Lecture 11: Introduction to Hash Tables
Administrivia

When you’re submitting your group writeup to gradescope, be sure to use the group submission option if you have a partner.

Project 1 part 2 due Thursday night.

Exercise 2 due Friday night.

Project 2 will come out tonight, and Exercise 3 will come out Friday.

Due in two weeks (Wednesday the 31st for Project 2, and Friday the 2nd for Exercise 3)

They “should” be one week assignments... but next Friday is the midterm!

We’re leaving it to you to decide how/when to study for the midterm vs. doing homework.
Aside: How Fast is $\Theta(\log n)$?

If you just looked at a list of common running times

<table>
<thead>
<tr>
<th>Class</th>
<th>Big O</th>
<th>If you double N...</th>
<th>Example algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant</td>
<td>$O(1)$</td>
<td>unchanged</td>
<td>Add to front of linked list</td>
</tr>
<tr>
<td>logarithmic</td>
<td>$O(\log n)$</td>
<td>Increases slightly</td>
<td>Binary search</td>
</tr>
<tr>
<td>linear</td>
<td>$O(n)$</td>
<td>doubles</td>
<td>Sequential search</td>
</tr>
<tr>
<td>“n log n”</td>
<td>$O(n\log n)$</td>
<td>Slightly more than doubles</td>
<td>Merge sort</td>
</tr>
<tr>
<td>quadratic</td>
<td>$O(n^2)$</td>
<td>quadruples</td>
<td>Nested loops traversing a 2D array</td>
</tr>
</tbody>
</table>

You might think this was a small improvement.

It was a HUGE improvement!
Logarithmic vs. Linear

If you double the size of the input,
- A linear time algorithm takes twice as long.
- A logarithmic time algorithm has a constant additive increase to its running time.

To make a logarithmic time algorithm take twice as long, how much do you have to increase $n$ by?

You have to square it $\log(n^2) = 2 \log(n)$.

A gigabyte worth of integer keys can fit in an AVL tree of height 60.

It takes a ridiculously large input to make a logarithmic time algorithm go slowly.

Log isn’t “that running time between linear and constant” it’s “that running time that’s barely worse than a constant.”
Logarithmic Running Times

This identity is so important, one of my friends made me a cross-stitch of it.

Two lessons:
1. Log running times are REALLY REALLY FAST.
2. $O(\log(n^3))$ is not simplified, it’s just $O(\log n)$
Aside: Traversals

What if the heights of subtrees were corrupted.

How could we calculate from scratch?

We could use a “traversal”
- A process that visits every piece of data in a data structure.

```java
int height(Node curr){
    if(curr==null) return -1;
    int h = Math.max(height(curr.left),height(curr.right));
    return h+1;
}
```
Three Kinds of Traversals

InOrder(Node curr) { 
    InOrder(curr.left);
    doSomething(curr);
    InOrder(curr.right);
}

PreOrder(Node curr) { 
    doSomething(curr);
    PreOrder(curr.left);
    PreOrder(curr.right);
}

PostOrder(Node curr) { 
    PostOrder(curr.left);
    PostOrder(curr.right);
    doSomething(curr);
}

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Traversals Practice

For each of the following scenarios, choose an appropriate traversal:

1. Print out all the keys in an AVL-Dictionary in sorted order.

2. Make a copy of an AVL tree

3. Determine if an AVL tree is balanced (assume height values are not stored)
For each of the following scenarios, choose an appropriate traversal:

1. Print out all the keys in an AVL-Dictionary in sorted order.
   - In order
2. Make a copy of an AVL tree
   - Pre-order
3. Determine if an AVL tree is balanced (assume height values are not stored)
   - Post-order
Traversals

If we have $n$ elements, how long does it take to calculate height? $\Theta(n)$ time.

The recursion tree (from the tree method) IS the AVL tree!

We do a constant number of operations at each node

In general, traversals take $\Theta(n \cdot f(k))$ time,

where $\text{doSomething}()$ takes $\Theta(f(k))$ time.

Common question on technical interviews!
Aside: Other Self-Balancing Trees

There are lots of flavors of self-balancing search trees

“Red-black trees” work on a similar principle to AVL trees.

“Splay trees”
- Get $O(\log n)$ amortized bounds for all operations.

“Scapegoat trees”

“Treaps” – a BST and heap in one (!)

B-trees (see other 373 versions) optimized for huge datasets.

If you have an application where you need a balanced BST that also [does something] it might already exist. Google first, you might be able to use a library.
Hashing
Review: Dictionaries

Dictionary ADT

**state**
- Set of items & keys
- Count of items

**behavior**
- `put(key, item)` add item to collection indexed with key
- `get(key)` return item associated with key
- `containsKey(key)` return if key already in use
- `remove(key)` remove item and associated key
- `size()` return count of items

ArrayDictionary\(<K, V>\>

**state**
- `Pair<K, V>[]` data

**behavior**
- `put` create new pair, add to next available spot, grow array if necessary
- `get` scan all pairs looking for given key, return associated item if found
- `containsKey` scan all pairs, return if key is found
- `remove` scan all pairs, replace pair to be removed with last pair in collection
- `size` return count of items in dictionary

LinkedDictionary\(<K, V>\>

**state**
- `front`
- `size`

**behavior**
- `put` if key is unused, create new pair, add to front of list, else replace with new value
- `get` scan all pairs looking for given key, return associated item if found
- `containsKey` scan all pairs, return if key is found
- `remove` scan all pairs, skip pair to be removed
- `size` return count of items in dictionary

AVLDictionary\(<K, V>\>

**state**
- `overallRoot`
- `size`

**behavior**
- `put` if key is unused, create new pair, place in BST order, rotate to maintain balance
- `get` traverse through tree using BST property, return item if found
- `containsKey` traverse through tree using BST property, return if key is found
- `remove` traverse through tree using BST property, replace or nullify as appropriate
- `size` return count of items in dictionary
Review: Dictionaries

Why are we so obsessed with Dictionaries? It’s all about data baby!

When dealing with data:
- Adding data to your collection
- Getting data out of your collection
- Rearranging data in your collection

SUPER common in comp sci
- Databases
- Network router tables
- Compilers and Interpreters

<table>
<thead>
<tr>
<th>Operation</th>
<th>ArrayList</th>
<th>LinkedList</th>
<th>BST</th>
<th>AVLTree</th>
</tr>
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<tbody>
<tr>
<td>put(key,value)</td>
<td>best</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>worst</td>
<td></td>
<td></td>
<td></td>
</tr>
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<td></td>
</tr>
<tr>
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<tr>
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Dictionary ADT

state
- Set of items & keys
- Count of items

behavior
- put(key, item) add item to collection indexed with key
- get(key) return item associated with key
- containsKey(key) return if key already in use
- remove(key) remove item and associated key
- size() return count of items

It's all about data baby!
Review: Dictionaries

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<td>Θ(1)</td>
</tr>
<tr>
<td></td>
<td>worst</td>
<td>Θ(n)</td>
<td>Θ(n)</td>
<td>Θ(n)</td>
</tr>
<tr>
<td>get(key)</td>
<td>best</td>
<td>Θ(1)</td>
<td>Θ(1)</td>
<td>Θ(1)</td>
</tr>
<tr>
<td></td>
<td>worst</td>
<td>Θ(n)</td>
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</tr>
<tr>
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<tr>
<td></td>
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Dictionary ADT

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put(key, item) add item to collection indexed with key
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3 Minutes
“In-Practice” Case

For Hash Tables, we’re going to talk about what you can expect “in-practice”
- Instead of just what the best and worst scenarios are.

Other resources (and previous versions of 373) use “average case”

There’s a lot of math (beyond the scope of the course) needed to make “average” statements precise.
- So we’re not going to do it that way.

For this class, we’ll just tell you what assumptions we’re making about how the “real world” usually works.

And then do worst-case analysis under those assumptions.
Can we do better?

What if we knew exactly where to find our data?

Implement a dictionary that accepts only integer keys between 0 and some value k

- -> Leverage Array Indices!

“Direct address map”

<table>
<thead>
<tr>
<th>Operation</th>
<th>Array w/ indices as keys</th>
</tr>
</thead>
<tbody>
<tr>
<td>put(key, value)</td>
<td>best O(1)</td>
</tr>
<tr>
<td></td>
<td>worst O(1)</td>
</tr>
<tr>
<td>get(key)</td>
<td>best O(1)</td>
</tr>
<tr>
<td></td>
<td>worst O(1)</td>
</tr>
<tr>
<td>remove(key)</td>
<td>best O(1)</td>
</tr>
<tr>
<td></td>
<td>worst O(1)</td>
</tr>
</tbody>
</table>

DirectAccessMap<Integer, V>

**state**
- Data[]
- size

**behavior**
- put put item at given index
- get get item at given index
- containsKey if data[] null at index, return false, return true otherwise
- remove nullify element at index
- size return count of items in dictionary
public V get(int key) {
    this.ensureIndexNotNull(key);
    return this.array[key];
}

public void put(int key, V value) {
    this.array[key] = value;
}

public void remove(int key) {
    this.ensureIndexNotNull(key);
    this.array[key] = null;
}

DirectAccessMap<Integer, V>

state
Data[]
size
behavior
put put item at given index
get get item at given index
containsKey if data[] null at index, return false, return true otherwise
remove nullify element at index
size return count of items in dictionary
Can we do this for any integer?

Idea 1:
Create a GIANT array with every possible integer as an index

Problems:
- Can we allocate an array big enough?
- Super wasteful

Idea 2:
Create a smaller array, but create a way to translate given integer keys into available indices

Problem:
- How can we pick a good translation?
**Review: Integer remainder with % “mod”**

The % operator computes the remainder from integer division.

```plaintext
14 \% 4 \ is \ 2
\[ \begin{array}{c}
3 \\
4 \ ) 14 \\
12 \\
2
\end{array} \]

218 \% 5 \ is \ 3
\[ \begin{array}{c}
43 \\
5 \ ) 218 \\
20 \\
18 \\
15 \\
3
\end{array} \]
```

Equivalently, to find \(a \% b\) (for \(a, b > 0\)):

```
while(a > b-1)
    a -= b;
return a;
```

Applications of % operator:

- Obtain last digit of a number: 230857 \% 10 is 7
- See whether a number is odd: 7 \% 2 is 1, 42 \% 2 is 0
- Limit integers to specific range: 8 \% 12 is 8, 18 \% 12 is 6

Limit keys to indices within array

For more review/practice, check out [https://www.khanacademy.org/computing/computer-science/cryptography/modarithmetic/a/what-is-modular-arithmetic](https://www.khanacademy.org/computing/computer-science/cryptography/modarithmetic/a/what-is-modular-arithmetic)
First Hash Function: % table size

<table>
<thead>
<tr>
<th>indices</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>elements</td>
<td>“foo”</td>
<td>“biz”</td>
<td></td>
<td></td>
<td></td>
<td>“bar”</td>
<td></td>
<td></td>
<td>“bop”</td>
<td></td>
</tr>
</tbody>
</table>

put(0, “foo”); 0 % 10 = 0
put(5, “bar”); 5 % 10 = 5
put(11, “biz”); 11 % 10 = 1
put(18, “bop”); 18 % 10 = 8
public V get(int key) {
    int newKey = getKey(key);
    this.ensureIndexNotNull(newKey);
    return this.data[newKey];
}

public void put(int key, int value) {
    this.array[getKey(key)] = value;
}

public void remove(int key) {
    int newKey = getKey(key);
    this.ensureIndexNotNull(newKey);
    this.data[newKey] = null;
}

public int getKey(int k) {
    return k % this.data.length;
}
### First Hash Function: % table size

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<th>0</th>
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<th>9</th>
</tr>
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<tbody>
<tr>
<td>elements</td>
<td>&quot;(&quot;</td>
<td>&quot;biz&quot;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>&quot;bar&quot;</td>
</tr>
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- put(0, "foo"); 0 % 10 = 0
- put(5, "bar"); 5 % 10 = 5
- put(11, "biz"); 11 % 10 = 1
- put(18, "bop"); 18 % 10 = 8
- put(20, "("); 20 % 10 = 0

Collision!
Hash Obsession: Collisions

Collision: multiple keys translate to the same location of the array

The fewer the collisions, the better the runtime!

Two questions:
1. When we have a collision, how do we resolve it?
2. How do we minimize the number of collisions?
Strategies to handle hash collisions
Strategies to handle hash collision

There are multiple strategies. In this class, we’ll cover the following ones:

1. Separate chaining

2. Open addressing
   - Linear probing
   - Quadratic probing
   - Double hashing
Handling Collisions

Solution 1: Chaining

Each space holds a “bucket” that can store multiple values. Bucket is often implemented with a LinkedList

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<tr>
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</tr>
<tr>
<td></td>
<td>In-practice: Θ(1)</td>
</tr>
<tr>
<td></td>
<td>worst: Θ(n)</td>
</tr>
<tr>
<td>get(key)</td>
<td>In-practice: Θ(1)</td>
</tr>
<tr>
<td></td>
<td>average: Θ(1)</td>
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<tr>
<td></td>
<td>worst: Θ(n)</td>
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<tr>
<td>remove(key)</td>
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</tr>
<tr>
<td></td>
<td>worst: Θ(n)</td>
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“In-Practice” Case:
Depends on average number of elements per chain

Load Factor λ
If n is the total number of key-value pairs
Let c be the capacity of array
Load Factor $\lambda = \frac{n}{c}$

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In-Practice

We’re going to make an **assumption** about how often collisions happen.

It’s not actually true, but it’s “close enough” to true that our big-O analyses will be pretty consistent with what you usually see in-practice.

**Our Hashing Assumption**

The hash function will distribute the input keys as evenly as possible across the buckets.

This is not true in the real-world.

But what is usually true in the real-world is pretty close is close enough that the big-O analyses are the same.
What is the worst-case under our hashing assumption?

We might have to go to the end of the linked list in one of the buckets. How long will that linked list be?

If we have $n$ keys and our hash table has $c$ buckets, it will be length $\frac{n}{c}$.

That number will come up so often, we give it a name. It’s the **load factor**.
- We denote it by $\lambda$.  

---

**In-Practice**

**Our Hashing Assumption**

The hash function will distribute the input keys as evenly as possible across the buckets.
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<td>$O(1 + \lambda)$</td>
</tr>
<tr>
<td>worst</td>
<td>$O(n)$</td>
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“In-Practice” Case:

Depends on average number of elements per chain

Load Factor $\lambda$

If $n$ is the total number of key-value pairs
Let $c$ be the capacity of array

$$\text{Load Factor } \lambda = \frac{n}{c}$$
Practice

Consider an IntegerDictionary using separate chaining with an internal capacity of 10. Assume our buckets are implemented using a LinkedList where we append new key-value pairs to the end.

Now, suppose we insert the following key-value pairs. What does the dictionary internally look like?

(1, a) (5, b) (11, a) (7, d) (12, e) (17, f) (1, g) (25, h)
What about non integer keys?

Hash Function

An algorithm that maps a given key to an integer representing the index in the array for where to store the associated value

Goals

Avoid collisions
- The more collisions, the further we move away from $O(1+\lambda)$
- Produce a wide range of indices, and distribute evenly over them

Low computational costs
- Hash function is called every time we want to interact with the data
How to Hash non Integer Keys

Implementation 1: Simple aspect of values
public int hashCode(String input) {
    return input.length();
}

Implementation 2: More aspects of value
public int hashCode(String input) {
    int output = 0;
    for(char c : input) {
        out += (int)c;
    }
    return output;
}

Implementation 3: Multiple aspects of value + math!
public int hashCode(String input) {
    int output = 1;
    for (char c : input) {
        int nextPrime = getNextPrime();
        out *= Math.pow(nextPrime, (int)c);
    }
    return Math.pow(nextPrime, input.length());
}
Consider a StringDictionary using separate chaining with an internal capacity of 10. Assume our buckets are implemented using a LinkedList. Use the following hash function:

```java
public int hashCode(String input) {
    return input.length() % arr.length;
}
```

Now, insert the following key-value pairs. What does the dictionary internally look like?

(“a”, 1) ("ab", 2) ("c", 3) ("abc", 4) ("abcd", 5) ("abcdabcd", 6) ("five", 7) ("hello world", 8)
Java and Hash Functions

Object class includes default functionality:
- equals
- hashCode

If you want to implement your own hashCode you should:
- Override BOTH hashCode() and equals()

If a.equals(b) is true then a.hashCode() == b.hashCode() MUST also be true

That requirement is part of the Object interface. Other people’s code will assume you’ve followed this rule.

Java’s HashMap (and HashSet) will assume you follow these rules and conventions for your custom objects if you want to use your custom objects as keys.
Resizing

Our running time in practice depends on $\lambda$. What do we do when $\lambda$ is big?

Resize the array!
- Usually we double, that’s not quite the best idea here
- Increase array size to next prime number that’s roughly double the array size
  - Prime numbers tend to redistribute keys, because you’re now modding by a completely unrelated number.
- If $\% \text{TableSize}$ gives you $k$ then $\% 2*\text{TableSize}$ gives either $k$ or $2k$.
- Rule of thumb: Resize sometime around when $\lambda$ is somewhere around 1 if you’re doing separate chaining.
  - When you resize, you have to rehash everything!