Lecture 10: AVL Trees

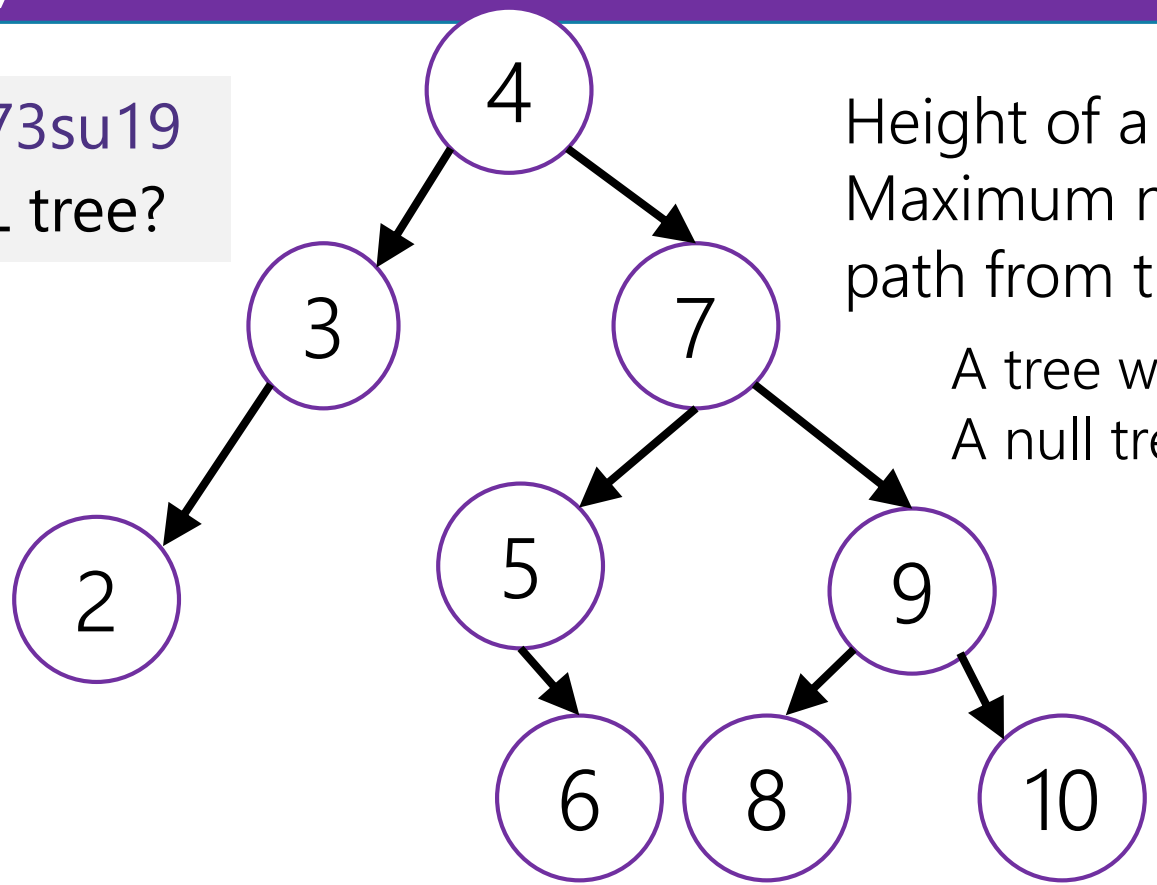
CSE 373 Data Structures and Algorithms

Warm-Up

An AVL tree is a binary search tree that also meets the following rule

AVL condition: For every node, the height of its left subtree and right subtree differ by at most 1.

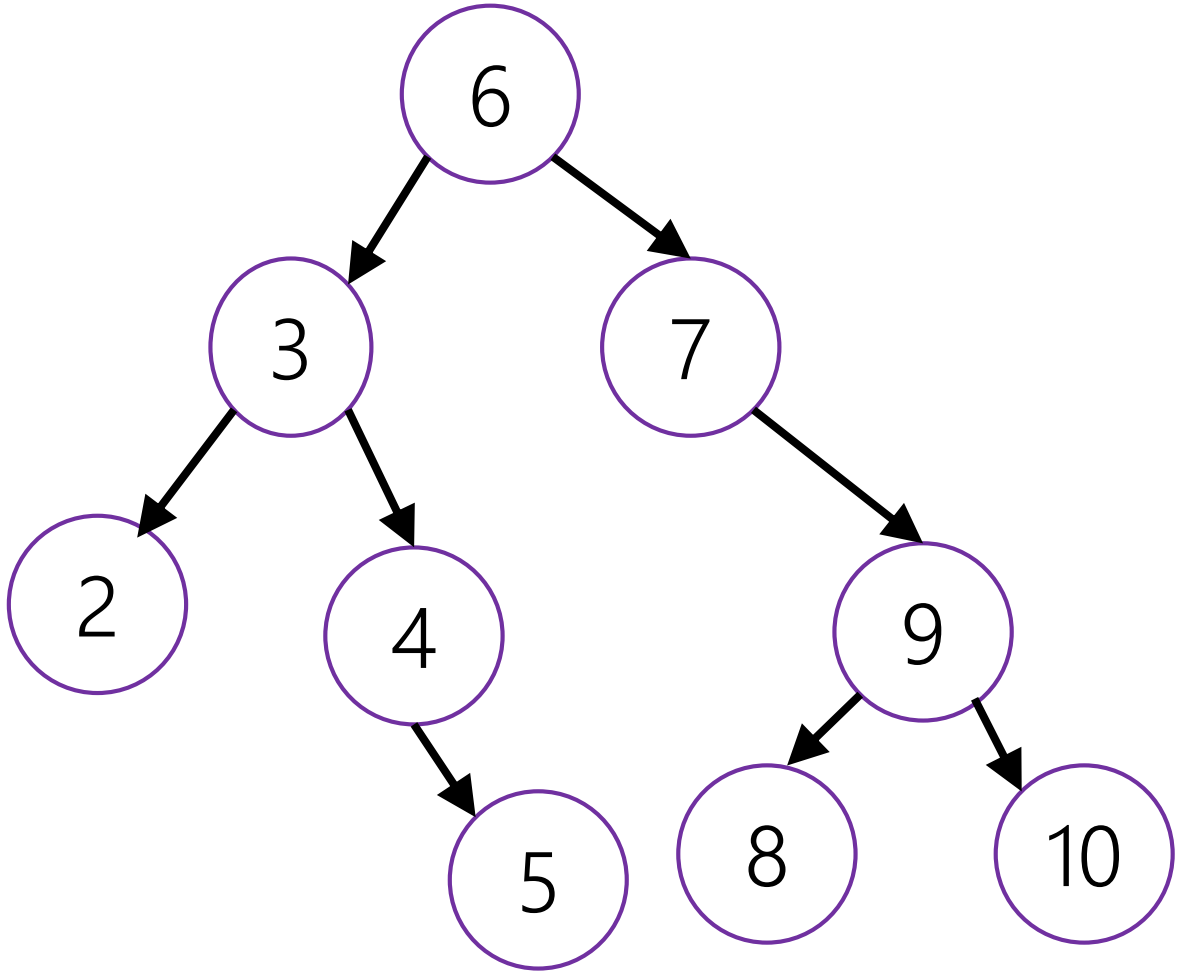
pollEV.com/cse373su19
Is this a valid AVL tree?

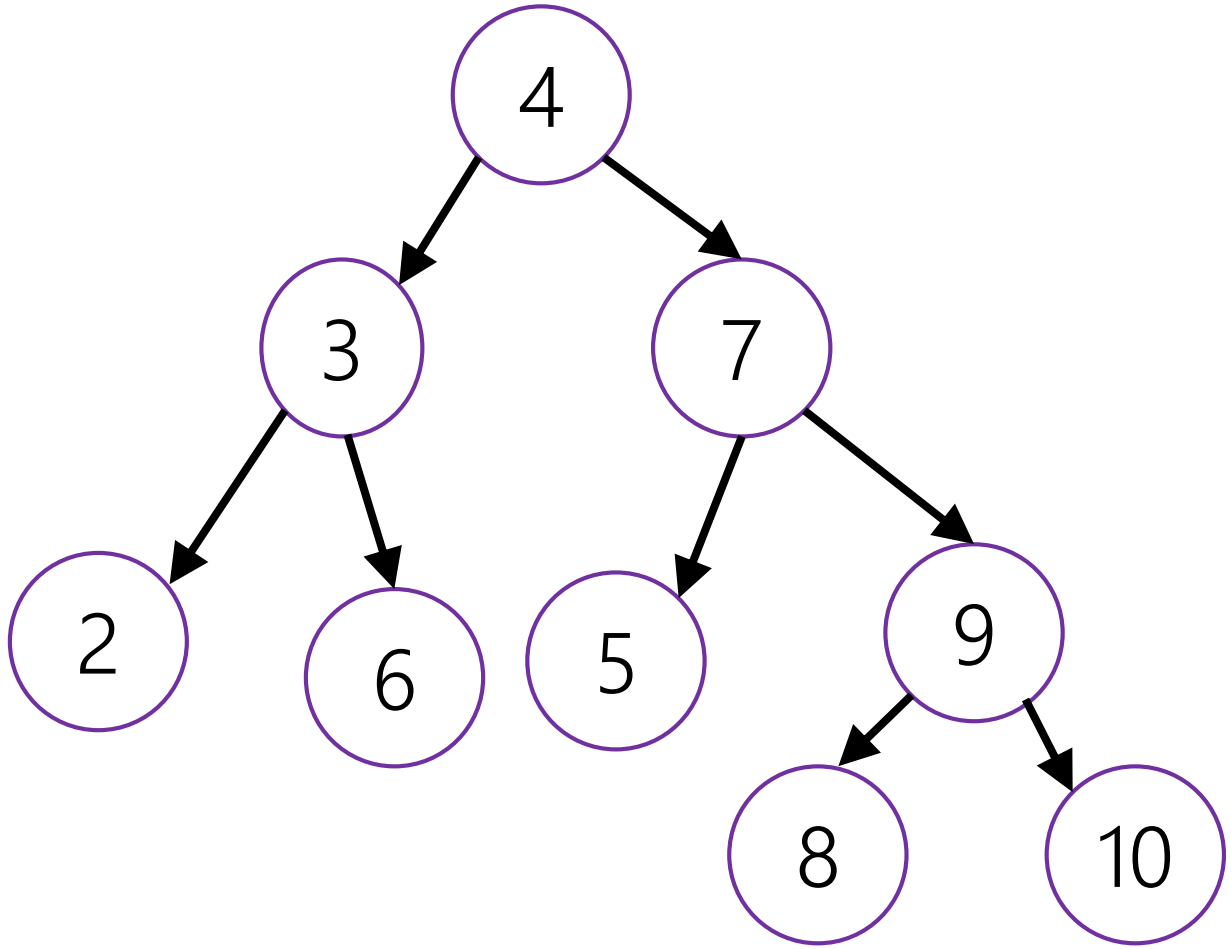


Height of a tree:
Maximum number of edges on a path from the root to a leaf.

A tree with one node has height 0
A null tree (no nodes) has height -1

Are These AVL Trees?





Avoiding the Degenerate Tree

An AVL tree is a binary search tree that also meets the following rule

AVL condition: For every node, the height of its left subtree and right subtree differ by at most 1.

This will avoid the $\Theta(n)$ behavior! We have to check:

1. We must be able to maintain this property when inserting/deleting
2. Such a tree must have height $O(\log n)$.

Bounding the Height

Let $m(h)$ be the minimum number of nodes in an AVL tree of height h.

If we can say $m(h)$ is big, we'll be able to say that a tree with n nodes has a small height.

So…what's $m(h)$?

$$m(h) = \begin{cases} 1 & \text{if } h = 0 \\ 2 & \text{if } h = 1 \\ m(h-1) + m(h-2) + 1 & \text{otherwise} \end{cases}$$

A simpler recurrence

Hey! That's a recurrence!

Recurrences can describe any kind of function, not just running time of code!

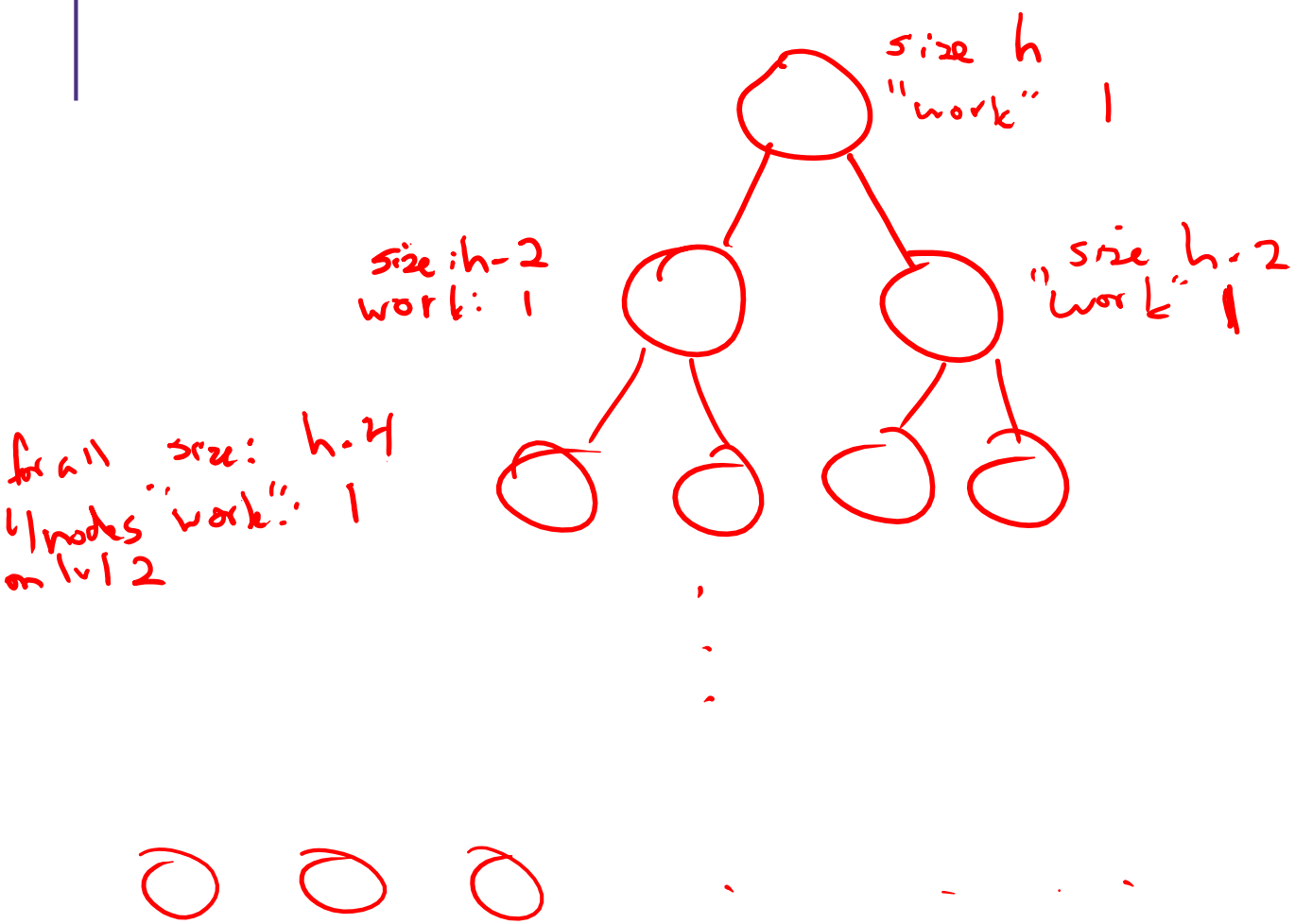
$$m(h) = \begin{cases} 1 & \text{if } h = 0 \\ 2 & \text{if } h = 1 \\ m(h-1) + m(h-2) + 1 & \text{otherwise} \end{cases}$$

We could use tree method, but it's a little…weird.

It'll be easier if we change things just a bit:

$$m(h) \geq \begin{cases} 1 & \text{if } h = 0 \\ 2 & \text{if } h = 1 \\ m(h-\mathbf{2}) + m(h-2) + 1 & \text{otherwise} \end{cases}$$

$$m(h) \geq \begin{cases} 1 & \text{if } h = 0 \\ 2 & \text{if } h = 1 \\ m(h-2) + m(h-2) + 1 & \text{otherwise} \end{cases}$$



Answer the following questions:

1. What is the size of the input on level i?
2. What is the work done by each node on the i^{th} recursive level
3. What is the number of nodes at level i?
4. What is the total work done at the i^th recursive level?
5. What value of i does the last level occur?
6. What is the total work across the base case level?

5 Minutes

Tree Method Practice

For simplicity, assume h is even
(algebra similar for odd case)

1. What is the size of the input on level i? $h-2i$

2. What is the work done by each node on the i^{th} recursive level? 1

3. What is the number of nodes at level i? 2^i

4. What is the total work done at the i^{th} recursive level? $2^i \cdot 1$

5. What value of i does the last level occur?

   $h-2i = 0 \rightarrow i = \frac{h}{2}$

6. What is the total work across the base case level? $2^{h/2}$

$$m(h) \geq \sum_{i=0}^{\frac{h}{2}-1} 2^i + 2^{h/2} = 2^{h/2} - 1 + 2^{h/2}$$

$$m(h) \geq 2^{h/2}$$

Finishing the Argument

Suppose I give you an AVL tree of height h, how many nodes, n can it have?

Every AVL tree of height h has at least $m(h)$ nodes (that's how $m()$ was defined)

So $n \geq m(h) \geq 2^{h/2}$

This inequality is true for all AVL trees of height h. So we can reverse the logic,

i.e. it's still true to say "if we have n nodes, what can the height be? It must satisfy:

$n \geq 2^{h/2}$

$\log_2 n \geq h/2$

$h \leq 2 \log_2 n$ AVL trees are always short, just like we wanted!

Avoiding the Degenerate Tree

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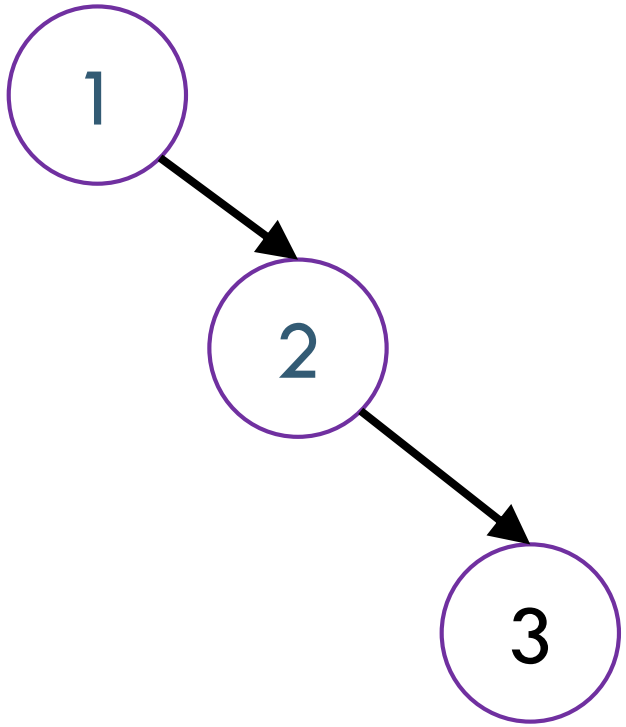
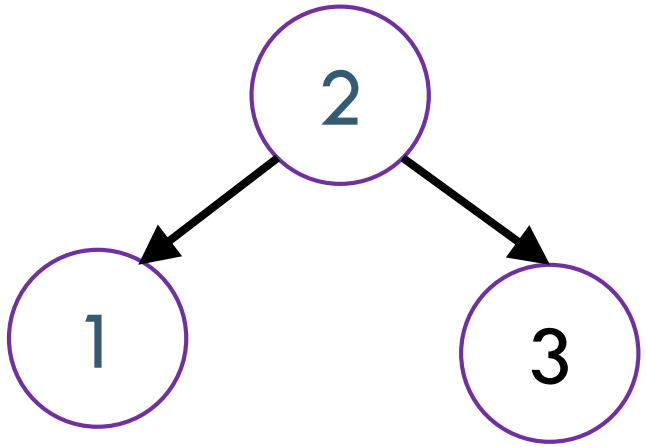
This will avoid the $\Theta(n)$ behavior! We have to check:

1. We must be able to maintain this property when inserting/deleting
2. Such a tree must have height $O(\log n)$. 

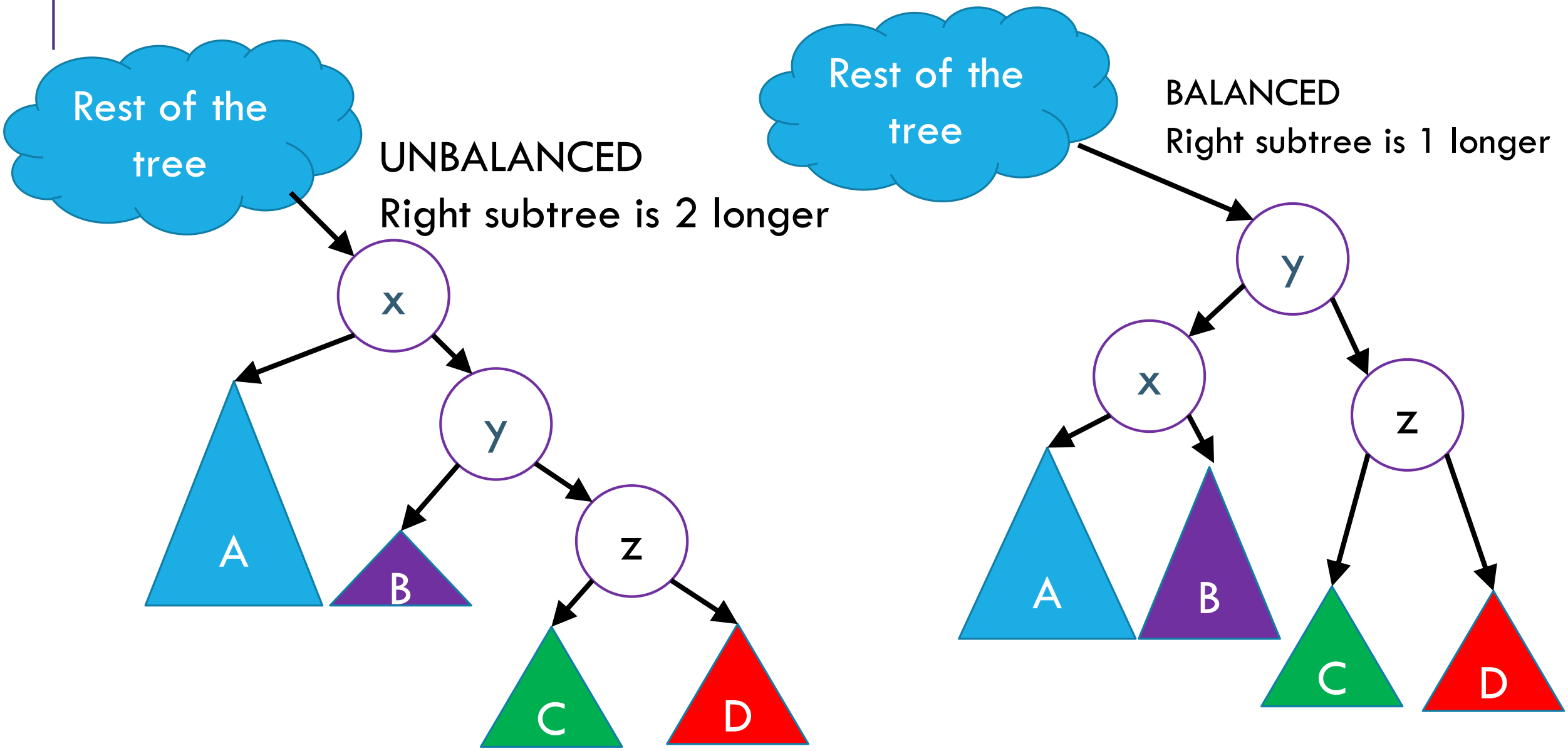
Insertion

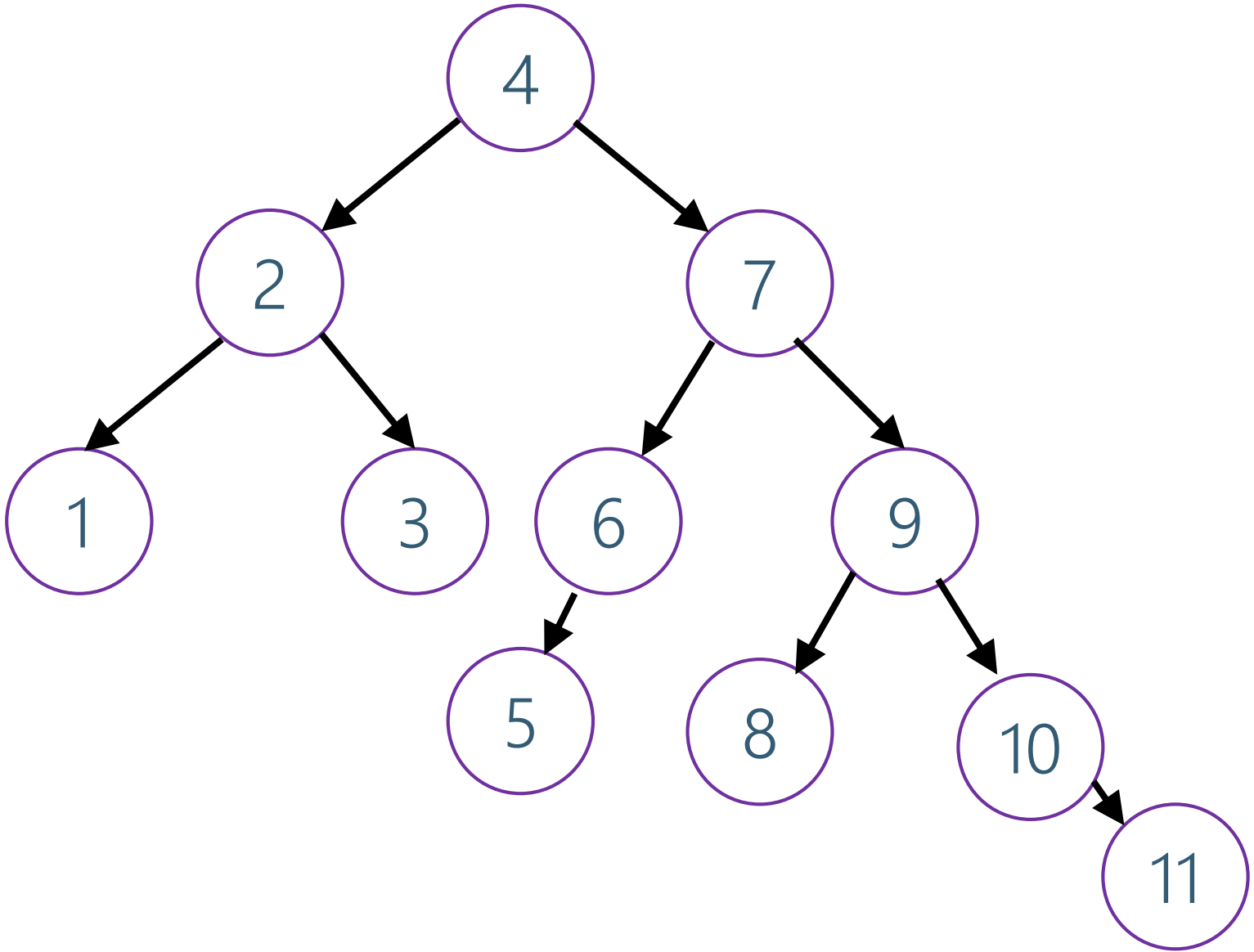
What happens if when we do an insertion, we break the AVL condition?





Left Rotation

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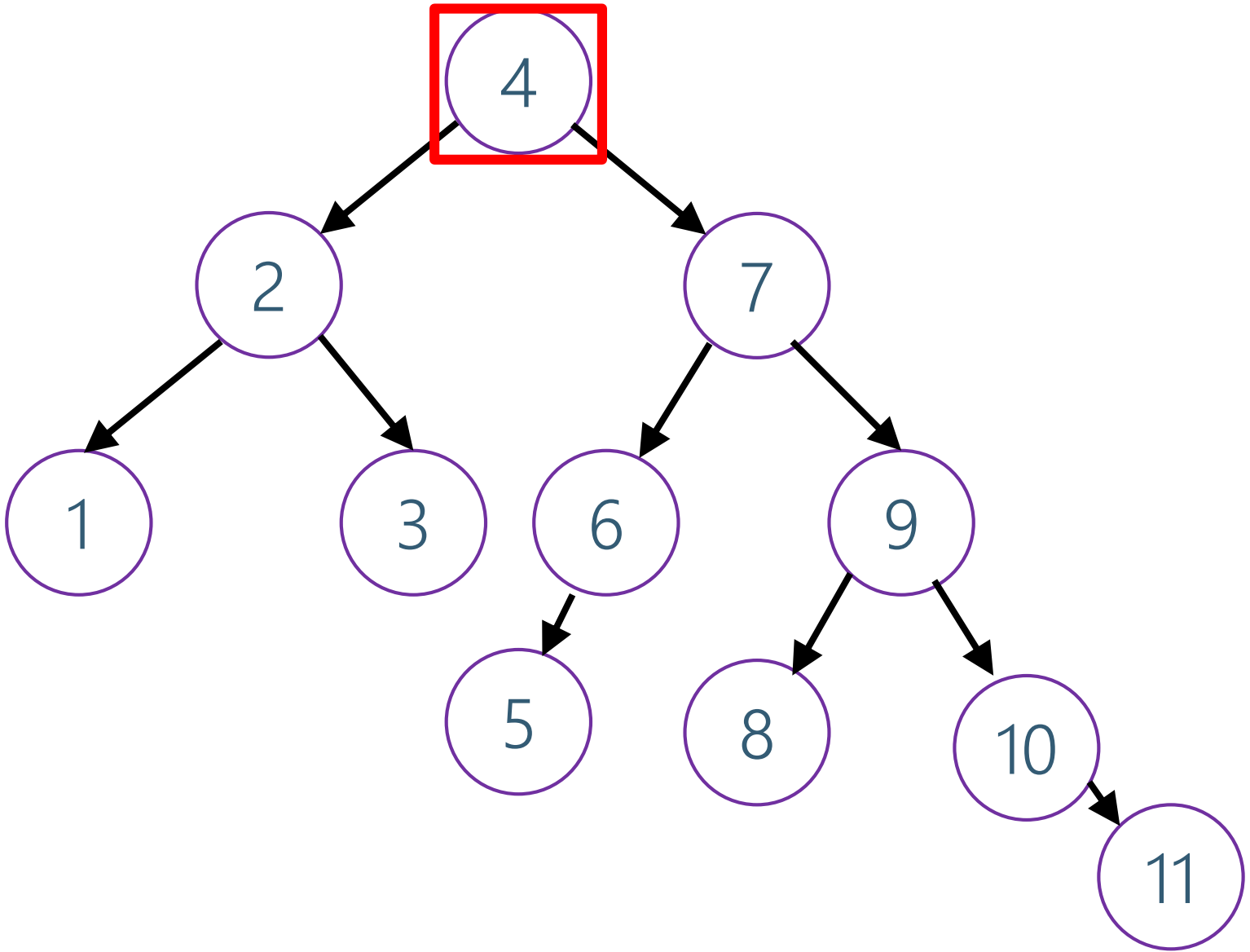
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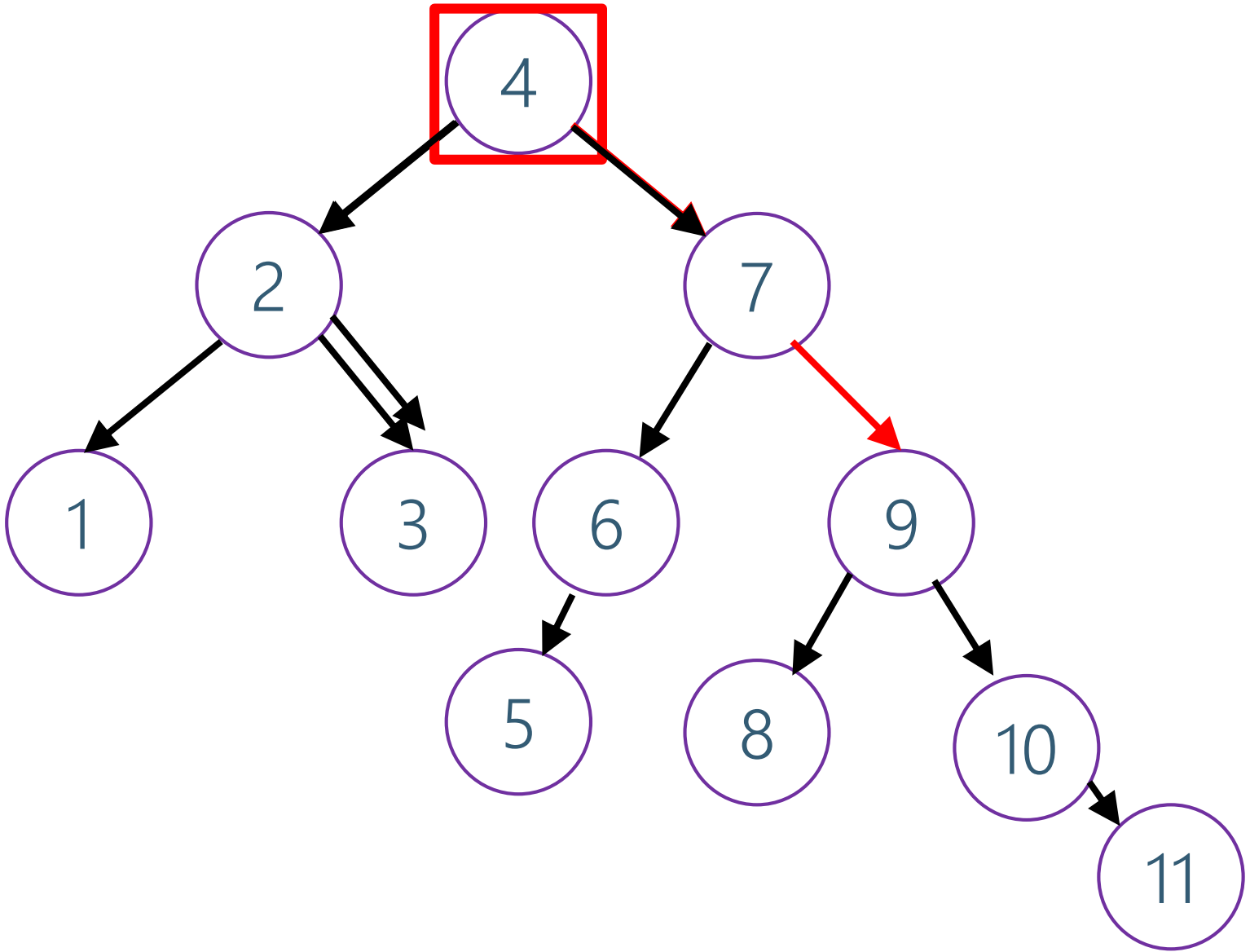
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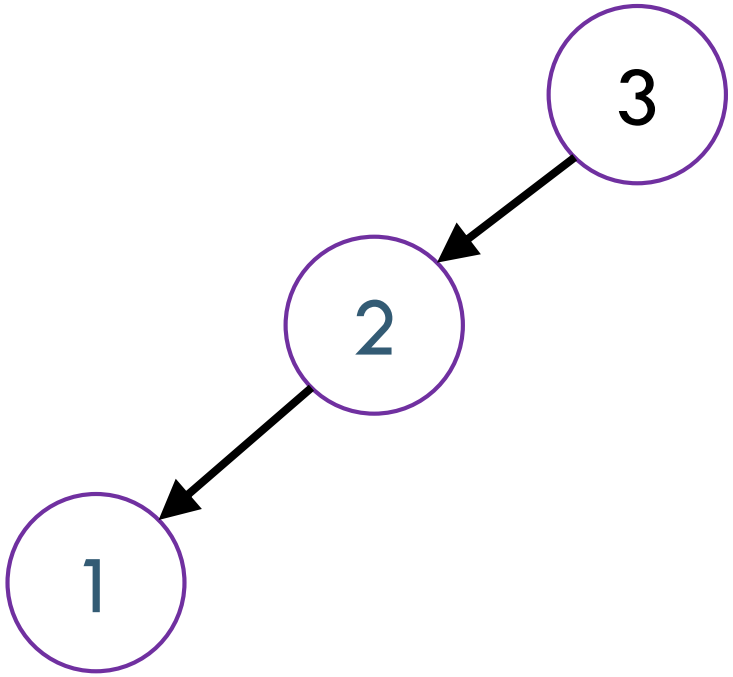
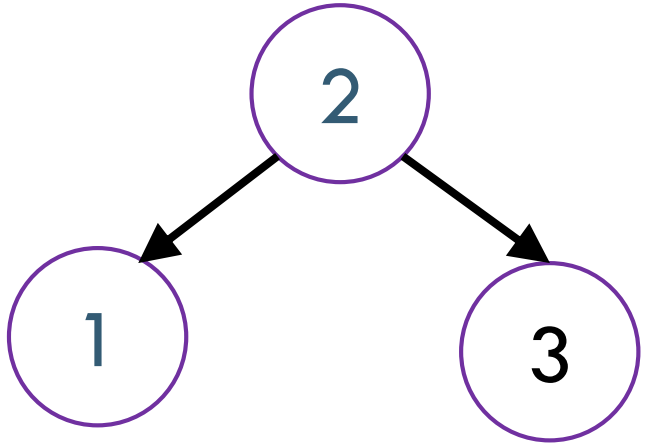
Left subtree
Right subtree whose height differs by at most 1
Somewhat
~~Perfectly~~ balanced...
...As all things should be

Right rotation

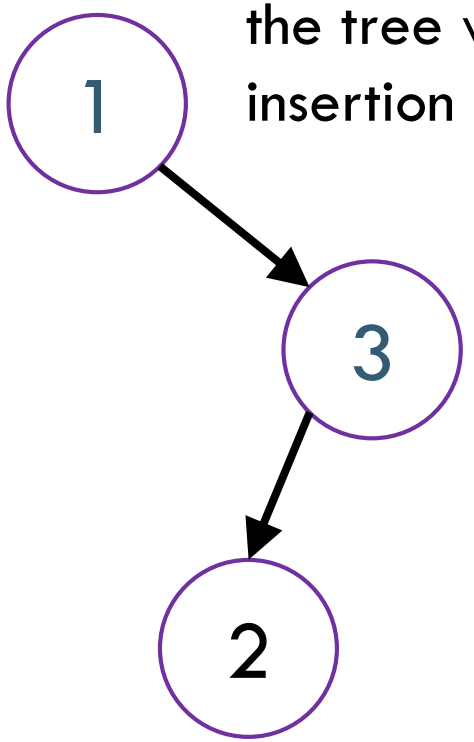




Just like a left roation, just reflected.

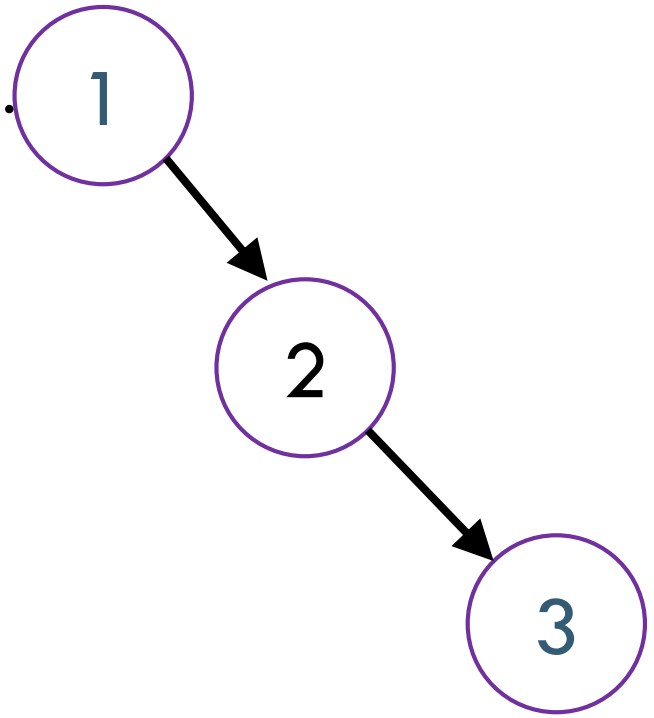
It Gets More Complicated

There's a "kink" in the tree where the insertion happened.

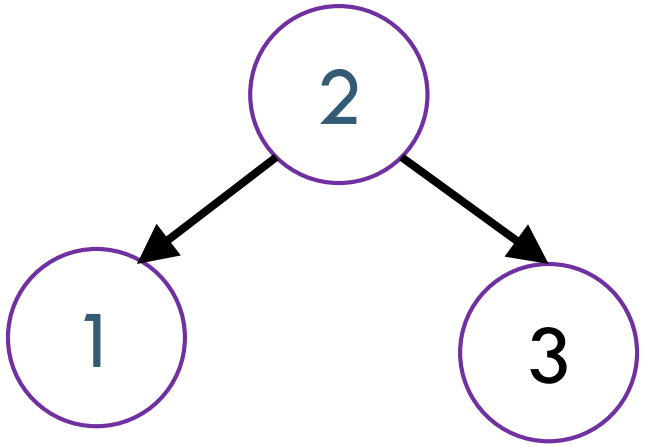


Can't do a left rotation

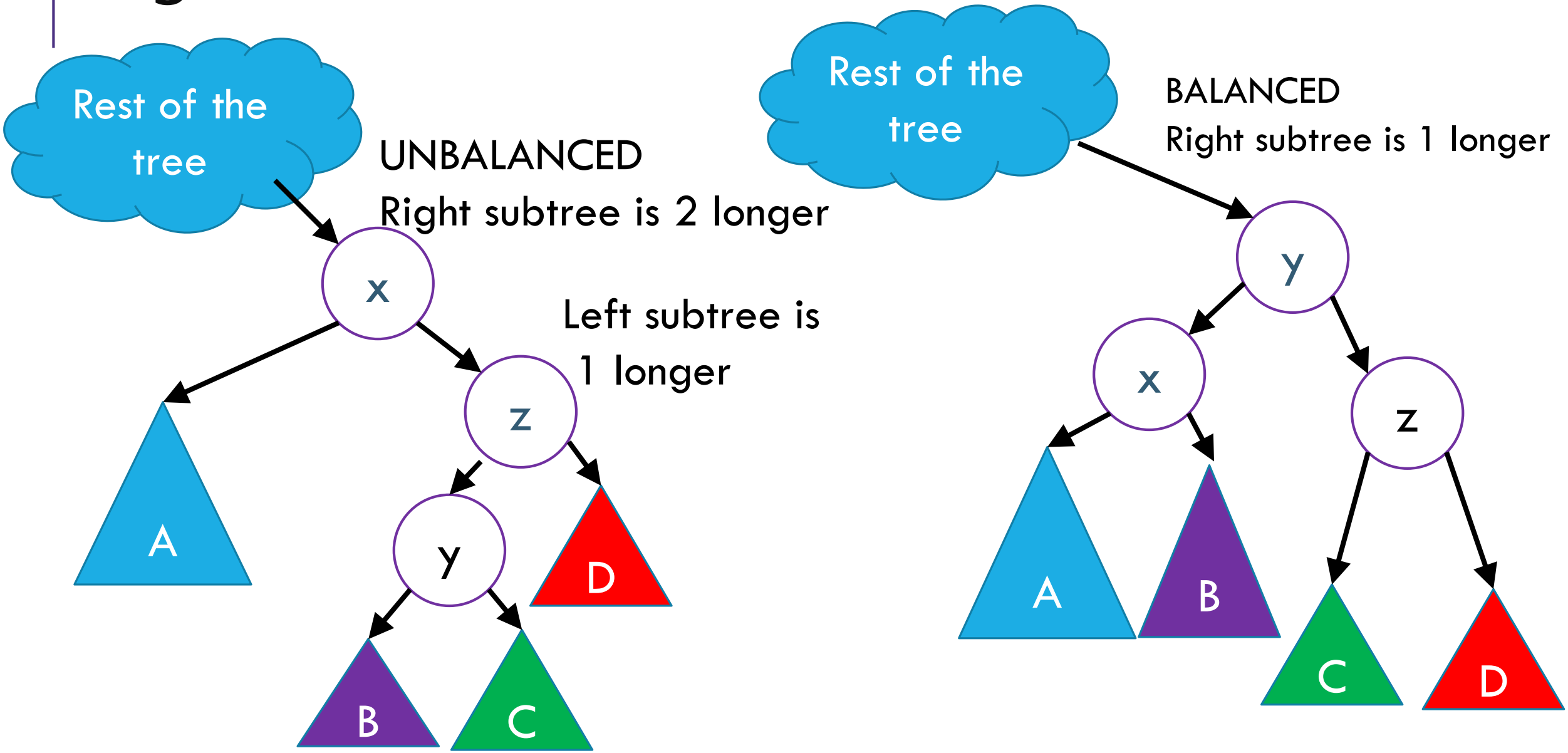
Do a "right" rotation around 3 first.



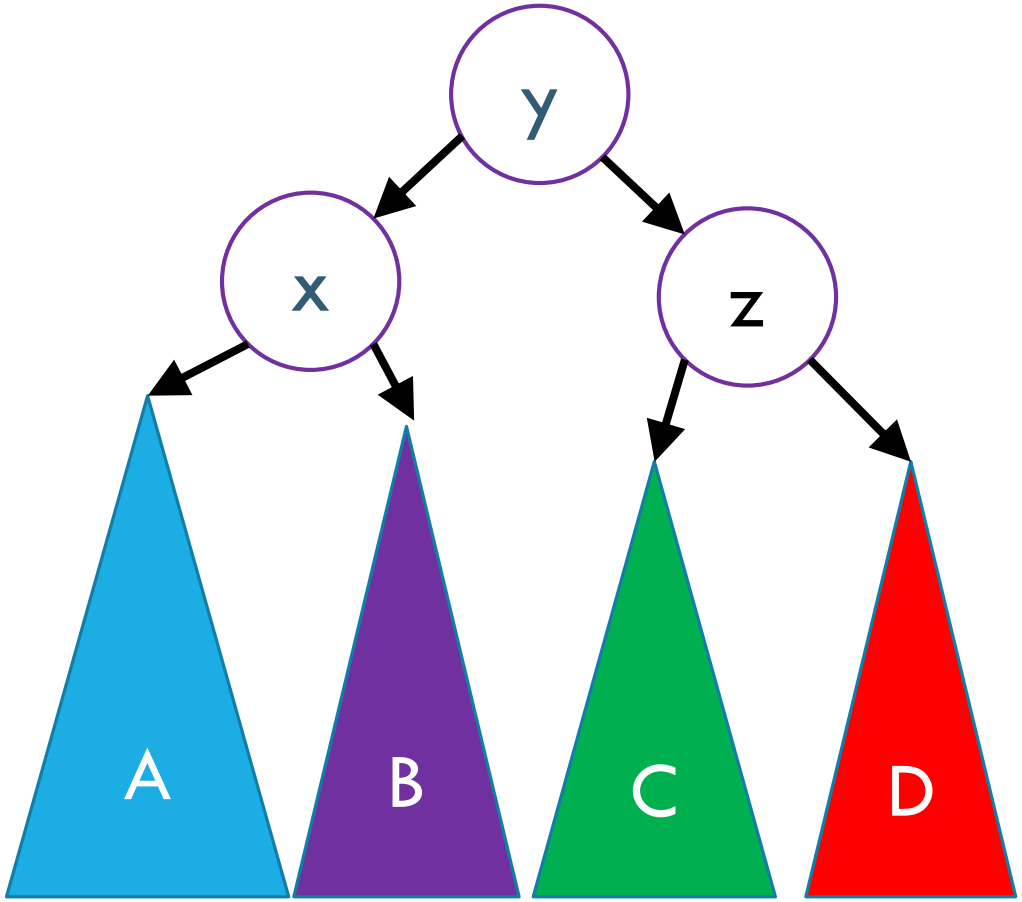
Now do a left rotation.



Right Left Rotation

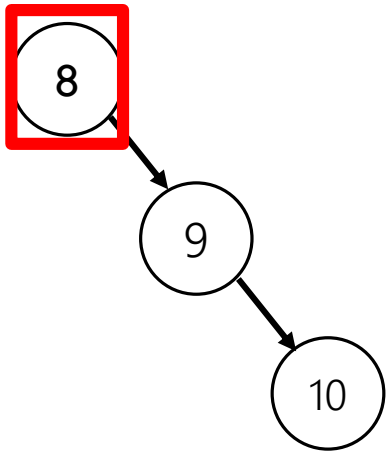


Four Types of Rotations

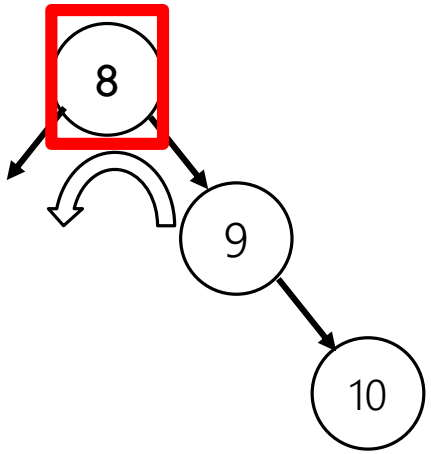


Insert location	Solution
Left subtree of left child (A)	Single right rotation
Right subtree of left child (B)	Double (left-right) rotation
Left subtree of right child (C)	Double (right-left) rotation
Right subtree of right child(D)	Single left rotation

AVL Example: 8,9,10,12,11



AVL Example: 8,9,10,12,11



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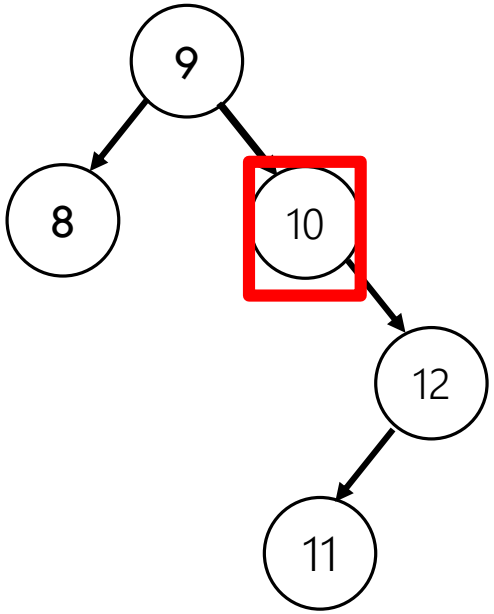

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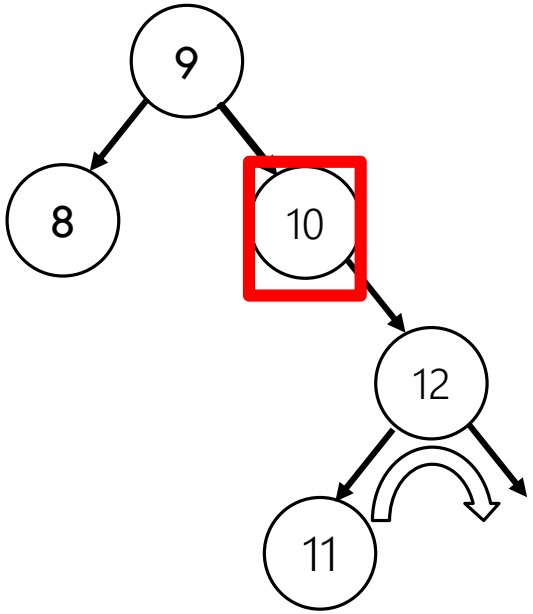

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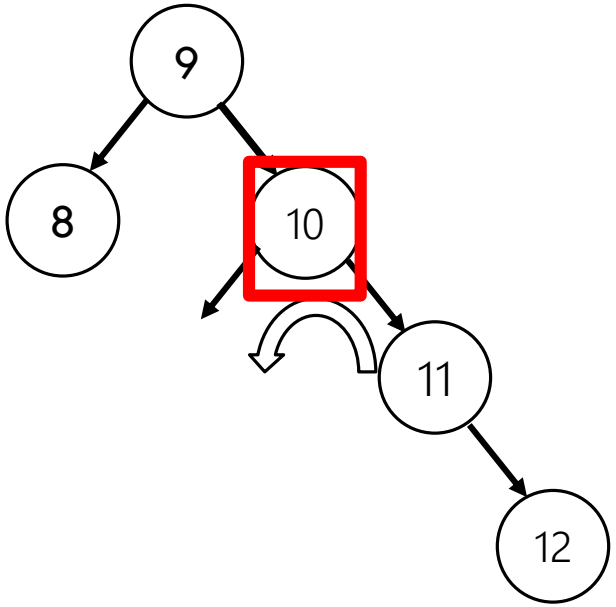


How Long Does Rebalancing Take?

Assume we store in each node the height of its subtree.

How do we find an unbalanced node?

How many rotations might we have to do?

How Long Does Rebalancing Take?

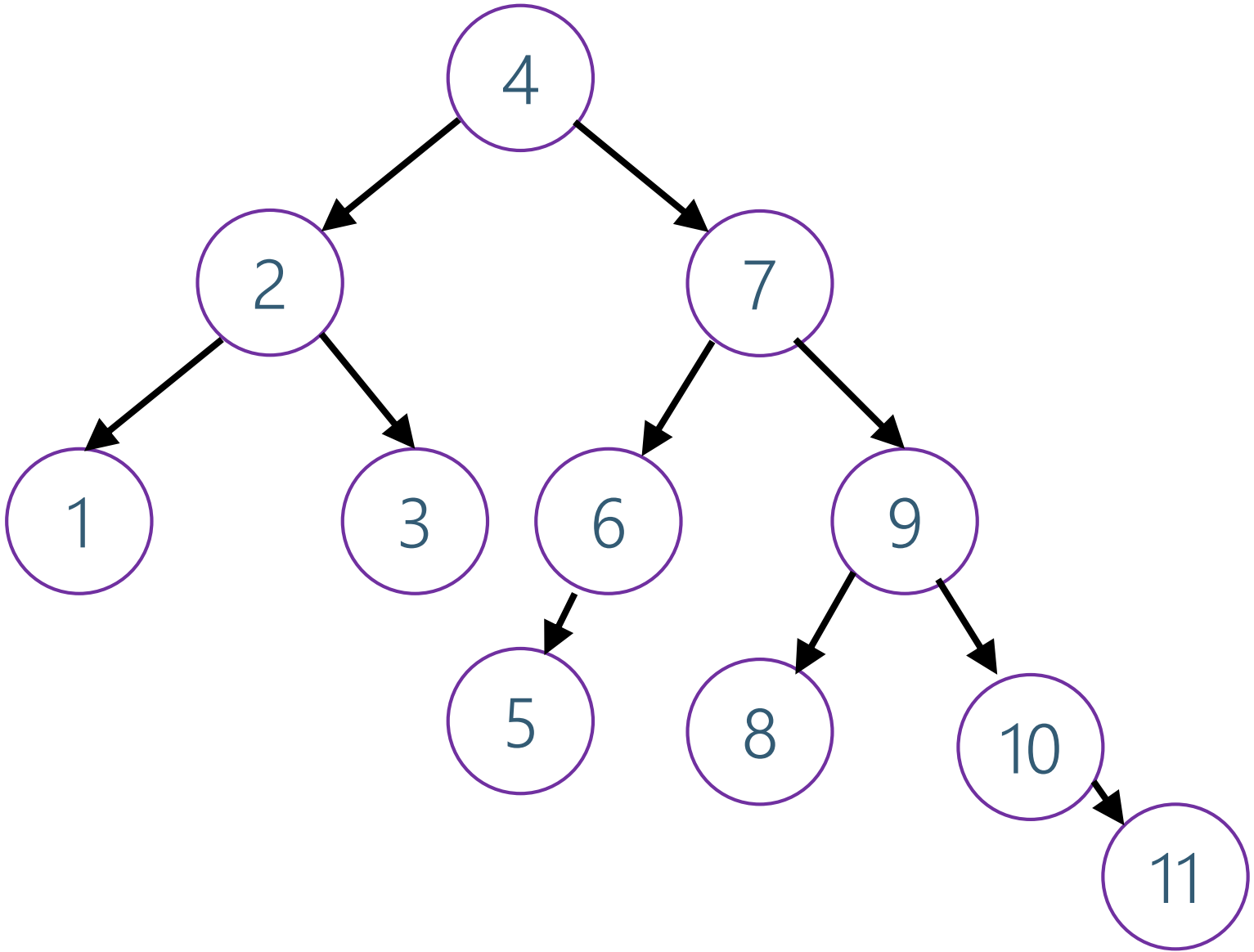
Assume we store in each node the height of its subtree.

How do we find an unbalanced node?

- Just go back up the tree from where we inserted.

How many rotations might we have to do?

- Just a single or double rotation on the lowest unbalanced node.
- A rotation will cause the subtree rooted where the rotation happens to have the same height it had before insertion.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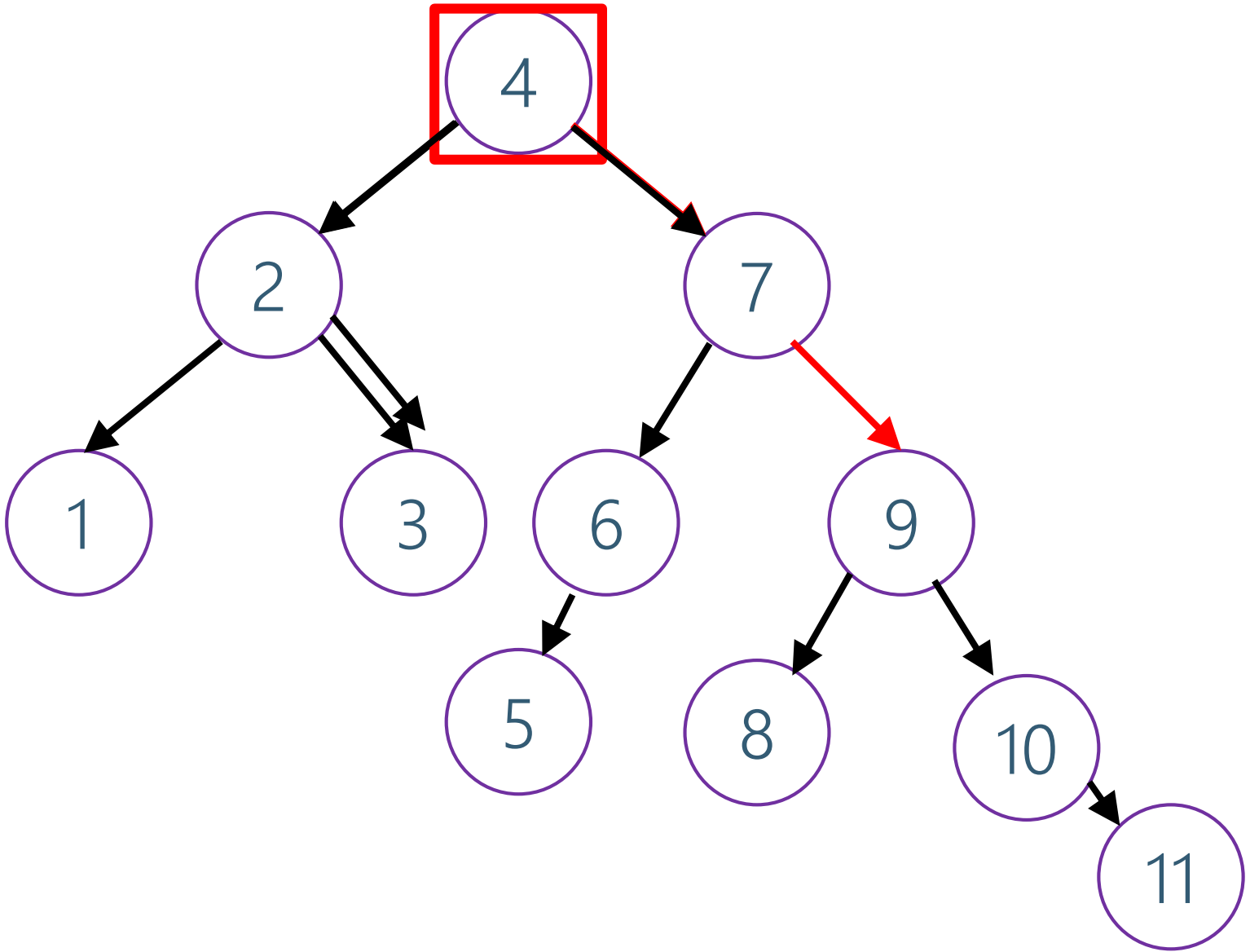
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Deletion

There is a similar set of rotations that will always let you rebalance an AVL tree after a deletion.

The textbook (or Wikipedia) can tell you more.

We won't test you on deletions, beyond the following facts:

- Deletion is similar to insertion.
- It takes $\Theta(\log n)$ time on a dictionary with n elements.
- We won't ask you to perform a deletion.

Avoiding the Degenerate Tree

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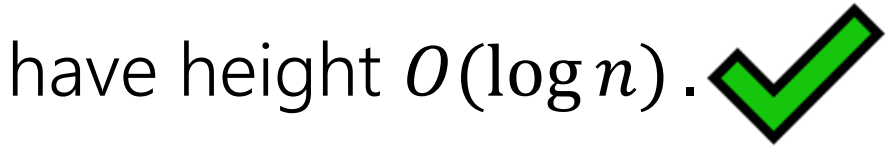
2. Such a tree must have height $O(\log n)$. 

Three Asides

Three related topics, we'll cover what we can:

Lazy deletion

How fast is $\Theta(\log n)$

Traversals

Aside: Lazy Deletion

Deleting things is hard. Let's be lazy.

Instead of actually removing elements from a data structure add a flag for whether the element is "really there."

Then when you call delete, just `find` the item you're looking for, and set the flag.

Should you "really delete" or just lazy delete?

This sounds like a **design decision**! What are the tradeoffs?

Aside: Lazy Deletion

Pros:

Much easier to write delete function.

Delete takes the same time as find (much faster for our data structures that shift everything!)

Cons:

All code now has to check that entries are really there

Running time of all operations is now in terms of elements ever inserted, not number of elements currently in the data structure.

Takeaways:

General technique (not just for dictionaries).

But only use it if you have a good reason.

Aside: How Fast is $\Theta(\log n)$?

If you just looked at a list of common running times

Class	Big O	If you double N…	Example algorithm
constant	O(1)	unchanged	Add to front of linked list
logarithmic	O(log n)	Increases slightly	Binary search
linear	O(n)	doubles	Sequential search
"n log n"	O(nlog n)	Slightly more than doubles	Merge sort
quadratic	$O(n^2)$	quadruples	Nested loops traversing a 2D array

You might think this was a small improvement.

It was a HUGE improvement!

Logarithmic vs. Linear

If you double the size of the input,

- A linear time algorithm takes twice as long.
- A logarithmic time algorithm has a constant **additive** increase to its running time.

To make a logarithmic time algorithm take twice as long, how much do you have to increase n by?

You have to square it $\log(n^2) = 2\log(n)$.

A gigabyte worth of integer keys can fit in an AVL tree of height 60.

It takes a ridiculously large input to make a logarithmic time algorithm go slowly.

Log isn't "that running time between linear and constant" it's "that running time that's barely worse than a constant.

Logarithmic Running Times

This identity is so important, one of my friends made me a cross-stitch of it.

Two lessons:

1. Log running times are REALLY REALLY FAST.
2. $O(\log(n^3))$ is not simplified, it's just $O(\log n)$

Aside: Traversals

What if the heights of subtrees were corrupted.

How could we calculate from scratch?

We could use a “traversal”

- A process that visits every piece of data in a data structure.

```
int height(Node curr){
      if(curr==null) return -1;
      int h = Math.max(height(curr.left),height(curr.right));
      return h+1;
}
```

Three Kinds of Traversals

```
InOrder(Node curr){
    InOrder(curr.left);
    doSomething(curr);
    InOrder(curr.right);
}
```

```
PreOrder(Node curr){
    doSomething(curr);
    PreOrder(curr.left);
    PreOrder(curr.right);
}
```

```
PostOrder(Node curr){
    PostOrder(curr.left);
    PostOrder(curr.right);
    doSomething(curr);
}
```

Traversal Practice

For each of the following scenarios, choose an appropriate traversal:

1. Print out all the keys in an AVL-Dictionary in sorted order.

2. Make a copy of an AVL tree

3. Determine if an AVL tree is balanced (assume height values are not stored)

Traversals

If we have n elements, how long does it take to calculate height?

$\Theta(n)$ time.

The recursion tree (from the tree method) IS the AVL tree!

We do a constant number of operations at each node

In general, traversals take $\Theta(n \cdot f(k))$ time,

where `doSomething()` **takes** $\Theta\big(f(k)\big)$ **time.**

Common question on technical interviews!

Aside: Other Self-Balancing Trees

There are lots of flavors of self-balancing search trees

"Red-black trees" work on a similar principle to AVL trees.

"Splay trees"

- Get $O(\log n)$ amortized bounds for all operations.

"Scapegoat trees"

"Treaps" – a BST and heap in one (!)

Similar tradeoffs to AVL trees.

Wednesday: A completely different idea for a dictionary

Goal: $O(1)$ operations in practice, in exchange for $\Theta(n)$ worst case.