Lecture 19: Disjoint Sets

CSE 373 Data Structures & Algorithms

A new ADT

We need a new ADT!

Disjoint-Sets (aka Union-Find) ADT

state

Family of Sets

- **sets are disjoint:** No element appears in more than one set
- No required order (neither within sets, nor between sets)
- Each set has a name (usually one of its elements)

behavior

makeSet(value) – creates a new set where the only member is the value. Picks a name

findSet(value) – looks up the name of the set containing value, returns the name of that set

union(x, y) – looks up set containing x and set containing y, combines two sets into one. All of the values of one set are added to the other, and the now empty set goes away. Chooses a name for combined set

A better idea

Here's a better idea:

We need to be able to combine things easily.

- Pointer based data structures are better at that.

But given a value, we need to be able to find the right set.

- Sounds like we need a dictionary somewhere

And we need to be able to find a certain element ("the representative") within a set quickly.

- Trees are good at that (better than linked lists at least)

The Real Implementation

Disjoint-Set ADT

state

Set of Sets
- **Disjoint:** Elements must be unique across sets
- No required order
- Each set has representative

Count of Sets

behavior

makeSet(x) – creates a new set within the disjoint set where the only member is x. Picks representative for set

findSet(x) – looks up the set containing element x, returns representative of that set

union(x, y) – looks up set containing x and set containing y, combines two sets into one. Picks new representative for resulting set

UpTreeDisjointSet<E>

state

`Collection<TreeSet> forest`

`Dictionary<NodeValues, NodeLocations> nodeInventory`

behavior

`makeSet(x)`-create a new tree of size 1 and add to our forest

`findSet(x)`-locates node with x and moves up tree to find root

`union(x, y)`-append tree with y as a child of tree with x

TreeSet<E>

state

`SetNode overallRoot`

behavior

`TreeSet(x)`

`add(x)`

`remove(x, y)`

`getRep()`-returns data of overallRoot

SetNode<E>

state

`E data`

`Collection<SetNode> children`

behavior

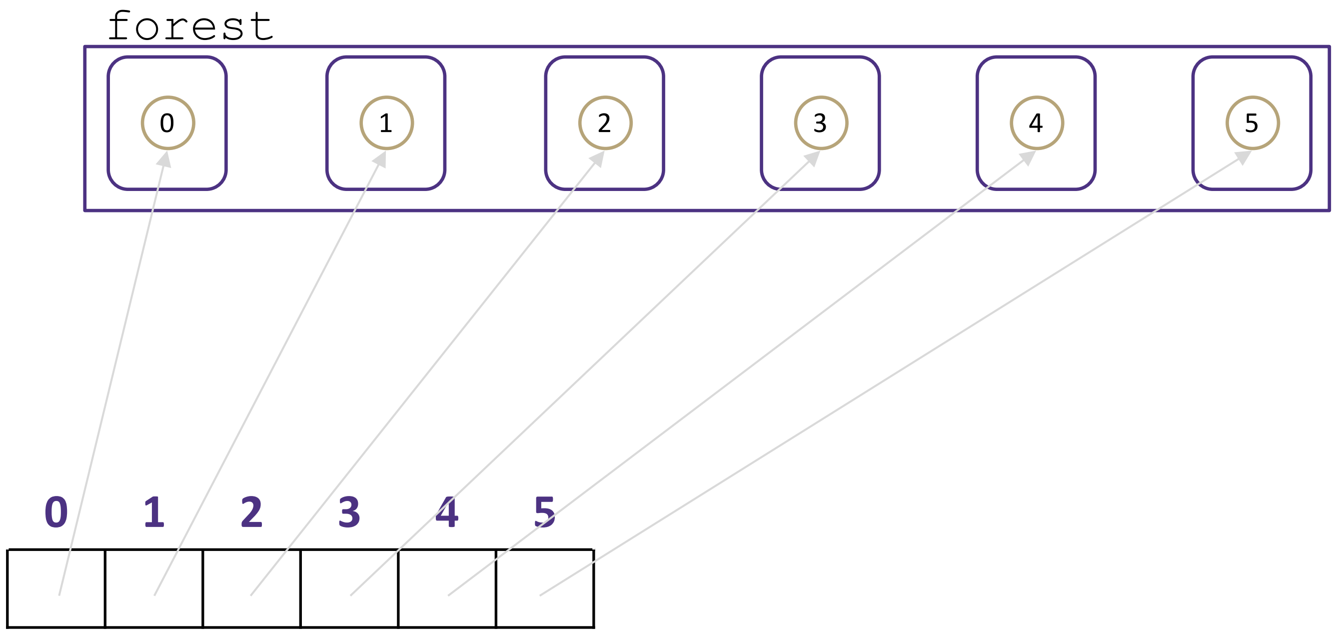
`SetNode(x)`

`addChild(x)`

`removeChild(x, y)`

Implement makeSet(x)

```
makeSet(0)
makeSet(1)
makeSet(2)
makeSet(3)
makeSet(4)
makeSet(5)
```



TreeDisjointSet<E>

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behavior

makeSet(x)-create a new tree of size 1 and add to our forest

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union(x, y)-append tree with y as a child of tree with x

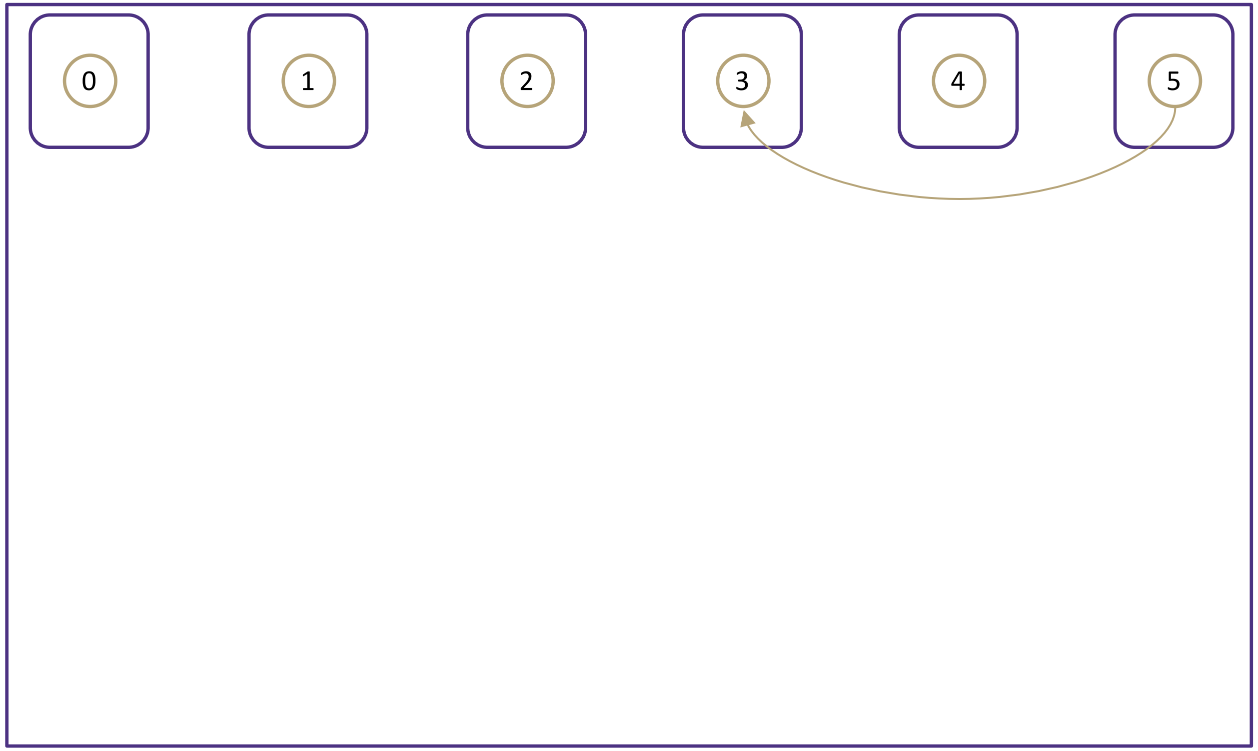
Worst case runtime? Just like with graphs, we're going to assume we have control over the dictionary keys and just say we'll always have $\Theta(1)$ dictionary behavior.

$O(1)$

Implement union(x, y)

`union(3, 5)`

forest



0	1	2	3	4	5
->	->	->	->	->	->

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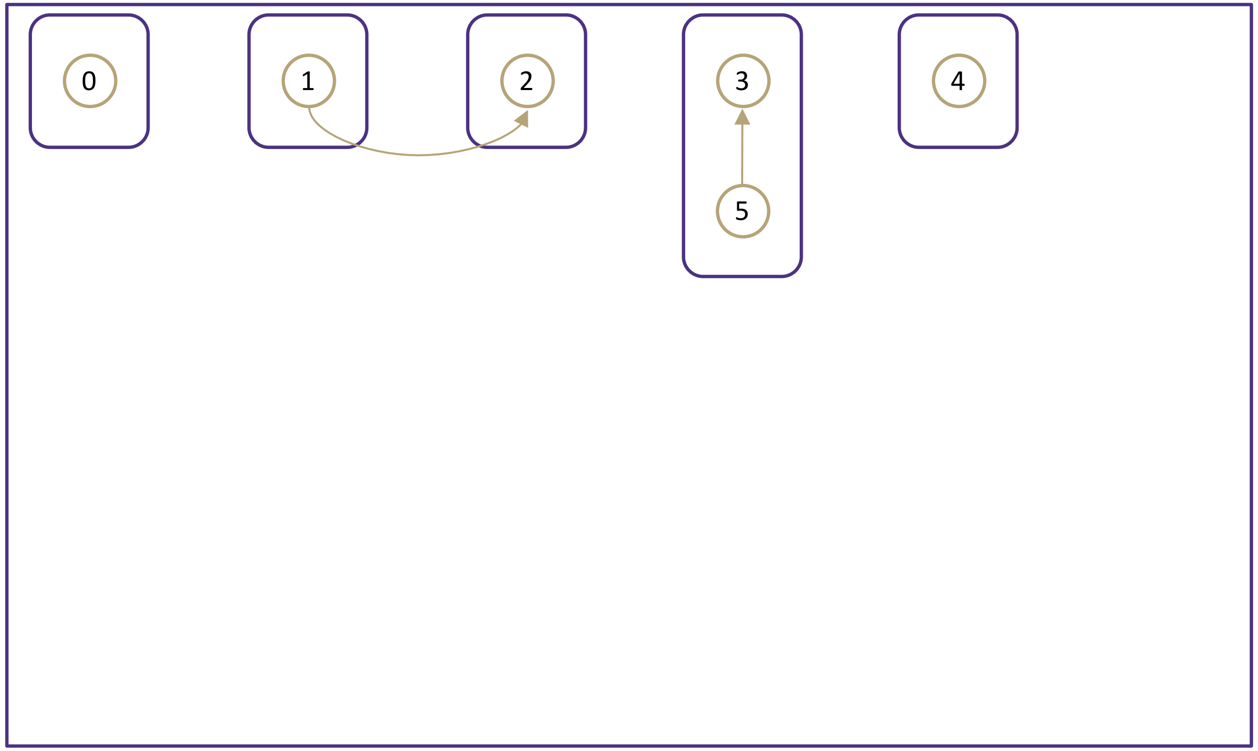
Implement union(x, y)

```
union(3, 5)

union(2, 1)
```

forest



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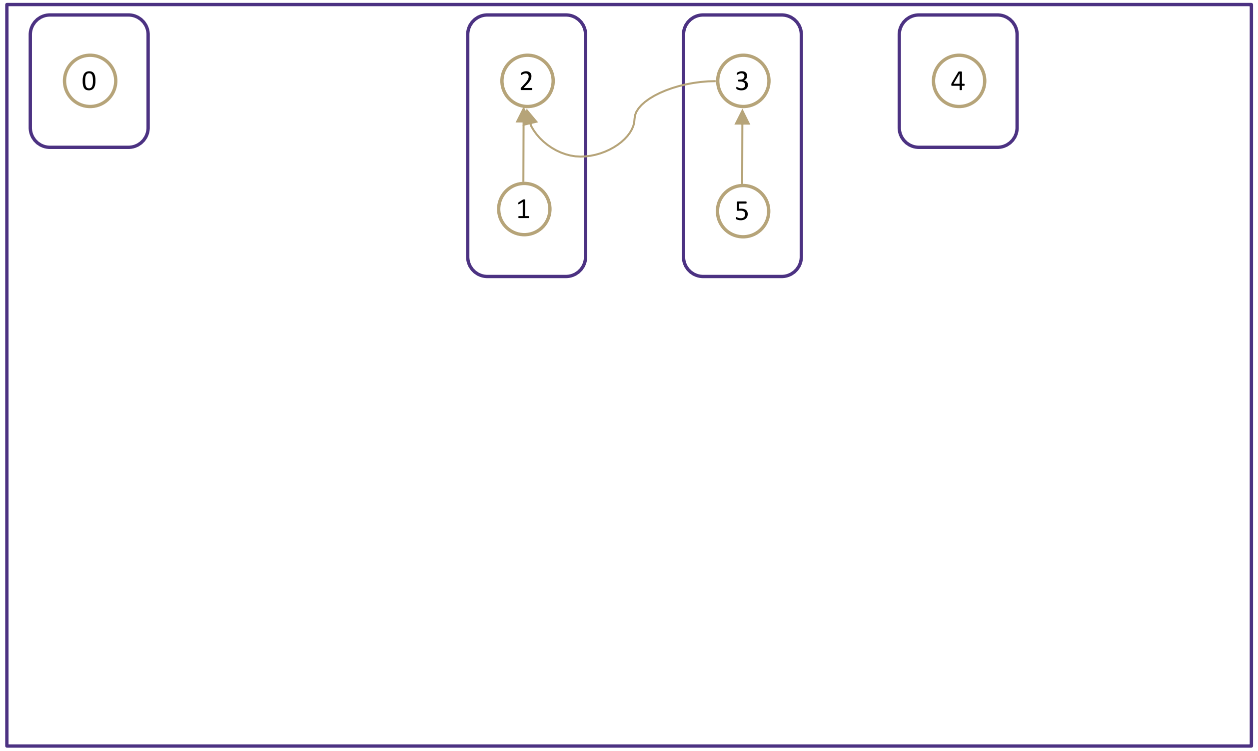

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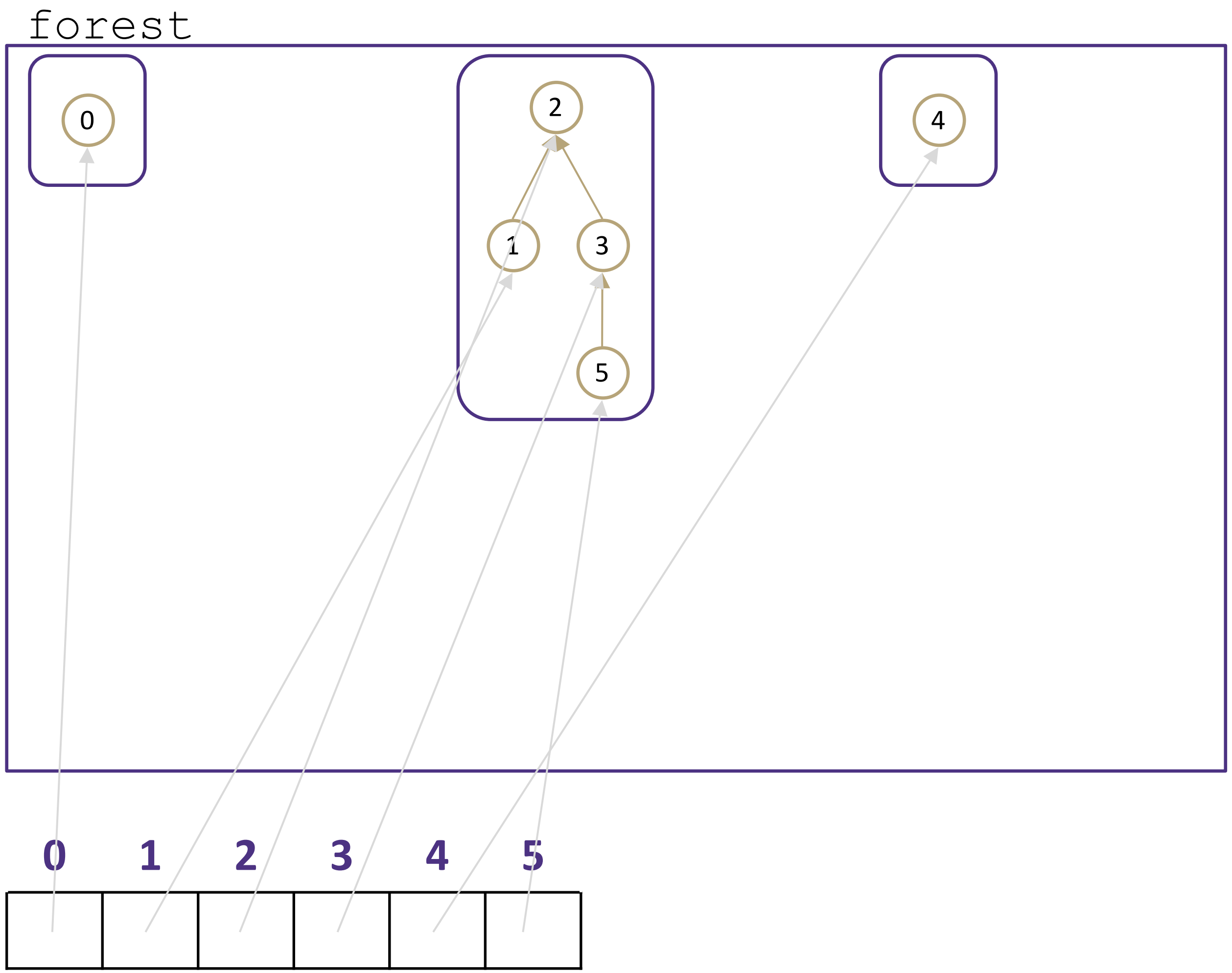

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Implement findSet(x)

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findSet(5)
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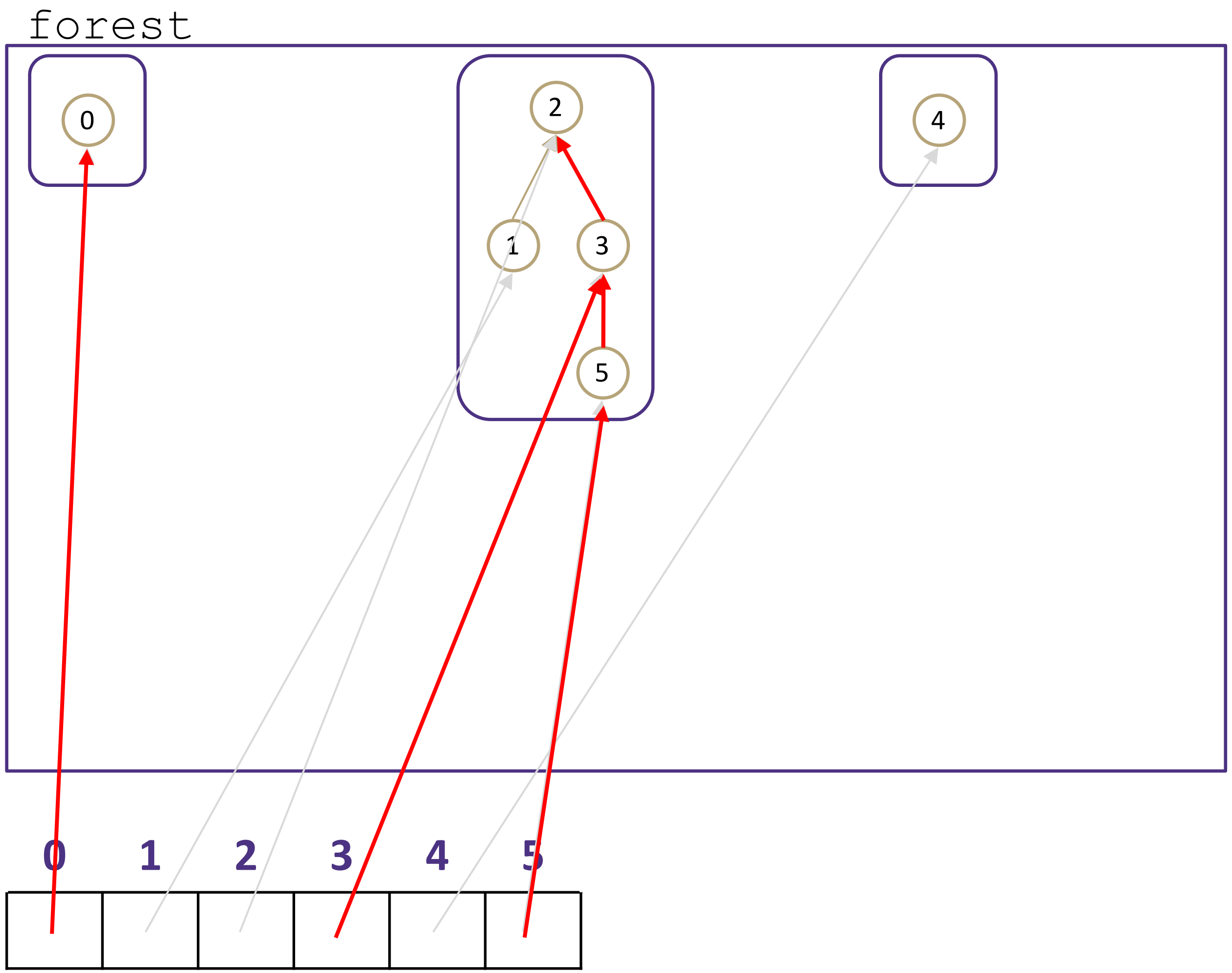

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Worst case runtime of findSet?

$\Theta(n)$

Worst case runtime of union?

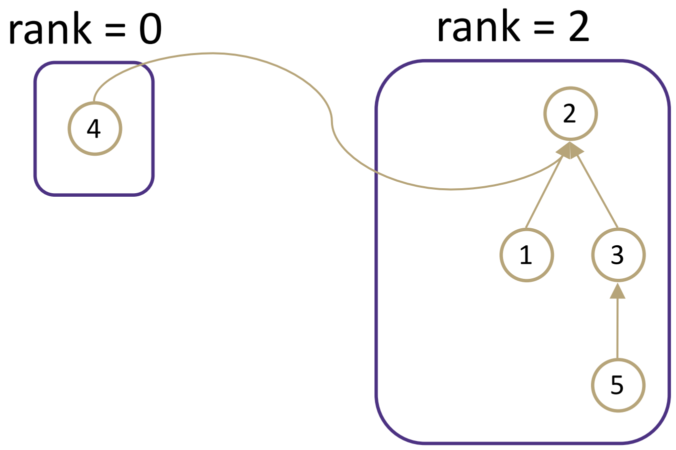
$\Theta(n)$ – union has to call find!

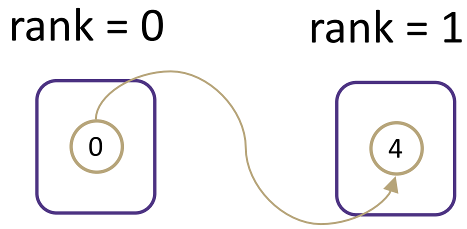
Improving union

Problem: Trees can be unbalanced

Solution: Union-by-rank!

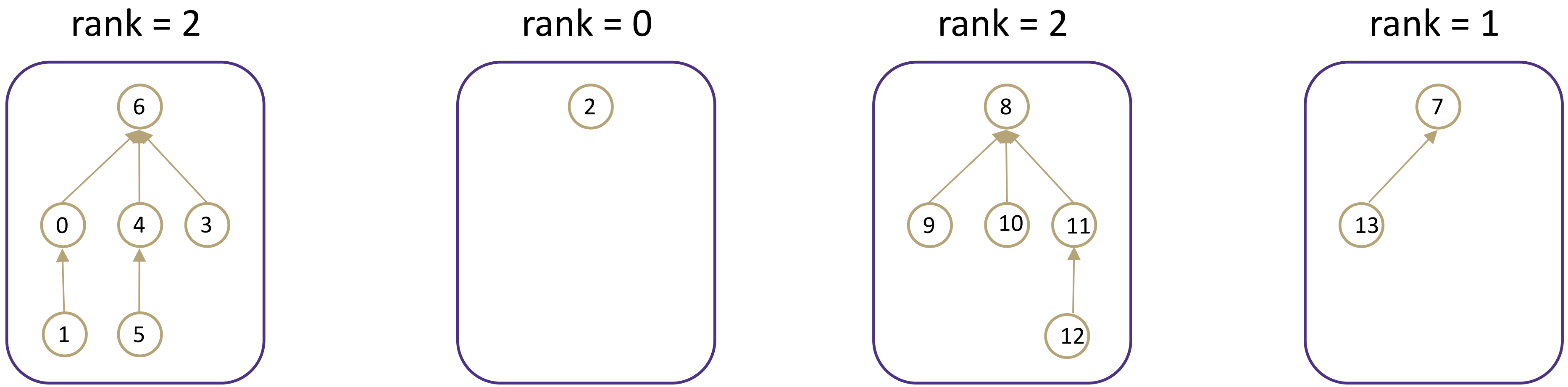
- rank is a lot like height (it's not quite height, for reasons we'll see soon)
- Keep track of rank of all trees
- makeSet creates a tree of rank 0.
- When unioning make the tree with larger rank the root. New rank is larger of two merged ranks.
- If it's a tie, pick one to be root arbitrarily and increase rank by one.





Practice

Given the following disjoint-set what would be the result of the following calls on union if we add the "union-by-rank" optimization. Draw the forest at each stage with corresponding ranks for each tree.




```
union(2, 13)
union(4, 12)
union(2, 8)
```

Practice

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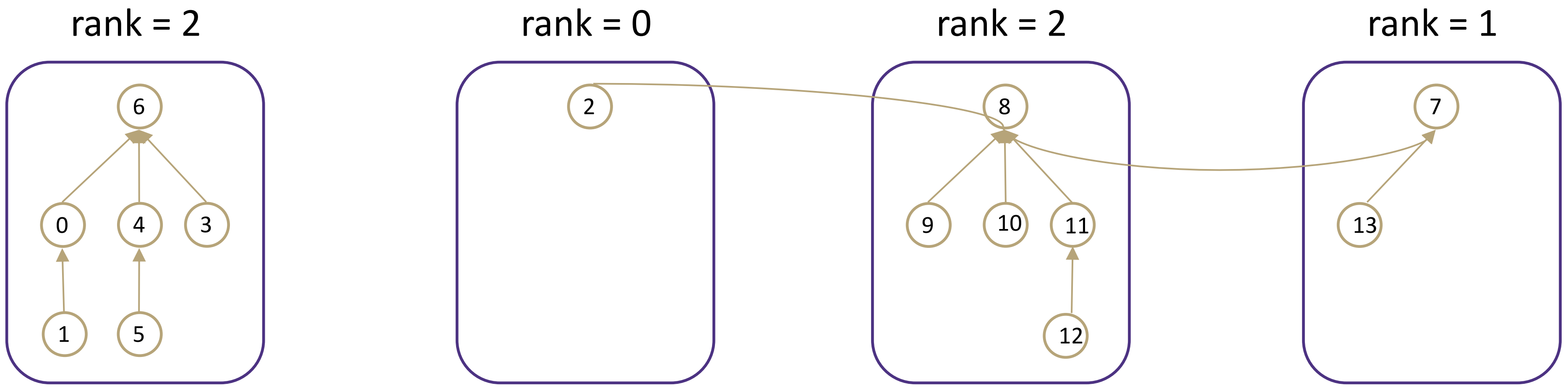

`union(2, 13)`

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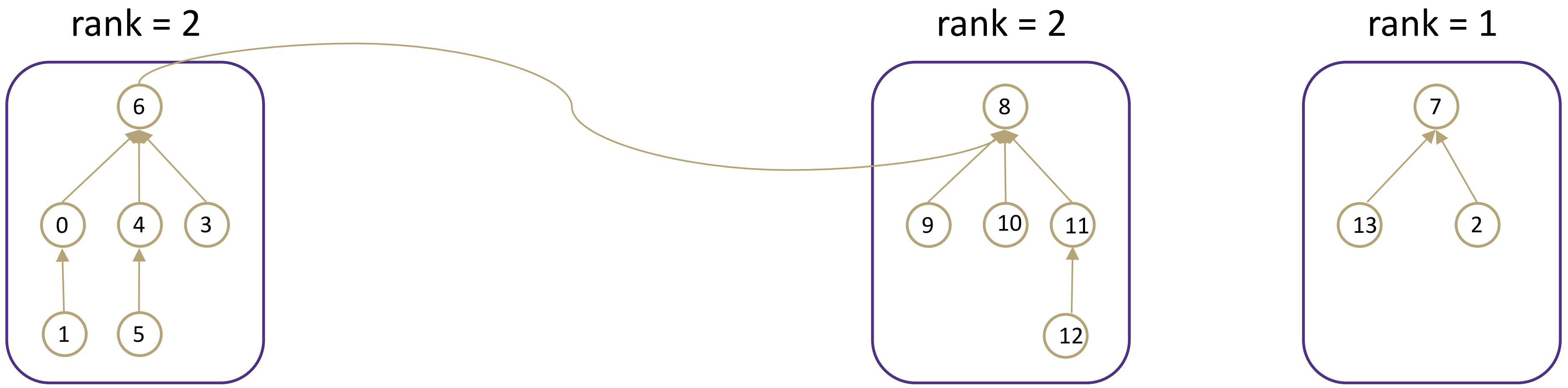


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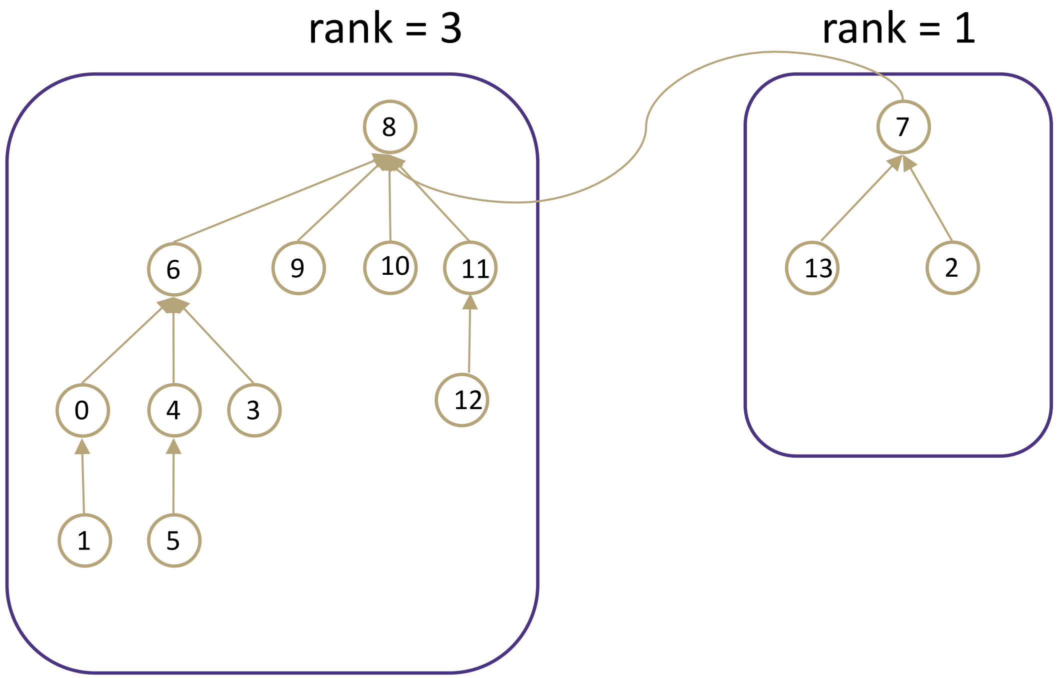


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Practice

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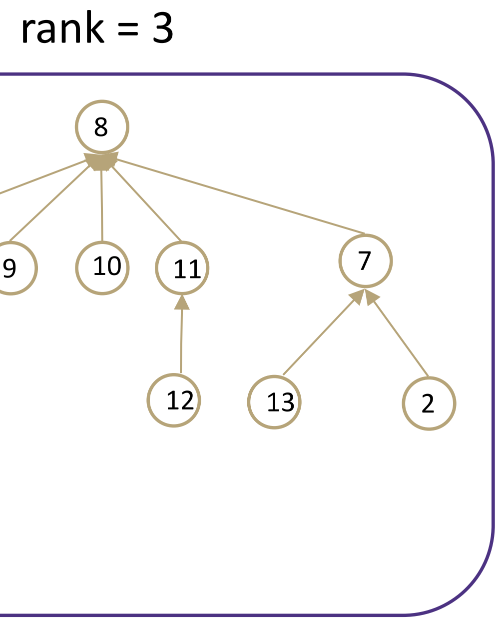



```
union(2, 13)
union(4, 12)
union(2, 8)
```

Does this improve the worst case runtimes?

findSet is $\Theta(\log(n))$ now, not $\Theta(n)$!

Improving findSet()

Problem: Every time we call findSet() you must traverse all the levels of the tree to find representative

Solution: Path Compression

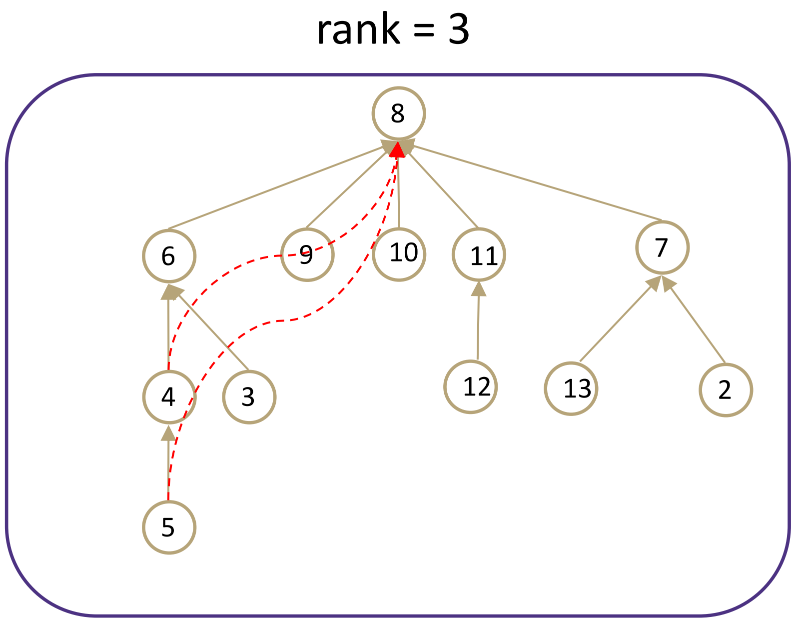
- Collapse tree into fewer levels by updating parent pointer of each node you visit
- Whenever you call findSet() update each node you touch's parent pointer to point directly to overallRoot

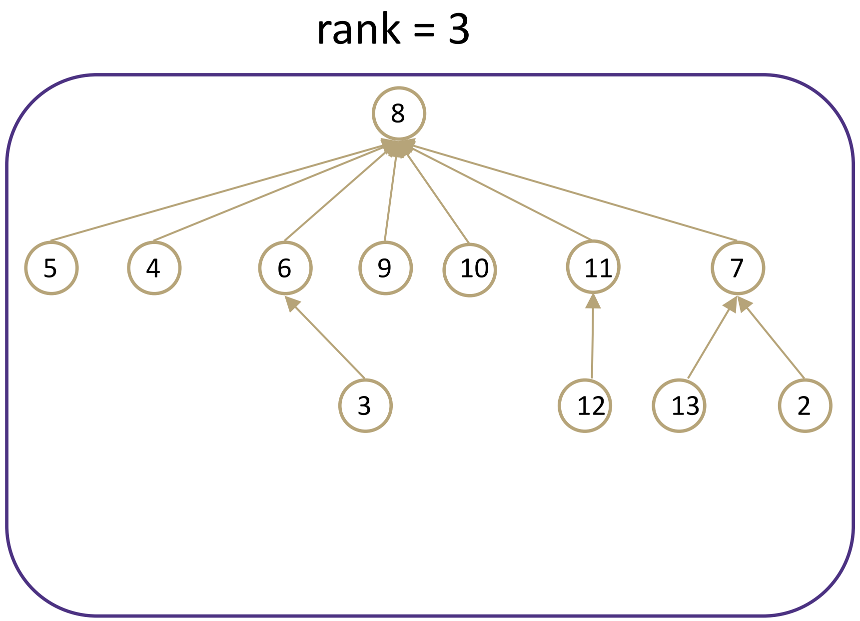
```
findSet(5)

findSet(4)
```

Does this improve the worst case runtimes?

Not the worst-case, but…in-practice it makes a big difference.





Example

Using the union-by-rank and path-compression optimized implementations of disjoint-sets draw the resulting forest caused by these calls:

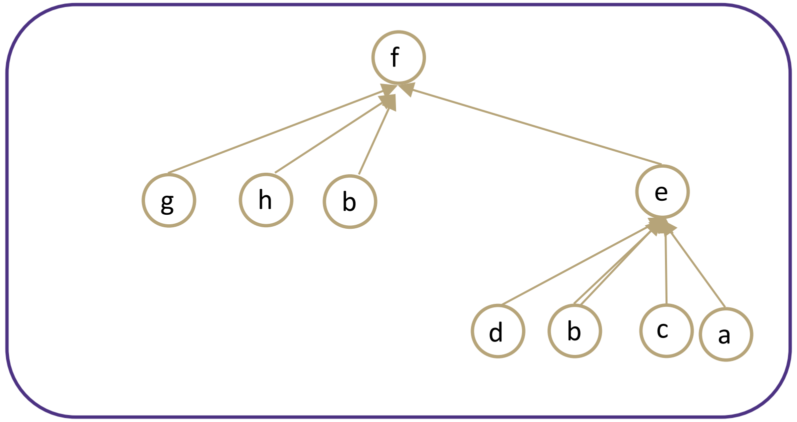
1. `makeSet(a)`
2. `makeSet(b)`
3. `makeSet(c)`
4. `makeSet(d)`
5. `makeSet(e)`
6. `makeSet(f)`
7. `makeSet(g)`
8. `makeSet(h)`
9. `union(c, e)`
10. `union(d, e)`
11. `union(a, c)`
12. `union(g, h)`
13. `union(b, f)`
14. `union(g, f)`
15. `union(b, c)`
16. `union(g, a)`

Example

Using the union-by-rank and path-compression optimized implementations of disjoint-sets draw the resulting forest caused by these calls:

1. makeSet(a)
2. makeSet(b)
3. makeSet(c)
4. makeSet(d)
5. makeSet(e)
6. makeSet(f)
7. makeSet(g)
8. makeSet(h)
9. union(c, e)
10. union(d, e)
11. union(a, c)
12. union(g, h)
13. union(b, f)
14. union(g, f)
15. union(b, c)
16. union(g, a)

rank = 2



Subtleties of Path Compression

Path compression is an optimization written into the `findSet` code.

It does not appear directly in the `union` code.

- It's not worth it; you'd have to rewrite the entire `findSet` code inside `union` to make it happen.

But `union` does make two `findSet` calls,

- So path compression will happen when you do a `union` call, just indirectly.

Optimized Up-trees Runtimes

	makeSet	findSet	Union
Worst-Case	$\Theta(1)$	$\Theta(\log n)$	$\Theta(\log n)$
Best-Case	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
In-Practice	$\Theta(1)$	$O(\log^* n)$	$O(\log^* n)$

Hey why are some of those $O()$ not $\Theta()$?
And…wait what's that * above the log?

$\log^*(n)$

$\log^*(n)$ is the "iterated logarithm"

It answers the question "how many times do I have to take the log of this to get a number at most 1?"

E.g. $\log^*(16) = 3$

$\log(16) = 4 \qquad \log(4) = 2 \qquad \log(2) = 1.$

$\log^* n$ grows ridiculously slowly.

$\log^*(10^{80}) \; = 5.$

10^{80} is the number of atoms in the observable universe. For all practical purposes these operations are constant time.

But they aren't $O(1)$.

Optimized Up-tree Runtimes

$\log^* n$ isn't tight – that's why those $\Theta()$ bounds became $O()$ bounds.

There is a tight bound. It's a function that grows even slower than $\log^* n$

- Google "inverse Ackerman function"

Kruskal's Algorithm

```
KruskalMST(Graph G)
   initialize each vertex to be its own component
   sort the edges by weight
   foreach(edge (u, v) in sorted order){
      if(u and v are in different components){
         add (u,v) to the MST
         Update u and v to be in the same component
      }
   }
```

What's the running time of Kruskal's?

For MST algorithms, assume that m dominates n
(if it doesn't, there is no spanning tree to find)

```
KruskalMST(Graph G)
   initialize new DisjointSets DS
   for(v : G.vertices) { DS.makeSet(v) }
   sort the edges by weight
   foreach(edge (u, v) in sorted order){
      if(DS.findSet(u) != DS.findSet(v)){
         add (u,v) to the MST
         DS.union(u,v)
      }
   }
```

$\Theta(n)$

$\Theta(m \log m)$

m calls, do we have to worry about the $\log n$ worst case?

n calls, do we have to worry about the $\log n$ worst case?

Intuition: We could make the $\log n$ running time happen once…but not really more than that.
Since we're counting total operations, we're actually going to see the "in-practice" behavior

Whether we hit worst-case or not: $\Theta(m \log m)$ is dominating term.

Running Time Notes

Intuition: We could make the bad case happen once…but not really more than that.

Since we're counting total operations, we're actually going to see "in-practice" behavior

This kind of statement is "amortized analysis"

- It's also the math behind why we always double the size of array-based data structures.

Some people write the running time as $\Theta(m \log n)$ instead of $\Theta(m \log m)$

They're assuming the graph doesn't have any **multi-edges**.

- I.e. there's at most one edge between any pair of vertices.

And they just think $\Theta(m \log n)$ looks better (even though it's just a constant factor)