Combining Graph Algorithms

Data Structures and Algorithms

Last Time

We described algorithms to find:

Topological Sort (aka Topological Ordering)

An ordering of the vertices so all edges go from left to right.

Strongly Connected Component

A subgraph C such that every pair of vertices in C is connected via some path **in both directions,** and there is no other vertex which is connected to every vertex of C in both directions.

Today: Use those algorithms to solve a bigger problem.

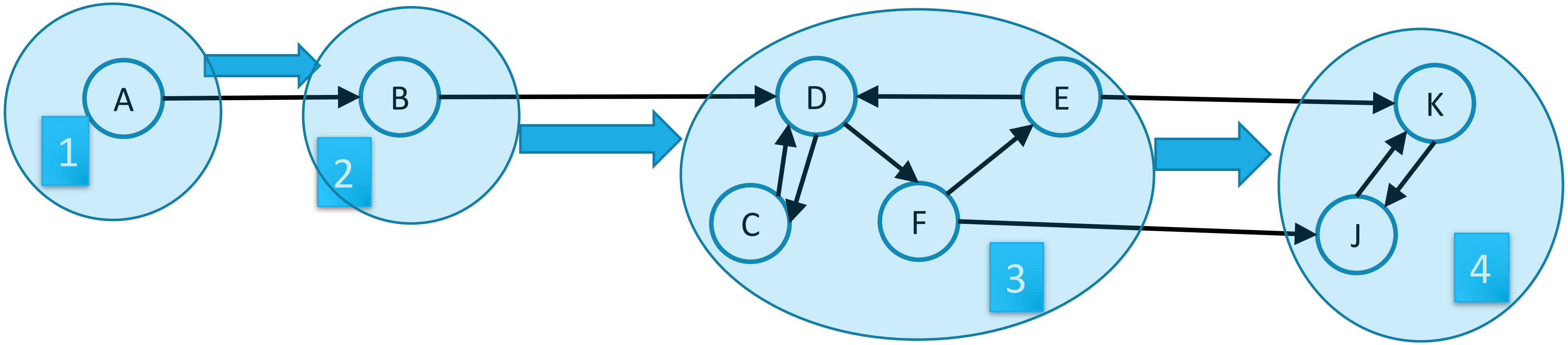
Why Find SCCs?

Graphs are useful because they encode relationships between arbitrary objects.

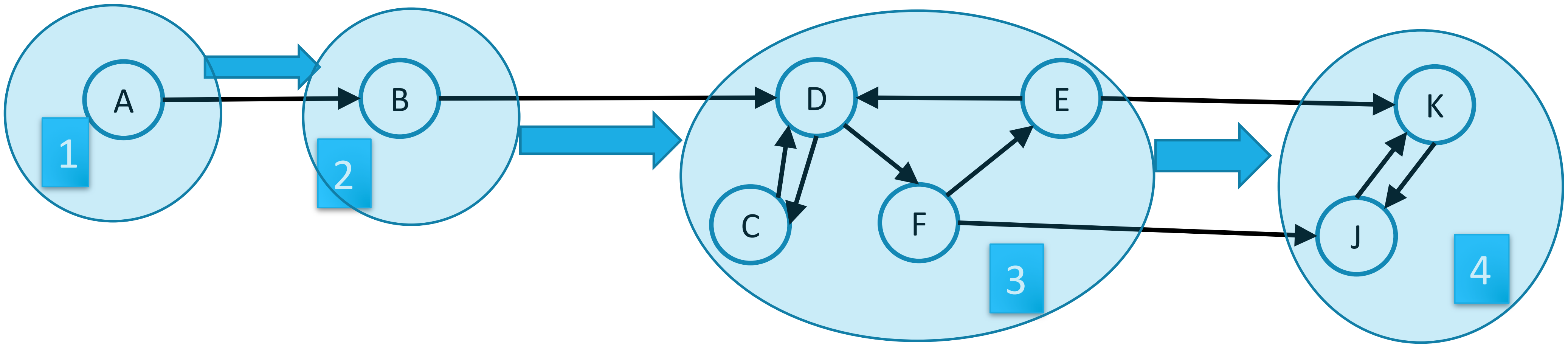
We've found the strongly connected components of G.

Let's build a new graph out of them! Call it H

- Have a vertex for each of the strongly connected components
- Add an edge from component 1 to component 2 if there is an edge from a vertex inside 1 to one inside 2.



Why Find SCCs?



That's awful meta. Why?

This new graph summarizes reachability information of the original graph.

- I can get from A (of G) in 1 to F (of G) in 3 if and only if I can get from 1 to 3 in H.

Why Must H Be a DAG?

H is always a DAG (i.e. it has no cycles). Do you see why?

If there were a cycle, I could get from component 1 to component 2 and back, but then they're actually the same component!

Takeaways

Finding SCCs lets you **collapse** your graph to the meta-structure.
If (and only if) your graph is a DAG, you can find a topological sort of your graph.

Both of these algorithms run in linear time.

Just about everything you could want to do with your graph will take at least as long.

You should think of these as **“almost free” preprocessing** of your graph.

- Your other graph algorithms only need to work on
  - topologically sorted graphs and
  - strongly connected graphs.

A Longer Example

The best way to really see why this is useful is to do a bunch of examples.

We don’t have time. The second best way is to see one example right now...

This problem doesn’t *look like* it has anything to do with graphs

- no maps
- no roads
- no social media friendships

Nonetheless, a graph representation is the best one.

I don’t expect you to remember the details of this algorithm.

I just want you to see

- graphs can show up anywhere.
- SCCs and Topological Sort are useful algorithms.

Example Problem: Final Review

We have a long list of types of problems we might want to put on the final.

- Heap insertion problem, big-O problems, finding closed forms of recurrences, graph modeling…
- What should Erik cover in the final review – what if we asked you?

To try to make you all happy, we might ask for your preferences. Each of you gives us two preferences of the form "I [do/don't] want a [] problem on the review" *

We'll assume you'll be happy if you get at least one of your two preferences.

Final Creation Problem

Given: A list of 2 preferences per student.
Find: A set of questions so every student gets at least one of their preferences (or accurately report no such question set exists).

*This is NOT how Erik is making the final review.

Review Creation: Take 1

We have Q kinds of questions and S students.

What if we try every possible combination of questions.

How long does this take? O($2^Q S$)

If we have a lot of questions, that's **really** slow.

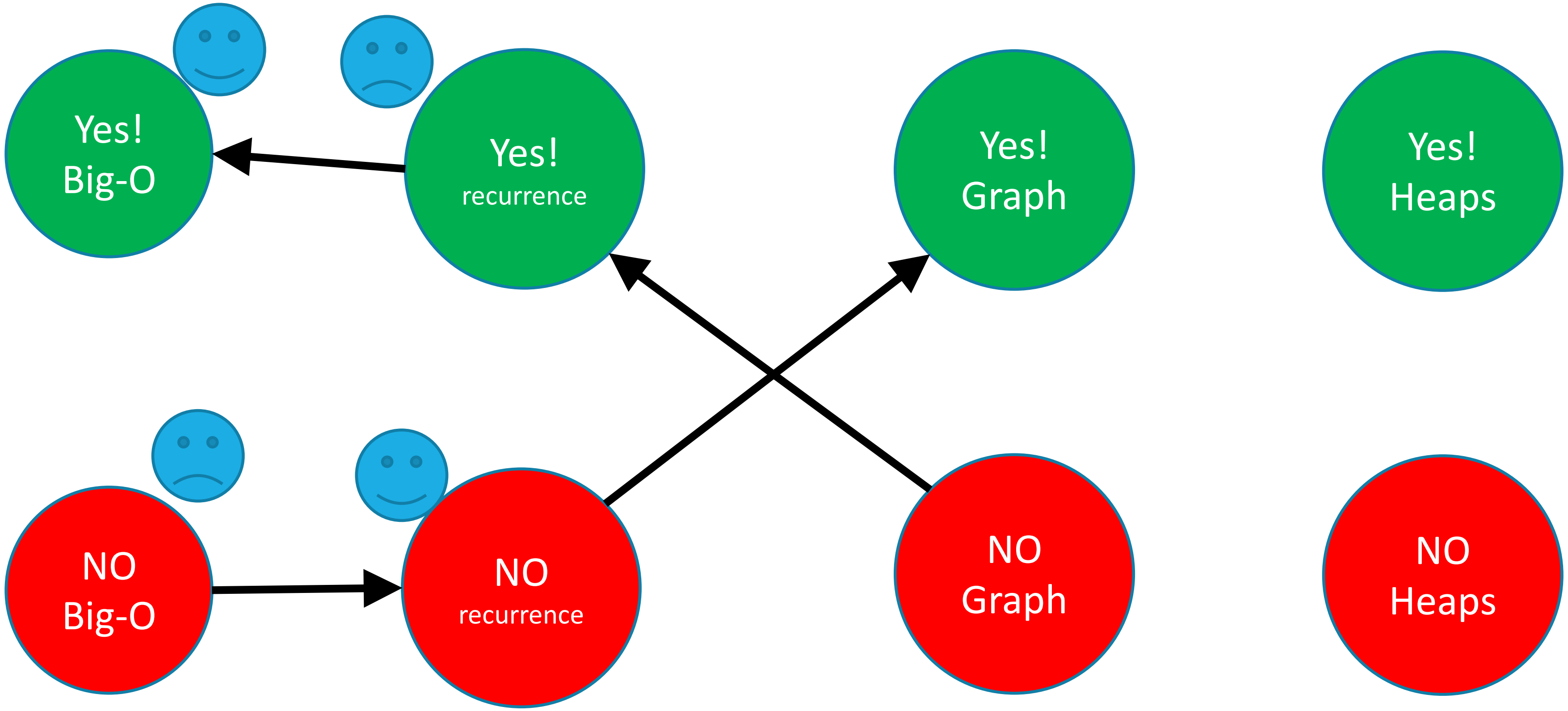
Instead we're going to use a graph.

What should our vertices be?

Review Creation: Take 2

Each student introduces new relationships for data:

Let's say your preferences are represented by this table:



Problem	YES	NO
Big-O	X	
Recurrence		X
Graph		
Heaps		

Problem	YES	NO
Big-O		
Recurrence	X	
Graph	X	
Heaps		

If we don't include a big-O proof, can you still be happy?
If we do include a recurrence can you still be happy?

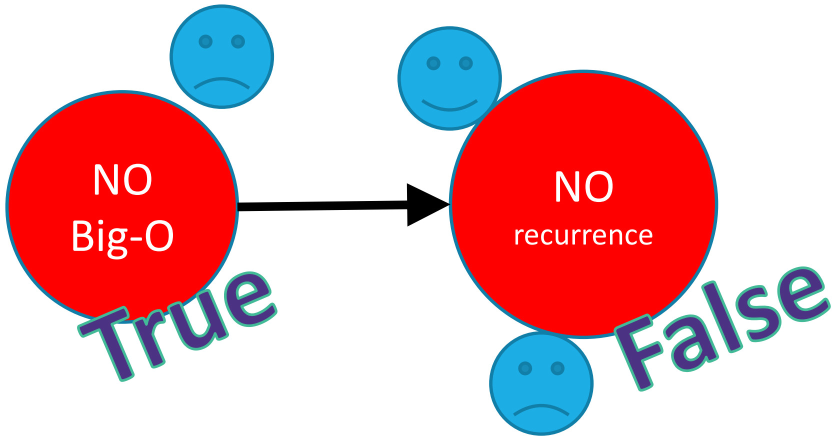
Review Creation: Take 2

Hey we made a graph!

What do the edges mean?

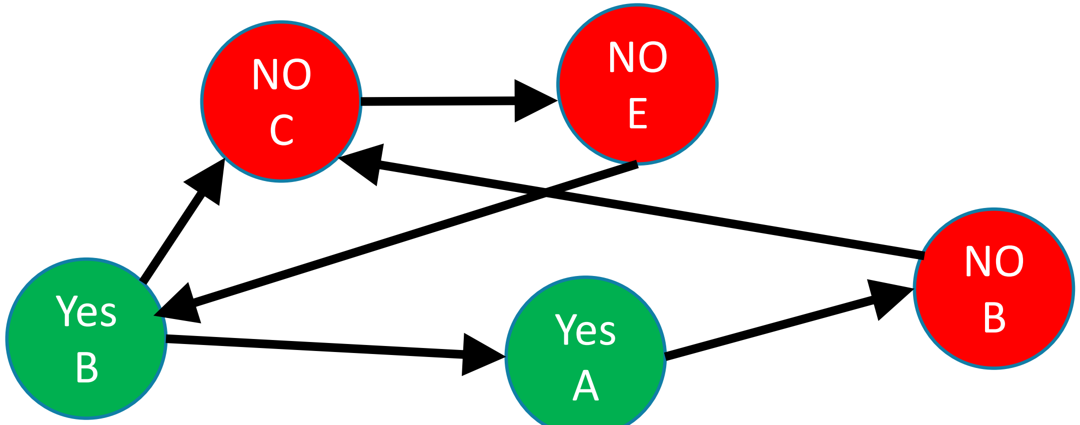
Each edge goes from something making someone unhappy, to the only thing that could make them happy.

- We need to avoid an edge that goes TRUE THING → FALSE THING



We need to avoid an edge that goes TRUE THING → FALSE THING

Let's think about a single SCC of the graph.



Can we have a true and false statement in the same SCC?

What happens now that Yes B and NO B are in the same SCC?

Final Creation: SCCs

The vertices of a SCC must either be all true or all false.

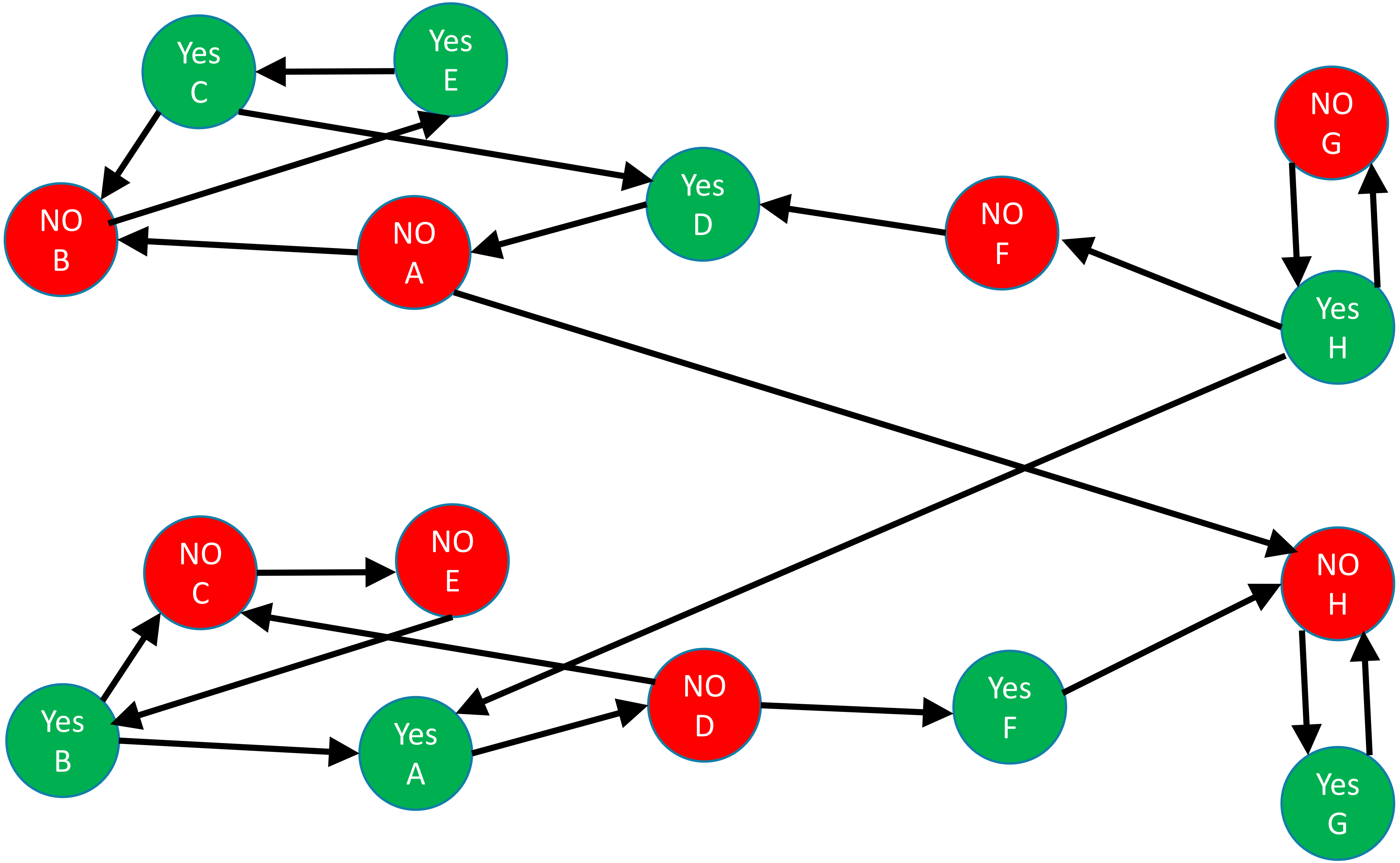
Algorithm Step 1: Run SCC on the graph. Check that each question-type-pair are in different SCC.

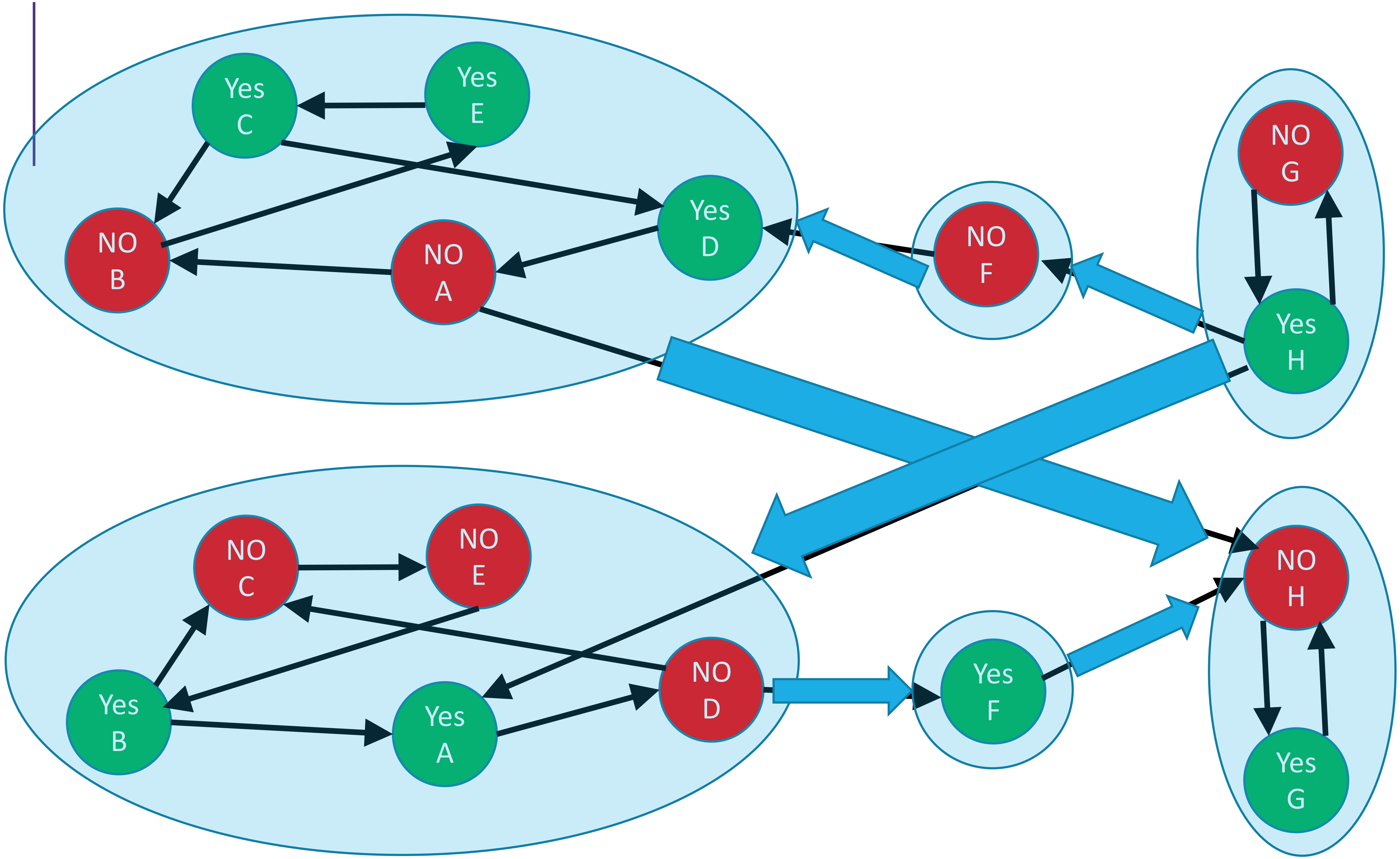
Now what? Every SCC gets the same value.
- Treat it as a single object!

We want to avoid edges from true things to false things.
- “Trues” seem more useful for us at the end.

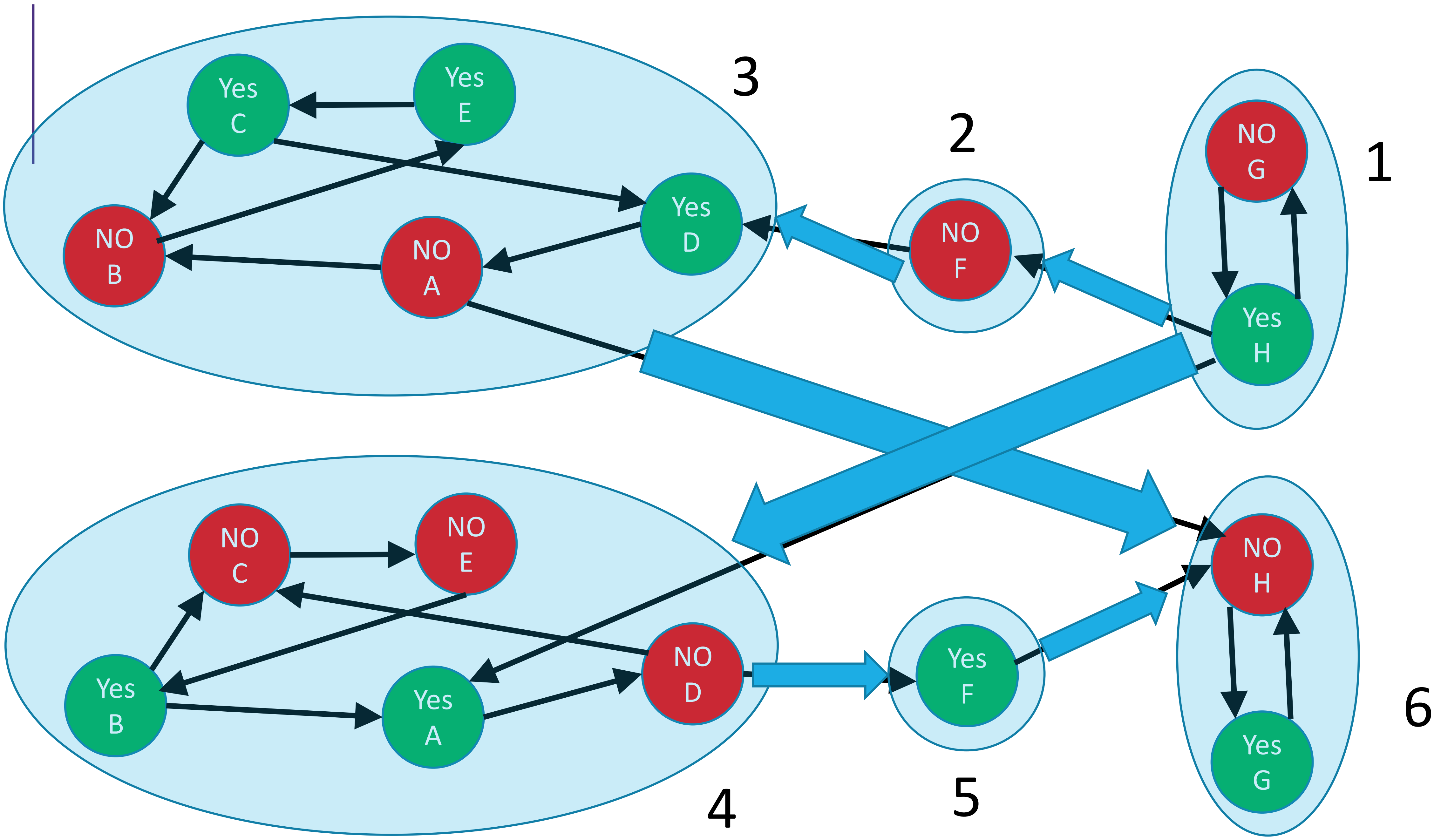
Is there some way to start from the end?

YES! Topological Sort

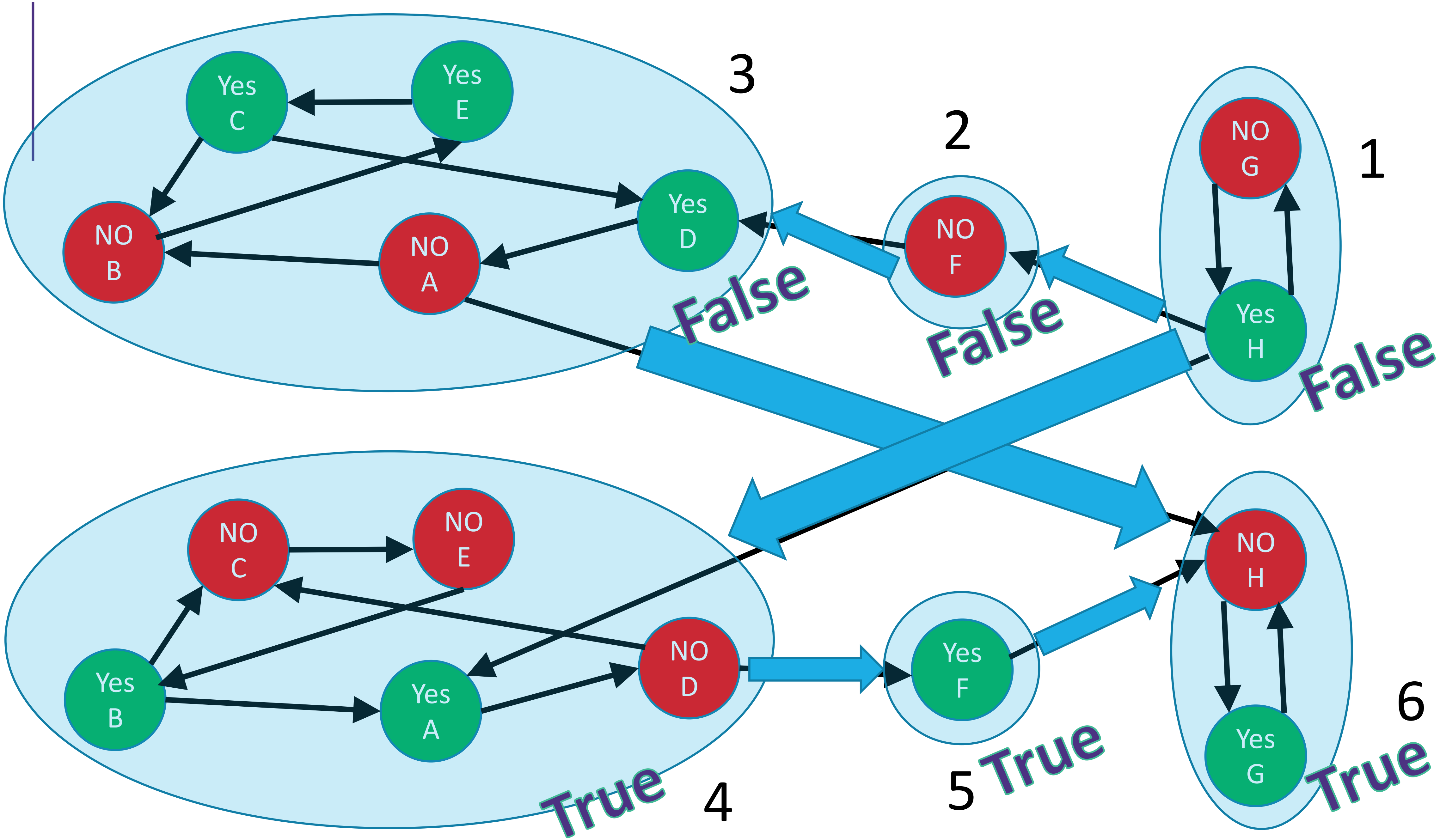

Yes C
Yes E
NO G
Yes D
NO F
NO B
NO A
Yes H
NO C
NO E
NO H
NO D
Yes F
Yes B
Yes A
Yes G



Yes
C
Yes
E
NO
G
Yes
D
NO
F
NO
B
NO
A
Yes
H
NO
C
NO
E
NO
H
NO
D
Yes
F
Yes
B
Yes
A
Yes
G



Yes C
Yes E
3
2
1
NO G
NO B
NO A
Yes D
NO F
Yes H
NO C
NO E
NO H
NO D
Yes F
Yes B
Yes A
Yes G
6
4
5


Yes C
Yes E
NO B
NO A
Yes D
3
False
2
NO F
False
1
NO G
Yes H
False
NO C
NO E
Yes B
Yes A
NO D
True
4
Yes F
5
True
NO H
Yes G
6
True

Making the Final

Algorithm:
Make the requirements graph.

Find the SCCs.

If any SCC has including and not including a problem, we can't make the final.

Run topological sort on the graph of SCC.

Starting from the end:

- if everything in a component is unassigned, set them to true, and set their opposites to false.

This works!!

How fast is it?

O(Q + S). That's a HUGE improvement.

Some More Context

The Final Making Problem was a type of "Satisfiability" problem.

We had a bunch of variables (include/exclude this question), and needed to satisfy everything in a list of requirements.

2-Satisfiability ("2-SAT")

Given: A set of Boolean variables, and a list of requirements, each of the form:

`variable1==[True/False] || variable2==[True/False]`

Find: A setting of variables to "true" and "false" so that **all** of the requirements evaluate to "true"

The algorithm we just made for Final Creation works for any 2-SAT problem.

Reductions, P vs. NP

What are we doing?

To wrap up the course we want to take a big step back.

This whole quarter we've been taking problems and solving them faster.

We want to spend the last few lectures going over more ideas on how to solve problems faster, and why we don't expect to solve everything extremely quickly.

We're going to

- Recall reductions – Robbie's favorite idea in algorithm design.
- Classify problems into those we can solve in a reasonable amount of time, and those we can't.
- Explain the biggest open problem in Computer Science

Reductions: Take 2

Reduction (informally)

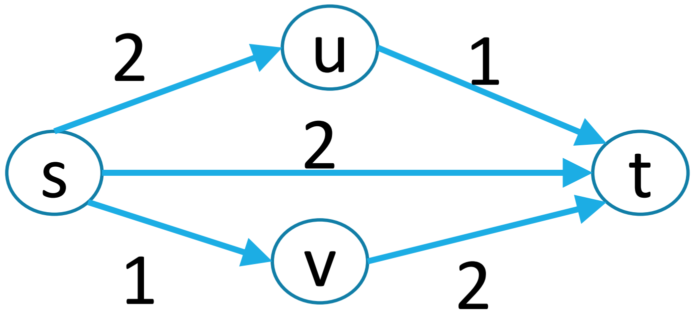
Using an algorithm for Problem B to solve Problem A.

You already do this all the time.

In Homework 3, you reduced implementing a hashset to implementing a hashmap.

Any time you use a library, you're reducing your problem to the one the library solves.

Weighted Graphs: A Reduction



Transform Input

Unweighted Shortest Paths

Transform Output

Reductions

It might not be too surprising that we can solve one shortest path problem with the algorithm for another shortest path problem.

The real power of reductions is that you can sometimes reduce a problem to another one that looks very very different.

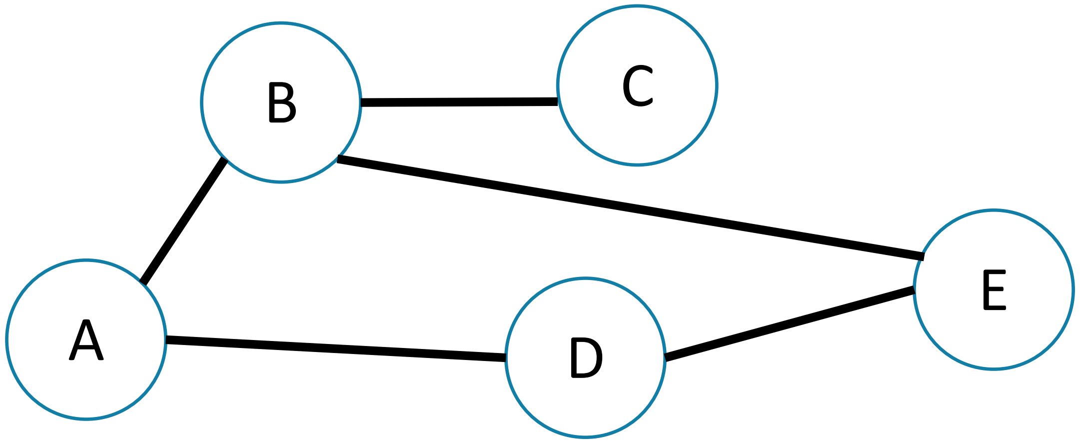
We're going to reduce a graph problem to 2-SAT.

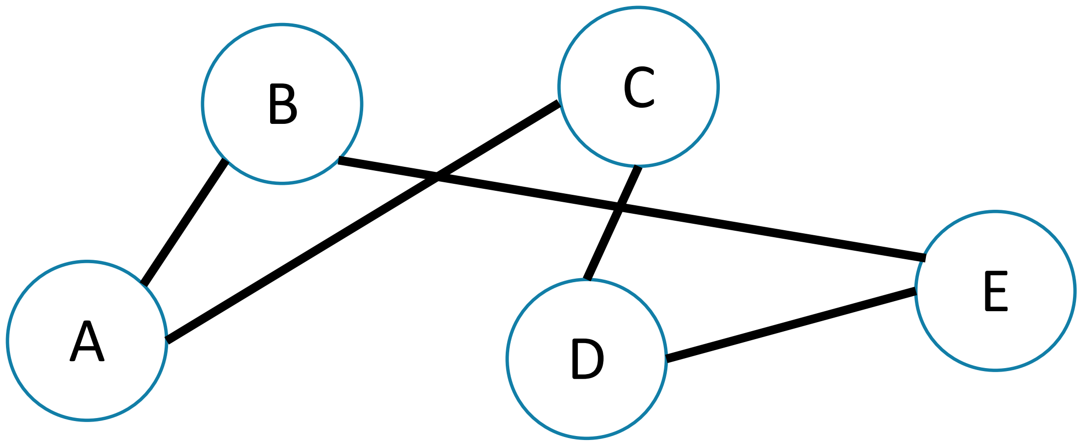
2-Coloring

Given an undirected, unweighted graph G, color each vertex "red" or "blue" such that the endpoints of every edge are different colors (or report no such coloring exists).

2-Coloring

Can these graphs be 2-colored? If so find a 2-coloring. If not try to explain why one doesn't exist.





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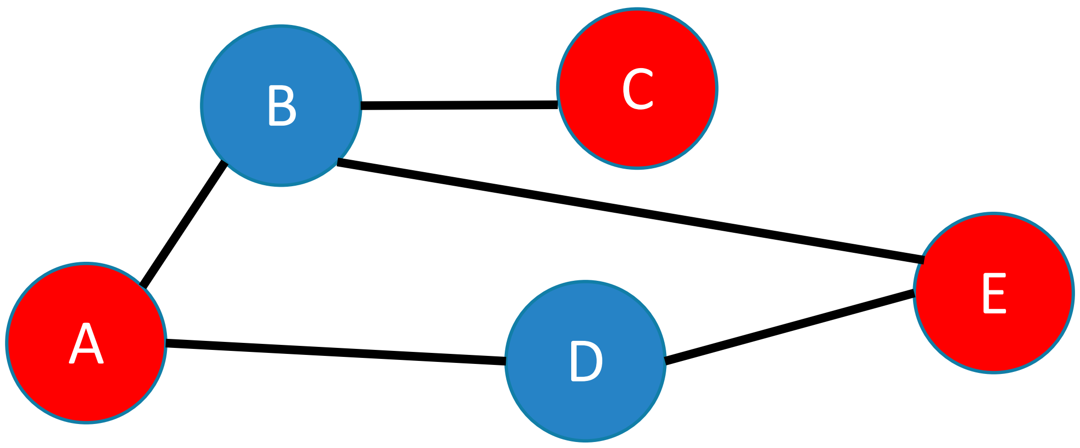
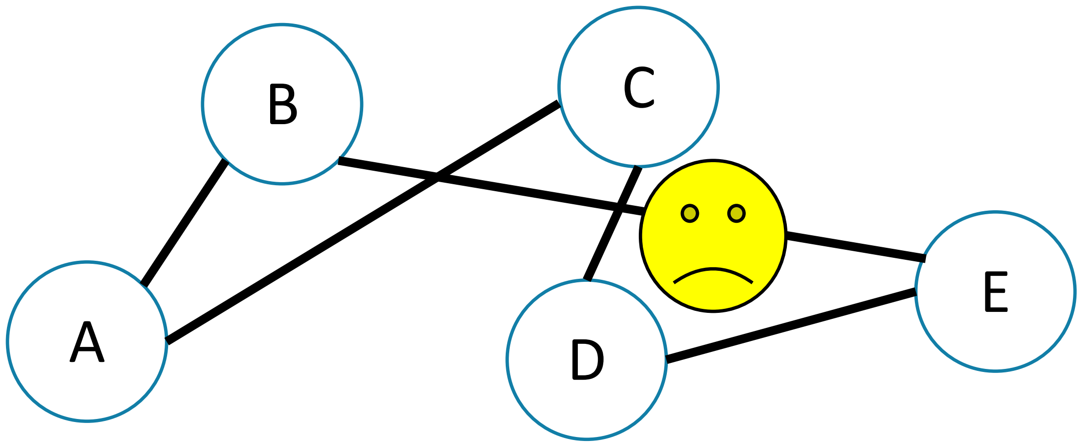




2-Coloring

Why would we want to 2-color a graph?

- We need to divide the vertices into two sets, and edges represent vertices that **can't** be together.

You can modify BFS to come up with a 2-coloring (or determine none exists)

- This is a good exercise!

But coming up with a whole new idea sounds like **work.**

And we already came up with that cool 2-SAT algorithm.

- Maybe we can be lazy and just use that!
- Let's **reduce** 2-Coloring to 2-SAT!

Use our 2-SAT algorithm
to solve 2-Coloring

A Reduction

We need to describe 2 steps

1. How to turn a graph for a 2-color problem into an input to 2-SAT

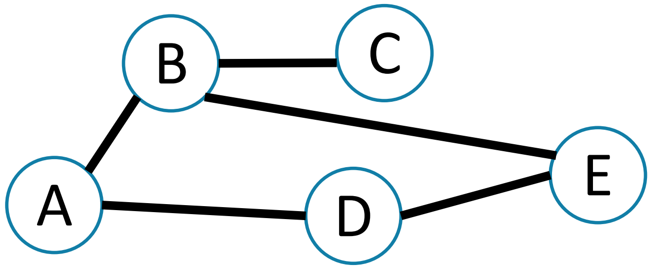
2. How to turn the ANSWER for that 2-SAT input into the answer for the original 2-coloring problem.

How can I describe a two coloring of my graph?

- Have a variable for each vertex – is it red?

How do I make sure every edge has different colors? I need one red endpoint and one blue one, so this better be true to have an edge from v1 to v2:

```
(v1IsRed || v2isRed) && (!v1IsRed || !v2IsRed)
```



Transform Input

2-SAT Algorithm

Transform Output

Efficient

We'll consider a problem "efficiently solvable" if it has a polynomial time algorithm.

I.e. an algorithm that runs in time $O(n^k)$ where k is a constant.

Are these algorithms always actually efficient?

Well………no

Your n^{10000} algorithm or even your $2^{2^{2^{2^{2}}}} \cdot n^3$ algorithm probably aren't going to finish anytime soon.

But these edge cases are rare, and polynomial time is good as a low bar

- If we can't even find an n^{10000} algorithm, we should probably rethink our strategy

Decision Problems

Let's go back to dividing problems into solvable/not solvable.
For today, we're going to talk about **decision problems**.

Problems that have a "yes" or "no" answer.

Why?

Theory reasons (ask me later).

But it's not too bad

- most problems can be rephrased as very similar decision problems.

E.g. instead of "find the shortest path from s to t" ask
Is there a path from s to t of length at most k?

P

P (stands for "Polynomial")

The set of all decision problems that have an algorithm that runs in time $O(n^k)$ for some constant k.

The decision version of all problems we've solved in this class are in P.

P is an example of a "complexity class"

A set of problems that can be solved under some limitations (e.g. with some amount of memory or in some amount of time).

I’ll know it when I see it.

Another class of problems we want to talk about.

“I’ll know it when I see it” Problems.

Decision Problems such that:

If the answer is YES, you can prove the answer is yes by

- Being given a “proof” or a “certificate”
- Verifying that certificate in polynomial time.

What certificate would be convenient for short paths?

- The path itself. Easy to check the path is really in the graph and really short.

I'll know it when I see it.

More formally,

NP (stands for "nondeterministic polynomial")

The set of all decision problems such that if the answer is YES, there is a proof of that which can be verified in polynomial time.

It's a common misconception that NP stands for "not polynomial"
Please never ever ever ever say that.

Please.

Every time you do a theoretical computer scientist sheds a single tear.

(That theoretical computer scientist is me)