Lecture 5: Algorithm Analysis and Modeling
Warm Up

Construct a mathematical function modeling the worst case runtime for the following functions

```java
public void mystery1(ArrayList<String> list) {
    for (int i = 0; i < 3000; i++) {
        for (int j = 0; j < 1000; j++) {
            int index = (i + j) % list.size();
            System.out.println(list.get(index));
        }
    }
}
```

Possible answer

\[ T(n) = 3000 \times (6000 + n) \]

```java
public void mystery2(ArrayList<String> list) {
    for (int i = 0; i < list.size(); i++) {
        for (int j = 0; j < list.size(); j++) {
            System.out.println(list.get(0));
        }
    }
}
```

Possible answer

\[ T(n) = n^2 \]

Approach

- start with basic operations, work inside out for control structures
- Each basic operation = +1
- Conditionals = worst case test operations + branch
- Loop = iterations (loop body)

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Adminstrivia

HW 1 Due Friday at 11:59pm
HW 2 goes live Friday
FILL OUT PARTNER FORM
1 partner
Netid NOT student number
Please fill out class survey
Assign me a partner form
Why don’t we care about exact constants?

Not enough information to compute precise constants

Depends on too many factors (underlying hardware, background processes, temperature etc...)

We really care about the growth of the function

Big O...
Comparing Functions
Asymptotic Analysis

asymptotic analysis – the process of mathematically representing runtime of a algorithm in relation to the number of inputs and how that relationship changes as the number of inputs grow

Two step process

1. **Model** – the process of mathematically representing how many operations a piece of code will run in relation to the number of inputs \( n \)

2. **Analyze** – compare runtime/input relationship across multiple algorithms
   - 1. Graph the model of your code where \( x = \) number of inputs and \( y = \) runtime
   - 2. For which inputs will one perform better than the other?
Imagine you have three possible algorithms to choose between. Each has already been reduced to its mathematical model.

\[
\begin{align*}
  f(n) &= n \\
  g(n) &= 4n \\
  h(n) &= n^2
\end{align*}
\]

The growth rate for \(f(n)\) and \(g(n)\) looks very different for small numbers of input. \(h(n)\) has a distinctly different growth rate.

...but since both are linear, eventually look similar at large input sizes.

But for very small input values, \(h(n)\) actually has a slower growth rate than either \(f(n)\) or \(g(n)\).
**Review: Complexity Classes**

A complexity class is a category of algorithm efficiency based on the algorithm’s relationship to the input size N.

<table>
<thead>
<tr>
<th>Class</th>
<th>Big O</th>
<th>If you double N...</th>
<th>Example algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant</td>
<td>O(1)</td>
<td>unchanged</td>
<td>Add to front of linked list</td>
</tr>
<tr>
<td>logarithmic</td>
<td>O(log₂N)</td>
<td>Increases slightly</td>
<td>Binary search</td>
</tr>
<tr>
<td>linear</td>
<td>O(N)</td>
<td>doubles</td>
<td>Sequential search</td>
</tr>
<tr>
<td>log-linear</td>
<td>O(Nlog₂N)</td>
<td>Slightly more than doubles</td>
<td>Merge sort</td>
</tr>
<tr>
<td>quadratic</td>
<td>O(N²)</td>
<td>quadruples</td>
<td>Nested loops traversing a 2D array</td>
</tr>
<tr>
<td>cubic</td>
<td>O(N³)</td>
<td>Multiplies by 8</td>
<td>Triple nested loop</td>
</tr>
<tr>
<td>polynomial</td>
<td>O(N^c)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>exponential</td>
<td>O(c^N)</td>
<td>Multiplies drastically</td>
<td></td>
</tr>
</tbody>
</table>

[http://bigocheatsheet.com/](http://bigocheatsheet.com/)
Moving from Model to Complexity Class

Say an algorithm runs \(0.4N^3 + 25N^2 + 8N + 17\) statements

17 is quickly dwarfed in the context of thousands of inputs

We ignore constants like 25 because they are tiny next to \(N\)

\(N^3\) is so powerful it dominates the overall runtime

\(O(N^3)\)

\(10n \log n + 3n\)

\(5n^2 \log n + 13n^3\)

\(20n \log \log n + 2n \log n\)

\(2^{3n}\)

Consider the runtime when \(N\) is extremely large

Lower order terms don’t matter – delete them.

Multiplying by constant factors has little effect on growth rate

Highest order term dominates the overall rate of change

Gives us a “simplified big-O”

\(O(n \log n)\)

\(O(n^3)\)

\(O(n \log n)\)

\(O(8^n)\)
**Definition: Big-O**

We wanted to find an upper bound on our algorithm’s running time, but
- We don’t want to care about constant factors.
- We only care about what happens as \( n \) gets large.

**Big-O**

\( f(n) \) is \( O(g(n)) \) if there exist positive constants \( c, n_0 \) such that for all \( n \geq n_0 \),
\[
f(n) \leq c \cdot g(n)
\]

We also say that \( g(n) \) “dominates” \( f(n) \)

\( O(g(n)) \) is the “family” or “set” of all functions that are dominated by \( g(n) \)

**Why \( n_0 \)?**

**Why \( c \)?**
Applying Big O Definition

Show that \( f(n) = 10n + 15 \) is \( O(n) \)

Apply definition term by term

\[
10n \leq c \cdot n \text{ when } c = 10 \text{ for all values of } n
\]

\[
15 \leq c \cdot n \text{ when } c = 15 \text{ for } n \geq 1
\]

Add up all your truths

\[
10n + 15 \leq 10n + 15n \leq 25n \text{ for } n \geq 1
\]

Select values for \( c \) and \( n_0 \) and prove they validate the definition

Take \( c = 25 \) and \( n_0 = 1 \)

\[
10n \leq 25n \text{ for all values of } n
\]

\[
15 \leq 25n \text{ for } n \geq 1
\]

Thus because a \( c \) and \( n_0 \) exist, \( f(n) \) is \( O(n) \)
Exercise: Proving Big O

Demonstrate that $5n^2 + 3n + 6$ is dominated by $n^3$ by finding a $c$ and $n_0$ that satisfy the definition of domination

$5n^2 + 3n + 6 \leq 5n^2 + 3n^2 + 6n^2$ when $n \geq 1$

$5n^2 + 3n^2 + 6n^2 = 14n^2$

$5n^2 + 3n + 6 \leq 14n^2$ for $n \geq 1$

$14n^2 \leq cn^3$ for $c = ?$ $n \geq ?$

$\frac{14}{n} \rightarrow c = 14$ & $n \geq 1$

Big-O

$f(n)$ is $O(g(n))$ if there exist positive constants $c, n_0$ such that for all $n \geq n_0$,

$f(n) \leq c \cdot g(n)$
Edge Cases

True or False: $10n^2 + 15n$ is $O(n^3)$

It’s true – it fits the definition

$$10n^2 \leq c \cdot n^3 \text{ when } c = 10 \text{ for } n \geq 1$$

$$15n \leq c \cdot n^3 \text{ when } c = 15 \text{ for } n \geq 1$$

$$10n^2 + 15n \leq 10n^3 + 15n^3 \leq 25n^3 \text{ for } n \geq 1$$

$$10n^2 + 15n \text{ is } O(n^3) \text{ because } 10n^2 + 15n \leq 25n^3 \text{ for } n \geq 1$$

Big-O is just an upper bound. It doesn’t have to be a good upper bound

If we want the best upper bound, we’ll ask you for a **tight** big-O bound. $O(n^2)$ is the tight bound for this example.

It is (almost always) technically correct to say your code runs in time $O(n!)$. DO NOT TRY TO PULL THIS TRICK ON AN EXAM. Or in an interview.
Why Are We Doing This?

You already intuitively understand what big-O means.

Who needs a formal definition anyway?
- We will.

Your intuitive definition and my intuitive definition might be different.

We’re going to be making more subtle big-O statements in this class.
- We need a mathematical definition to be sure we’re on the same page.

Once we have a mathematical definition, we can go back to intuitive thinking.
- But when a weird edge case, or subtle statement appears, we can figure out what’s correct.
Function comparison: exercise

f(n) = n \leq g(n) = 5n + 3? \quad \text{True} – all linear functions are treated as equivalent
f(n) = 5n + 3 \leq g(n) = n? \quad \text{True}
f(n) = 5n + 3 \leq g(n) = 1? \quad \text{False}
f(n) = 5n + 3 \leq g(n) = n^2? \quad \text{True} – quadratic will always dominate linear
f(n) = n^2 + 3n + 2 \leq g(n) = n^3? \quad \text{True}
f(n) = n^3 \leq g(n) = n^2 + 3n + 2 ? \quad \text{False}
**O, Omega, Theta [oh my?]**

Big-O is an **upper bound**
- My code takes at most this long to run

Big-Omega is a **lower bound**

**Big-Omega**

\[ f(n) \in \Omega(g(n)) \text{ if } \exists \text{ positive constants } c,n_0 \text{ such that for all } n \geq n_0, \quad f(n) \geq c \cdot g(n) \]

Big Theta is “**equal to**”

**Big-Theta**

\[ f(n) \in \Theta(g(n)) \text{ if } f(n) \in \mathcal{O}(g(n)) \text{ and } f(n) \in \Omega(g(n)). \]

\[ \Omega(f(n)) \leq f(n) = \theta(f(n)) \leq O(f(n)) \]

- Is dominated by \( f(n) \in \mathcal{O}(g(n)) \)
- Dominates \( f(n) \in \Omega(g(n)) \)

\[ f(n), \quad O(1), \quad O(\log n), \quad O(n), \quad O(n^2), \quad O(n^3) \]
Viewing $O$ as a class

Sometimes you’ll see big-O defined as a family or set of functions.

<table>
<thead>
<tr>
<th>Big-O (alternative definition)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(g(n))$ is the set of all functions $f(n)$ such that there exist positive constants $c, n_0$ such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$</td>
</tr>
</tbody>
</table>

For that reason, some people write $f(n) \in O(g(n))$ where we wrote “$f(n)$ is $O(g(n))$”. Other people write “$f(n) = O(g(n))$” to mean the same thing.

The set of all functions that run in linear time (i.e. $O(n)$) is a “complexity class.”

We never write $O(5n)$ instead of $O(n)$ – they’re the same thing!

It’s like writing $\frac{6}{2}$ instead of 3. It just looks weird.
Examples

$4n^2 \in \Omega(1)$

true

$4n^2 \in \Omega(n)$

false

$4n^2 \in \Omega(n^2)$

true

$4n^2 \in \Omega(n^3)$

false

$4n^2 \in \Omega(n^4)$

false

$4n^2 \in O(1)$

false

$4n^2 \in O(n)$

true

$4n^2 \in O(n^2)$

true

$4n^2 \in O(n^3)$

true

$4n^2 \in O(n^4)$

true

---

**Big-O**

$f(n) \in O(g(n))$ if there exist positive constants $c, n_0$ such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$

**Big-Omega**

$f(n) \in \Omega(g(n))$ if there exist positive constants $c, n_0$ such that for all $n \geq n_0$, $f(n) \geq c \cdot g(n)$

**Big-Theta**

$f(n) \in \Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$. 
Practice

5n + 3 ∈ O(n) True
n ∈ O(5n + 3) True
5n + 3 = O(n) True
O(5n + 3) = O(n) True
O(n^2) = O(n) False

Big-O

\[ f(n) \in O(g(n)) \text{ if there exist positive constants } c, n_0 \text{ such that for all } n \geq n_0, \]
\[ f(n) \leq c \cdot g(n) \]

Big-Omega

\[ f(n) \in \Omega(g(n)) \text{ if there exist positive constants } c, n_0 \text{ such that for all } n \geq n_0, \]
\[ f(n) \geq c \cdot g(n) \]

n^2 ∈ O(1) False
n^2 ∈ O(n) False
n^2 ∈ O(n^2) True
n^2 ∈ O(n^3) True
n^2 ∈ O(n^{100}) True

Big-Theta

\[ f(n) \in \Theta(g(n)) \text{ if } f(n) \text{ is } O(g(n)) \text{ and } f(n) \text{ is } \Omega(g(n)). \]