Lecture 8:Tree Method

CSE 373: Data Structures and Algorithms

Administrivia

Project 1 Part 1 due tonight

- Fill out the late day form on the project page if you need to use late days.

Project 1 Part 2 out tonight

- Fix bugs from part 1 to get half of your missed points back.
- Run experiments on your code (connect the programming project to things learned in lecture).

Exercise 1 due Friday

Don’t Panic

We couldn’t apply Master Theorem to this recurrence:

$$T(n) = \begin{cases} T(n-1) + 1 \text{ if } n > 1 \\ 3 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

The books don’t have a nice theorem;

They do have methods for figuring out the big-O.

Unrolling

$$T(n) = \begin{cases} T(n-1) + 1 \text{ if } n > 1 \\ 3 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

Idea: keep plugging the definition of $T()$ into itself.
Until you find the pattern and can hit the base case.

Unrolling

$$T(n) = \begin{cases} T(n-1)+1 \text{ if } n>1 \\ 3 \qquad\qquad \text{otherwise} \end{cases}$$

$T(n) = \ T(n-1) + \ 1$

$[T(n-1-1)+1] \ +1 = T(n-2)+1+1$

$[T(n-2-1)+1] \ +1+1 = T(n-3)+1+1+1$

$[T(n-3-1)+1]+1+1+1 = T(n-4)+1+1+1+1$

$T(n-i)+i$ for any i.

The thing we don't understand is $T()$. We can get rid of it by hitting the base case.

Set i so that $n-i=1$. → $i=n-1$

$T\big(n-(n-1)\big)+(n-1)$

$T(1)+(n-1) = 3+(n-1) = n+2$

$$T(n) = n+2$$

We did it!

For BSTs:

If we’re in the case where everything is balanced, we have a much better dictionary.

But if have that degenerate BST, we’re no better off than with an array or linked list.

For analyzing code:

We didn’t just get the big-Θ, we actually got an exact expression too!

Let’s try another one!

More Practice

Write a recurrence to describe the running time of this function, then find the big-Θ for the running time.

```
public int dumbFindMax(int[] arr, int hi){
     if(hi == 0)
          return arr[0];
     int maxInd = 0;
     for(int i=0; i<hi; i++){
          if(arr[i] > arr[maxInd])
               maxInd=i;
     }
     return Math.max(arr[maxInd], dumbFindMax(arr, hi-1));
}
```

$$T(n) = \begin{cases} T(n-1) + n & \text{if } n \geq 2 \\ 1 & \text{otherwise} \end{cases}$$

You probably had some lower-order terms when you wrote this recurrence.
When we’re solving recurrences we usually ignore lower-order terms in non-recursive work.
They make the algebra a lot more complicated, and don’t affect the big-O.

We’ll tell you to ignore lower-order terms when we want you to.

$$T(n) = \begin{cases} T(n-1) + n & \text{if } n \geq 2 \\ 1 & \text{otherwise} \end{cases}$$

$$T(n-1) + n$$
$$T(n-1-1) + (n-1) + n = \mathrm{T(n-2) + (n-1) + n}$$
$$T(n-3) + (n-2) + (n-1) + n$$
$$T(n-4) + (n-3) + (n-2) + (n-1) + n$$
$$T(n-i) + \sum_{j=0}^{i-1} n - j$$

Plug in i so $n - i$ is 1

$$T(n-(n-1)) + \sum_{j=0}^{n-1-1} n - j =$$

$$T(n) = \begin{cases} T(n-1) + n & \text{if } n \geq 2 \\ 1 & \text{otherwise} \end{cases}$$

$$1 + \sum_{j=0}^{n-2} n - j = 1 + \sum_{j=0}^{n-2} n - \sum_{j=0}^{n-2} j$$

$$= 1 + n(n-1) - \sum_{j=0}^{n-2} j$$

$$= 1 + n(n-1) - \frac{(n-1)(n-2)}{2}$$

$$= n^2 - n - \frac{n^2}{2} + \frac{3n}{2} - 1$$

$$\in \Theta(n^2)$$

$$T(n) = \begin{cases} 3T\left(\frac{n}{4}\right) + n^2 \text{ if } n > 1 \\ 4 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

We can unroll to get the answer here, but it’s really easy to make a small algebra mistake.

If that happens we might not be able to find the pattern

- Or worse find the wrong pattern.

There’s a way to organize our algebra so it’s easier to find the pattern.

Tree Method

Idea: We'll do the same algebra, but let's give ourselves a visual to make the organization easier.

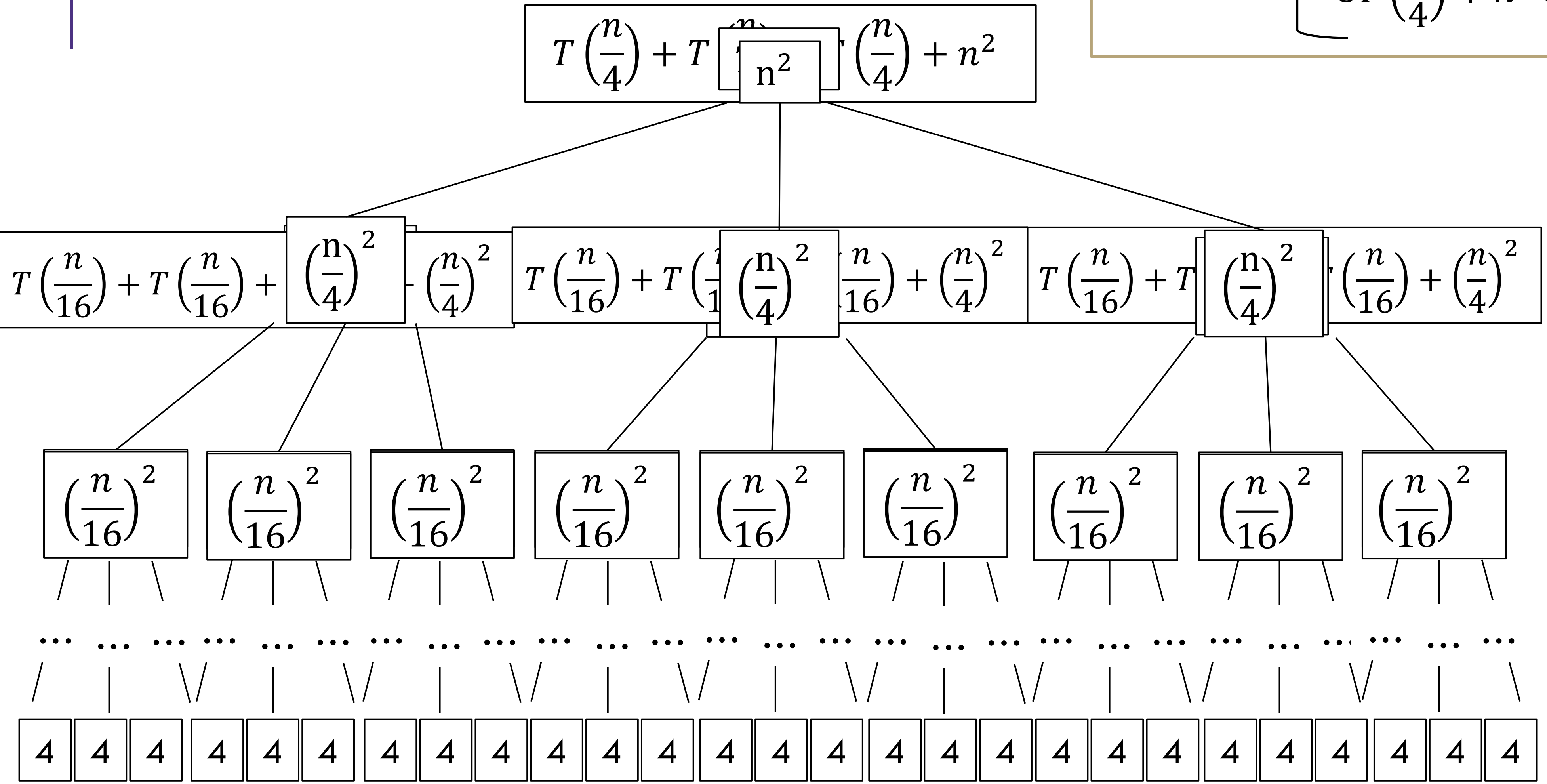
We'll make a **tree.**

Each node of the tree represents one recursive call

- The children of that node are the new recursive calls made

Tree Method Practice

$$T(n) = \begin{cases} 4 \; when \; n \leq 1 \\ 3T\left(\frac{n}{4}\right) + n^2 \; otherwise \end{cases}$$



Answer the following questions:

1. What is the size of the input on level i?
2. What is the work done by each node on the i^{th} recursive level
3. What is the number of nodes at level i?
4. What is the total work done at the i^th recursive level?
5. What value of i does the last level occur?
6. What is the total work across the base case level?

EXAMPLE MODIFIED FROM PROVIDED BY CS 161 – JESSICA SU
HTTPS://WEB.STANFORD.EDU/CLASS/ARCHIVE/CS/CS161/CS161.1168/LECTURE3.PDF

Tree Method Practice

$$T(n) = \begin{cases} 4 \text{ when } n \le 1 \\ 3T\left(\frac{n}{4}\right) + n^2 \text{ otherwise} \end{cases}$$

5 Minutes

1. What is the size of the input on level i? $\frac{n}{4^i}$
2. What is the work done by each node on the i^{th} recursive level? $\left(\frac{n}{4^i}\right)^2$
3. What is the number of nodes at level i? 3^i
4. What is the total work done at the i^{th} recursive level?
   $$3^i\left[\left(\frac{n}{4^i}\right)\right]^2 = \left(\frac{3}{16}\right)^i n^2$$
5. What value of i does the last level occur?
   $$\frac{n}{4^i} = 1 \rightarrow n = 4^i \rightarrow i = \log_4 n$$
6. What is the total work across the base case level? $3^{\log_4 n} \cdot 4$

power of a log
$x^{\log_b y} = y^{\log_b x}$

$4 \cdot n^{\log_4 3}$

Level (i)	Number of Nodes	Work per Node	Work per Level
0	1	n^2	n^2
1	3	$\left(\frac{n}{4}\right)^2$	$\frac{3}{4^2}n^2$
2	9	$\left(\frac{n}{4^2}\right)^2$	$\frac{3^2}{4^4}n^2$
base	$3^{\log_4 n}$	4	$4 * 3^{\log_4 n}$

Combining it all together…

$$T(n) = \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i n^2 + 4n^{\log_4 3}$$

Tree Method Practice

$$T(n) = \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i n^2 + 4n^{\log_4 3}$$

factoring out a constant

$$\sum_{i=a}^{b} cf(i) = c\sum_{i=a}^{b} f(i)$$

$$T(n) = n^2 \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i + 4n^{\log_4 3}$$

Identities are on the webpage.
You don't need to memorize them.

finite geometric series

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}$$

Closed form:

$$T(n) = n^2 \left(\frac{\left(\frac{3}{16}\right)^{\log_4 n} - 1}{\frac{3}{16} - 1}\right) + 4n^{\log_4 3}$$

So what's the big-Θ…

$$T(n) = n^2 \left(-\frac{16}{13}\right)\left(\frac{3}{16}\right)^{\log_4 n} + \left(\frac{16}{13}\right) n^2 + 4n^{\log_4 3}$$

$$T(n) = n^2 \left(-\frac{16}{13}\right)(n)^{\log_4 \frac{3}{16}} + \left(\frac{16}{13}\right) n^2 + 4n^{\log_4 3}$$

$$T(n) \in \Theta(n^2)$$

More Tree Method

$$T(n) = \begin{cases} 6T\left(\frac{n}{2}\right) + 2n \text{ if } n > 8 \\ 3 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

Tree Method Practice

$$T(n) = \begin{cases} 6T\left(\frac{n}{2}\right) + 2n \text{ if } n > 8 \\ 3 \qquad\qquad \text{otherwise} \end{cases}$$

Answer the following questions:

1. What is the size of the input on level i?
2. What is the work done by each node on the i^{th} recursive level
3. What is the number of nodes at level i?
4. What is the total work done at the i^th recursive level?
5. What value of i does the last level occur?
6. What is the total work across the base case level?

Tree Method Practice

5 Minutes

$$T(n) = \begin{cases} 6T\left(\frac{n}{2}\right) + 2n \text{ if } n > 2 \\ 3 \quad \text{otherwise} \end{cases}$$

1. What is the size of the input on level i? $\frac{n}{2^i}$

2. What is the work done by each node on the i^{th} recursive level? $2\frac{n}{2^i}$

3. What is the number of nodes at level i? 6^i

4. What is the total work done at the i^{th}recursive level?

   $$6^i\left[2\frac{n}{2^i}\right] = 2 \cdot 3^i \cdot n$$

5. What value of i does the last level occur?

   $\frac{n}{2^i} = 2 \rightarrow n = 2^{i+1} \rightarrow i = \log_2(n) - 1$

6. What is the total work across the base case level?

   $6^{\log_2(n)\,-1} \cdot 3$

   power of a log

   $x^{\log_b y} = y^{\log_b x}$

   $\frac{3 \cdot 6^{\log_2 n}}{6} = \frac{1}{2} \cdot n^{\log_2 6} = \frac{1}{2} \cdot n^{\log_2 6}$

Level (i)	Number of Nodes	Work per Node	Work per Level
0	1	$2n$	$2n$
1	2	$\frac{2n}{8}$	$\frac{n}{2}$
2	4	$2\left(\frac{n}{8^2}\right)$	$\frac{n}{8}$
base	$2^{\log_8 n - 1}$	3	$\frac{3}{2}n^{1/3}$

Combining it all together…

$$T(n) = \sum_{i=0}^{\log_2(n)\,-2} 2 \cdot 3^i n + \frac{1}{2} n^{\log_2 6}$$

$$T(n) = \sum_{i=0}^{\log_2(n)-2} 2 \cdot 3^i n + \frac{1}{2} n^{\log_2 6}$$

$$= 2n \sum_{i=0}^{\log_2(n)-2} 3^i + \frac{1}{2} n^{\log_2 6}$$

$$= 2n \frac{3^{\log_2(n)-1}}{3-1} + \frac{1}{2} n^{\log_2 6}$$

$$= n \cdot \frac{n^{\log_2(3)}}{3} + \frac{1}{2} n^{\log_2 6}$$

$$= \frac{n^{\log_2(3)+1}}{3} + \frac{1}{2} n^{\log_2 6}$$

$$= \frac{n^{\log_2(6)}}{3} + \frac{1}{2} n^{\log_2 6} = \frac{5}{6} n^{\log_2 6}$$

finite geometric series

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}$$

power of a log

$$x^{\log_b y} = y^{\log_b x}$$

$$1 = \log_2 2$$

$$\log_a b + \log_a c = \log_a(bc)$$