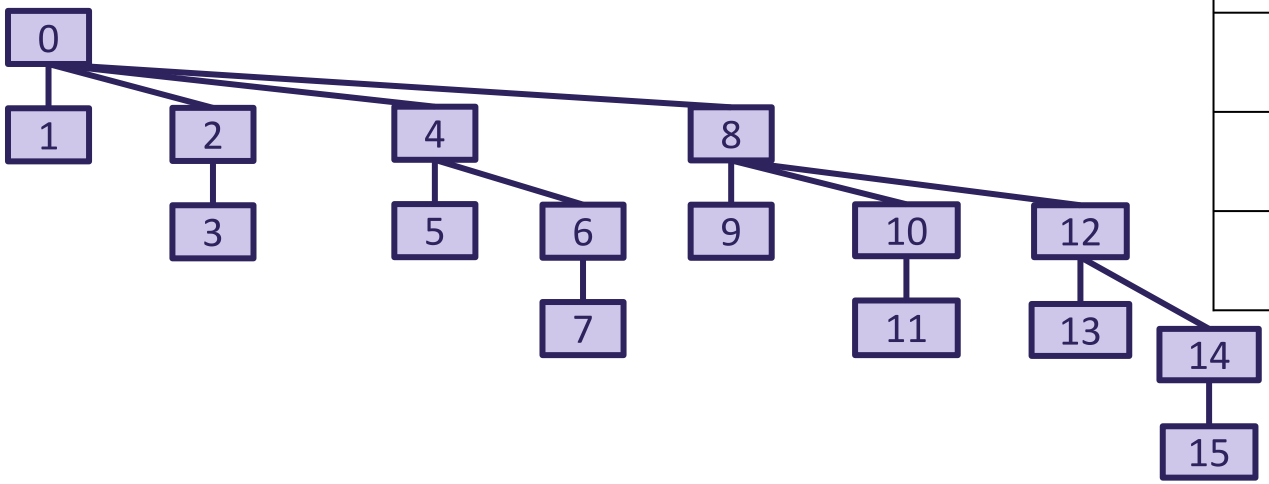


N	H
1	0
2	1
4	2
8	3

WeightedQuickUnion: Performance

- Consider the worst case where the tree height grows as fast as possible
- Worst case tree height is Θ(log N)

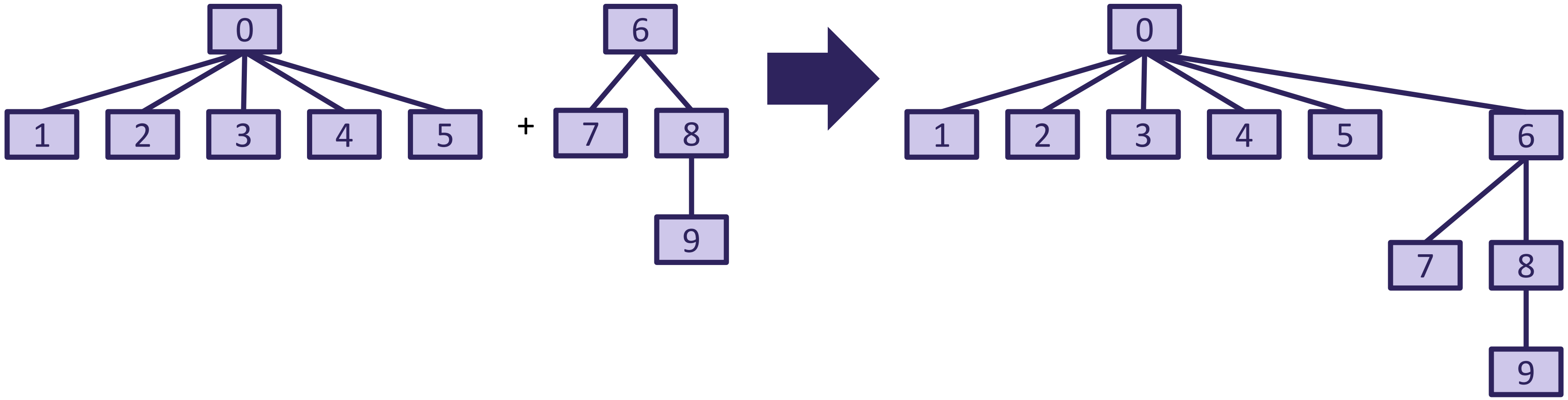
N	H
1	0
2	1
4	2
8	3
16	4



Why Weights Instead of Heights?

- We used the number of items in a tree to decide upon the root

- Why not use the height of the tree?
  - HeightedQuickUnion's runtime is asymptotically the same: Θ(log(n))
  - It's easier to track weights than heights, even though WeightedQuickUnion can lead to some suboptimal structures like this one:



WeightedQuickUnion Runtime

	Maps to Sets	QuickFind	QuickUnion	WeightedQuickUnion
`makeSet(value)`	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
`find(value)`	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$	$\Theta(\log n)$
`union(x, y)` `assuming root args`	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$
`union(x, y)`	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$	$\Theta(\log n)$

- This is pretty good! But there's one final optimization we can make: **path compression**

Lecture Outline

DISJOINT SETS ADT

1
QuickFind

2
QuickUnion
Optimizes for the Union operation

3
Weighted QuickUnion
Avoids the worst case runtime for Find

4
Weighted QuickUnion + Path Compression
Makes future Find operations faster

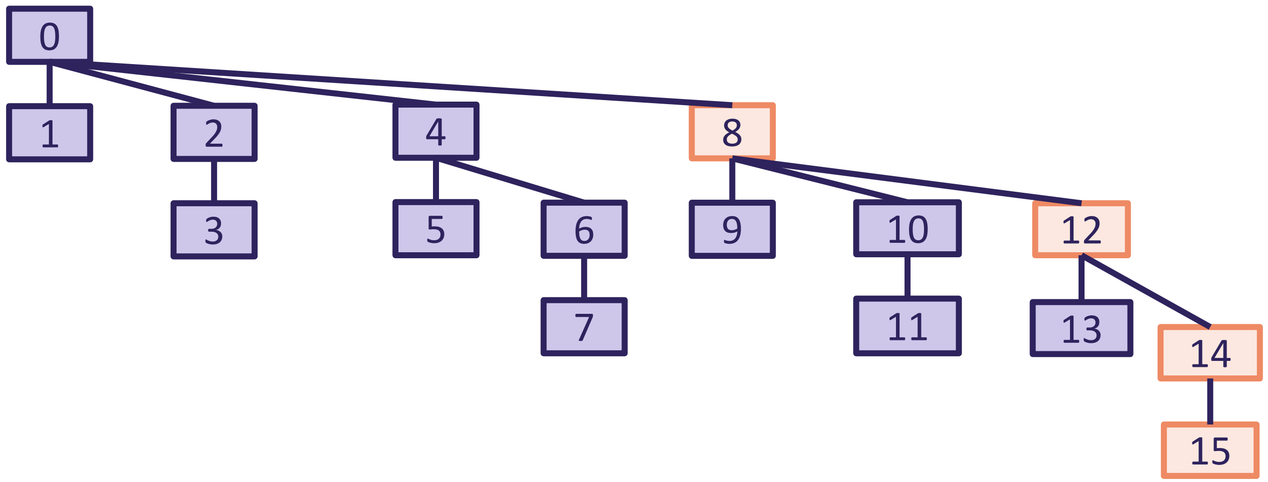
Modifying Data Structures for Future Gains

- Thus far, the modifications we've studied are designed to *preserve invariants*
  - E.g. Performing rotations to preserve the AVL invariant
  - We rely on those invariants always being true so every call is fast

- Path compression is entirely different: we are modifying the tree structure to *improve future performance*
  - Not adhering to a specific invariant
  - The first call may be slow, but will optimize so future calls can be fast

Path Compression: Idea

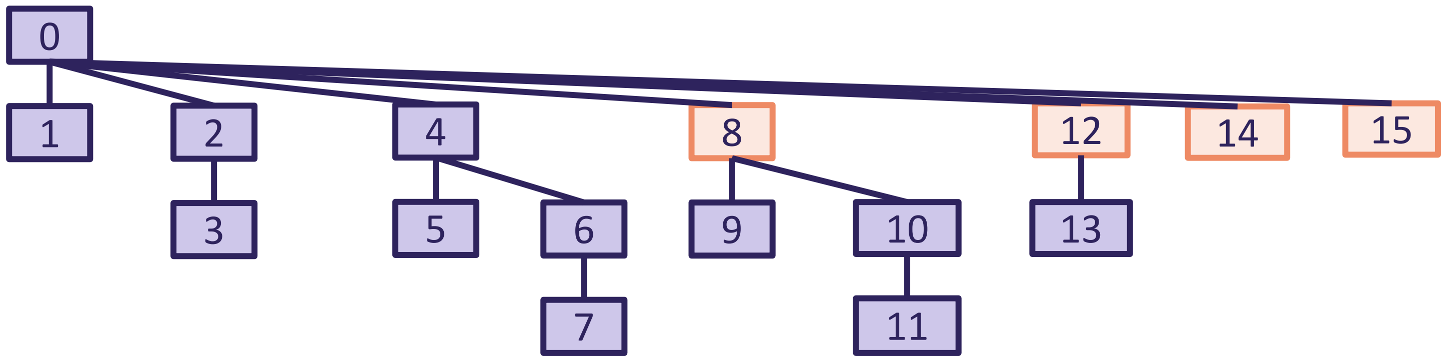
- This is the worst-case topology if we use WeightedQuickUnion



- Idea: When we do find(15), move all **visited nodes** under the root
  - Additional cost is insignificant (we already have to visit those nodes, just constant time work to point to root too)

Path Compression: Idea

- This is the worst-case topology if we use WeightedQuickUnion



- Idea: When we do find(15), move all **visited nodes** under the root
  - Additional cost is insignificant (we already have to visit those nodes, just constant time work to point to root too)
- Perform Path Compression on every find(), so future calls to find() are faster!