Section 10: Finals review

1. Finding big-O

For each of the following, find a tight, simplified big-O bound using Master Theorem, or state that Master Theorem doesn't apply to that recurrence.

(a) $A(x) = \begin{cases} 1 & \text{if } x = 1 \\ 3A(\frac{x}{3}) + 3x^2 & \text{otherwise} \end{cases}$

(b) $B(n) = \begin{cases} 1 & \text{if } n = 1 \\ 8B(\frac{n}{3}) + 2n & \text{otherwise} \end{cases}$

(c) $C(t) = \begin{cases} 3 & \text{if } t = 1 \\ 4C(\frac{t}{5}) + t^3 + 2 & \text{otherwise} \end{cases}$

(d) $D(m) = \begin{cases} 3 & \text{if } m = 1 \\ 3D(m-1) + 2 & \text{otherwise} \end{cases}$

2. Memory, locality, and dictionaries

(a) In lecture, we discussed three different optimizations for disjoint sets: union-by-rank, path compression, and the array representation.

If we implement disjoint sets using `Node` objects with a "data" and "parent" field and implement the first two optimizations, our `find` and `union` methods will have a nearly-constant average-case runtime.

In that case, why do we bother with the array representation?

(b) Suppose you implement a dictionary with a sorted array and another with an AVL tree. Consider the time needed to iterate over the key-value pairs of a `SortedArrayDictionary` vs an `AvlDictionary`. It turns out that iterating over the `SortedDictionary` is nearly 10 times faster than iterating over the `AvlDictionary`. Think about why that might be.

Now, suppose we take those same dictionaries and try repeatedly calling the `get(...)` method a few hundred thousand times, picking a different random key each time.

Surprisingly, we no longer see such an extreme difference in performance. The `SortedArrayDictionary` is at most only about twice as fast as the `AvlDictionary`.

Why do you suppose that is? Be sure to discuss both (a) why the difference in performance is much less extreme and (b) why `SortedArrayDictionary` is still a little faster.

3. Code Modeling

Consider the following problems:

(a) What is the simplified tight O bound for the runtime of each of the loops below? What is the simplified tight O bound for the runtime of method1?

```
public void method1(int n) {
    for (int i = 10; i < n; i++) {
        System.out.println("373");
    }

    for (int i = n; i >= 1; i /= 2) {
        System.out.println("is");
    }

    for (int i = 0; i < 9999999; i++) {
        System.out.println("cool");
    }

    for (int i = n; i >= n - 4; i-=1) {
        System.out.println("I guess");
    }
}
```

(b) This time, write out the summation of method2 as well as give the simplified tight oh bound for the runtime of this method in terms of n.

```
public void method2(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < i; j++) {
            for (int k = 0; k < i; k++) {
                System.out.println("Hi, my name is " + n);
            }
        }
    }
}
```

(c) What is the simplified tight oh bound for the runtime of each of the loops below? Ignore the context of the if conditions for now. Finally, give the overall simplified runtime of method3.

```
public void method3(int n) {
    if (n < 10000) {
        for (int i = 0; i < n * n * n; i++) {
            System.out.println(":0");
        }
    } else if (n == 10001) {
        for (int i = 0; i < n * n * n * n; i++) {
            System.out.println(":/");
        }
    } else {
        for (int i = 0; i < 2 * n; i++) {
            System.out.println(":D");
        }
    }
}
```

(d) What is the simplified tight oh bound for the runtime of each of the loops above? Ignore the context of the if conditions for now. Finally, give the overall simplified runtime of method4.

```java
public void method4(int n) {
    for (int i = 0; i < n; i++) {
        if (i == 0) {
            for (int j = 0; j < n; j++) {
                System.out.println("lol");
            }
        } else if (n < 1000) {
            for (int j = 0; j < n * n; j++) {
                System.out.println("rofl");
            }
        } else {
            for (int j = 0; j < 10000; j++) {
                System.out.println("haha");
            }
        }
    }
}
```

4. Code Modeling 2: Electric Boogaloo

For each scenario, provide a Θ bound for both the **best** and **worst** case runtimes for the function. Make sure your bounds are as tight/simplified as possible. Give your answers in terms of n, the number of strings in `input`.

You may assume that all strings are of constant length, but can make no other assumptions about the input. You should also assume that all iterators are efficient.

(a) Consider the following snippet of Java code:

```java
public static IDictionary<String, Integer> countStrings(ISet<String> input) {
    IDictionary<String, Integer> dict = new BSTDictionary<String, Integer>();

    for (String curString : input)
    {
        dict.put(curString, 1 + dict.getOrDefault(curString, 0));
    }

    return dict;
}
```

(i) Describe the runtime of this function in terms of n, the number of strings contained in `input`. Provide bounds for both the **best** and **worst** case runtimes. Note that `BSTDictionary` uses a BST to implement a dictionary.

(ii) Do the same thing, but assume now that `dict` is of type `AVLDictionary`. This dictionary is implemented using an AVL Tree, just like in the experiments for HW5.

(iii) Do the same thing, but assume now that `dict` is of type `ArrayDictionary`. This is the data structure you implemented in HW2. You can ignore the time it takes to resize.

(iv) Do the same thing, but assume now that `dict` is of type `ChainedHashDictionary`. This is the data structure you implemented in HW4. You can ignore the time it takes to resize. You can also assume that `.hashCode()` takes constant time.

(b) For the same four `IDictionary` implementations, provide best and worst case runtime bounds for **this** code snippet:

```
public static IDictionary<String, Integer> countStrings(IList<String> input) {
    IDictionary<String, Integer> dict = new BSTDictionary<String, Integer>();

    for (String curString : input)
    {
        dict.put(curString, 1 + dict.getOrDefault(curString, 0));
    }

    return dict;
}
```

Hint 1: You should be able to do this problem very quickly by looking at your answers for the previous snippet.
Hint 2: What's different about this function?

(i)

(ii)

(iii)

(iv)

5. Heaps

Consider the following list of numbers:

`[1, 5, 2, -6, 7, 12, 3, -5, 6, 2, 1, 6, 7, 2, 1, -2]`

(a) Insert these numbers into a min 2-heap (into a min-heap with up to two children per node). Show both the final tree and the array representation.

(b) Insert these numbers into a max 3-heap (a max heap with up to three children per node). Show both the final tree and the array representation.

(c) Insert these numbers into a min 4-heap using Floyd's `buildHeap` algorithm. Show both the final tree and the array representation.

(d) Insert these numbers into a max 2-heap using Floyd's `buildHeap` algorithm. Show both the final tree and the array representation.

(e) Suppose we modify Floyd's `buildHeap` algorithm so we start from the front of the array, iterate forward, and call `percolateDown(...)` on each element. Why is this a bad idea?

6. Sorting

(a) During lecture, we focused on five different sorting algorithms: insertion sort, merge sort, quick sort, selection sort, and heap sort.

For each of these five algorithms, state:

- The best and worst-case runtimes
- Whether the sorting algorithm is stable
- Whether the sorting algorithm is in-place
- Whether the sorting algorithm is a general-purpose one, or if there are any restrictions on how it can or should be used.

(b) Suppose we want to sort an array containing 50 strings. Which of the above four algorithms is the best choice?

(c) Suppose we have an array containing a few hundred elements that is almost entirely sorted, apart from a one or two elements that were swapped with the previous item in the list. Which of the algorithms is the best way of sorting this data?

(d) Suppose we are writing a website named "sort-my-numbers.com" which accepts and sorts numbers uploaded by the user. Keeping in mind that users can be malicious, which of the above algorithms is the best choice?

(e) Suppose we want to sort an array of ints, but also want to minimize the amount of extra memory we want to use as much as possible. Which of the above algorithms is the best choice?

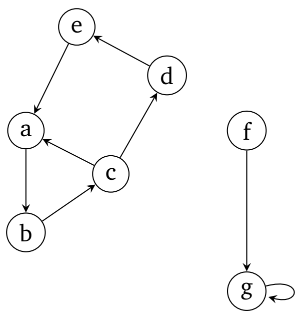
(f) Suppose we have a version of quick sort which selects pivots randomly and creates partitions in the manner described in lecture. Explain how you would build an input array that would cause this version of quick sort to always run in $\mathcal{O}\left(n^2\right)$ time.

Your answer should explain on a high level what your array would look like and what happens when you try running quick sort on it. You do not need to give a specific example of such a array, though you may if you think it will help make your explanation more clear.

(g) How can you modify both versions of quicksort so that they no longer display $\mathcal{O}\left(n^2\right)$ behavior given the same inputs?

7. Graph basics

Consider the following graph:



(a) Draw this graph as an adjacency matrix.

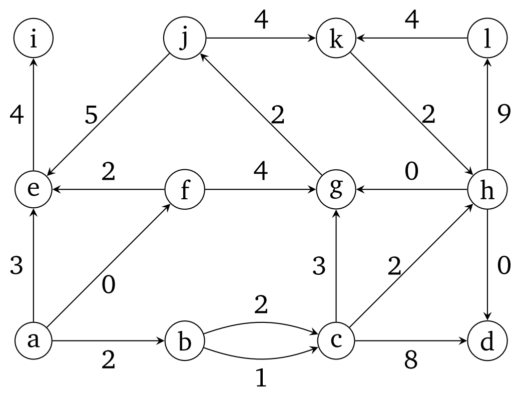
(b) Draw this graph as an adjacency list.

(c) Suppose we run BFS on this list, starting on node a. In what order do we visit each node? Assume we break ties by selecting the node that's alphabetically lower.

(d) Suppose we run DFS on this list, starting on node a. In order do we visit each node? Assume we break ties in the same way as above.

8. Dijkstra's algorithm

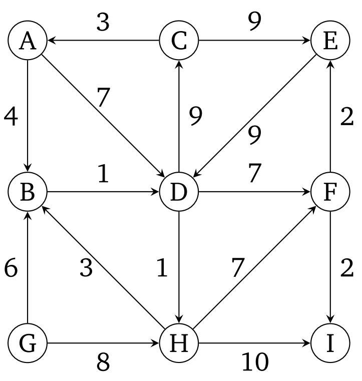
(a) Consider the following graph.



(i) Run Dijkstra's algorithm on the following graph starting on node a. List the final costs of each node, the edges selected by Dijkstra's algorithm, and whether or not Dijkstra's algorithm returned the correct result. In the case of ties, select the node that is the smallest alphabetically.

(ii) In general, is it possible to run Dijkstra's from a single node in a graph in order to recover the shortest path between **any** two pairs of nodes? Why or why not?

(b) Consider this following graph.



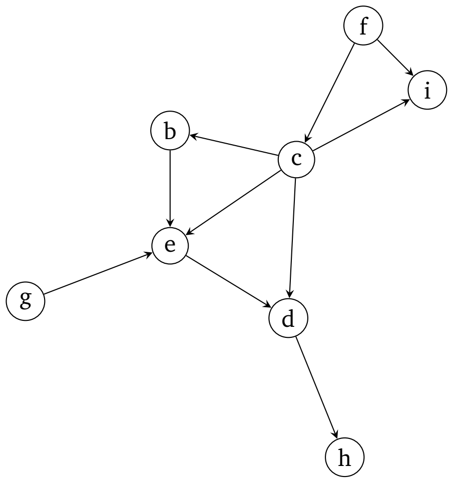
(i) Run Dijkstra's algorithm on the following graph starting on node G. List the final costs of each node, the edges selected by Dijkstra's algorithm, and whether or not Dijkstra's algorithm returned the correct result. In the case of ties, select the node that is the smallest alphabetically.

(ii) When solving this problem, you had a choice when picking the shortest path from G to H. Do ties matter in Dijkstra's? Why or why not? If so, provide another set of shortest paths.

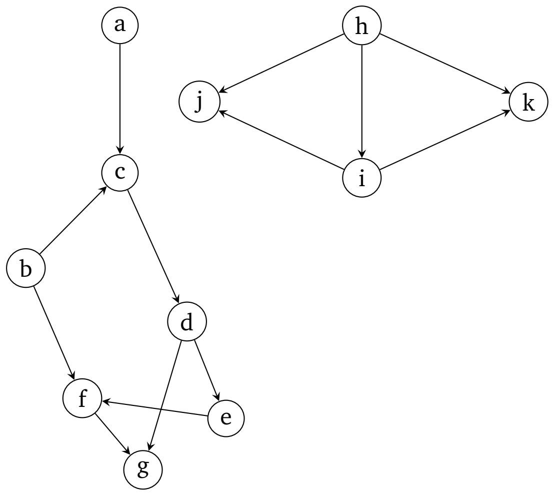
(c) More generally, in any graph, what are some of the reasons why running Dijkstra's might not work correctly?

9. Topological sort

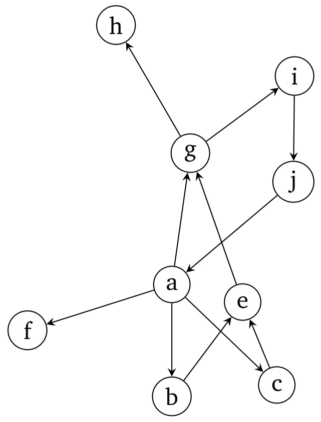
(a) List three different topological orderings of the following graph. If no ordering exists, briefly explain why.



(b) List three different topological orderings of the following graph. If no ordering exists, briefly explain why.

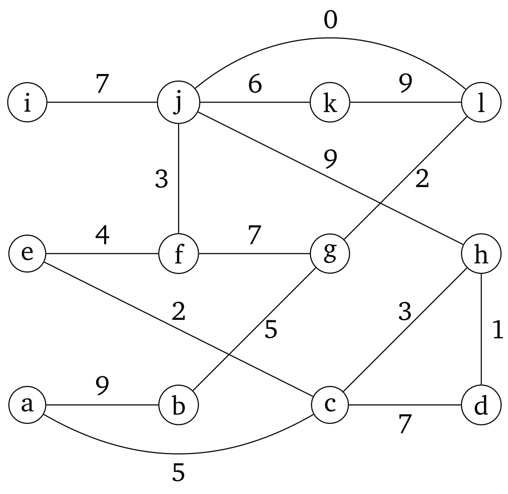


(c) List three different topological orderings of the following graph. If no ordering exists, briefly explain why.



10. Minimum spanning trees

Consider the following graph:



(a) Run Prim's algorithm on the above graph starting on node a to find a minimum spanning tree.

Draw the final MST and the costs per each node. In the case of ties, select the node that is the smallest alphabetically.

(b) Run Kruskal's algorithm on the above graph to find an MST. In the case of ties, select the edge containing the node that is the smallest alphabetically.

Draw the final MST.

(c) Suppose we have the following disjoint set. What happens when we run `union(7, 8)`, which uses union-by-rank optimization? Draw both the new trees as well as the array representation of the disjoint set.

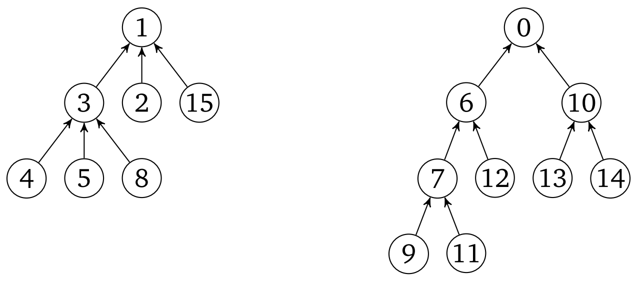


Figure 1: Assume that the rank of the left tree is 4 and the rank of the right tree is 3.

(d) Suppose we have a disjoint set containing 16 elements. Assuming our disjoint set implements the union-by-rank and path-compression algorithm, what is the height of the largest possible internal tree we can construct? Draw what this tree looks like, and what sequence of calls to `union(...)` and `findSet(...)` creates this tree.

11. Graphs and design

Consider the following problems, which we can all model and solve as a graph problem.

For each problem, describe (i) what your vertices and edges are and (ii) a short (2-3 sentence) description of how to solve the problem.

We will also include more detailed pseudocode to make solutions .

Your description does not need to explain how to implement any of the algorithms we discussed in lecture. However, if you *modify* any of the algorithms we discussed, you must discuss what that modification is.

(a) A popular claim is that if you go to any Wikipedia page and keep clicking on the first link, you will eventually end up at the page about "Philosophy". Suppose you are a given some Wikipedia page as a random starting point. How would you write an algorithm to verify this claim for the given starting point?

(b) Suppose you have a bunch of computers networked together (haphazardly) using wires. You want to send a message to every other computer as fast as possible. Unfortunately, some wires are being monitored by some shadowy organization that wants to intercept your messages.

After doing some reconnaissance, you were able to assign each wire a "risk factor" indicating the likelihood that the wire is being monitored. For example, if a wire has a risk factor of zero, it is extremely unlikely to be monitored; if a wire has a risk factor of 10, it is more likely to be monitored. The smallest possible risk factor is 0; there is no largest possible risk factor.

Implement an algorithm that selects wires to send your message such that (a) every computer receives the message and (b) you minimize the total risk factor. The total risk factor is defined as the sum of the risks of all of the wires you use.

(c) Explain how you would implement an algorithm that uses your predictions to find any computers where sending a message would force you to transmit a message over a wire with a risk factor of k or higher.

(d) Suppose you have a graph containing $|V|$ nodes. What is the maximum number of edges you can add while ensuring the graph is always a DAG? Assume you are not permitted to add parallel edges.

(e) Suppose you were walking in a field and unexpectedly ran into an alien. The alien, startled by your presence, dropped a book, ran into their UFO, and flew off.

This book ended up being a dictionary for the alien language – e.g. a book containing a bunch of alien words, with corresponding alien definitions.

You observe that the alien's language appears to be character based. Naturally, the first and burning question you have is what the alphabetical order of these alien characters are.

For example, in English, the character "a" comes before "b". In the alien language, does the character " ☽" come come before or after character "ϱ"? The world must know.

Assuming the dictionary is sorted by the alien character ordering, design an algorithm that prints out all plausible alphabetical orderings of the alien characters.

12. P and NP

Consider the following decision problem:

`ODD-CYCLE`: Given some input graph G, does it contain a cycle of odd length?

You want to show this decision problem is in NP.

(a) What is a convincing certificate a solver could return for this problem? Remember a certificate is a short way of proving that the G really has a cycle of odd length.

(b) Describe, in pseudocode, how you would check that the certificate is a valid one. Remember that a verifier takes in a supposed certificate and checks (in polynomial time) that the certificate is valid.

(c) What is the worst-case runtime of your verifier?

(d) Do you think it's likely this problem also happens to be in P? Why or why not?

13. Short answer

For each of the following questions, answer "true", "false", or "unknown" and justify your response. Your justification should be short – at most 2 or 3 sentences.

(a) If we implement Kruskal's algorithm using a general-purpose sort, Kruskal's algorithm will run in nearly-constant time.

(b) It is possible to reduce all problems in P to some problem in NP.

(c) Dijkstra's algorithm will always return the incorrect result if the graph contains negative-length edges.

(d) Suppose we want to find a MST of some arbitrary graph. If we run Prim's algorithm on any arbitrary node, we will always get back the same result.

(e) $\mathcal{O}\left(n^2 \log(3) + 4\right) = \mathcal{O}\left(4n + n^2\right)$

(f) There is an efficient way of solving the `3-SAT` decision problem.

(g) Iterating over a list using the iterator is always faster than iterating by repeatedly calling the `get(...)` method.

(h) We can always sort some list of length n in $\Theta\left(n\right)$ time.

(i) In a simple graph, if there are $|E|$ edges, the maximum number of possible vertices is $|V| \in \mathcal{O}\left(|E|^2\right)$.

(j) Consider the following question:

SHORT-PATH

input: Graph G with non-negative edge weights, vertices u, v and a number k

output: YES if there is a u, v path of length at most k and no otherwise.

Is SHORT-PATH in NP? In P?

(k) The `.get(...)` method of hash tables has a worst-case runtime of $\mathcal{O}(n)$, where n is the number of key-value pairs.

(l) A hash table implemented using open addressing is likely to have suboptimal performance when $\lambda > 0.5$.

(m) The `peekMin(...)` method in heaps has a worst-case runtime of $\mathcal{O}(\log(n))$.

(n) If a problem is in NP, that means it must take exponential time to solve.

(o) For any given graph, there exists exactly one unique MST.

14. Debugging

Suppose we are in the process of implementing a hash map that uses open addressing and quadratic probing and want to implement the `delete` method.

(a) Consider the following implementation of `delete`. List every bug you can find.

Note: You can assume that the given code compiles. Focus on finding run-time bugs, not compile-time bugs.

```java
public class QuadraticProbingHashTable<K, V> {
    private Pair<K, V>[] array;
    private int size;

    private static class Pair<K, V> {
        public K key;
        public V value;
    }

    // Other methods are omitted, but functional.

    /**
     * Deletes the key-value pair associated with the key, and
     * returns the old value.
     *
     * @throws NoSuchKeyException if the key-value pair does not exist in the method.
     */
    public V delete(K key) {
        int index = key.hashCode() % this.array.length;

        int i = 0;
        while (this.array[index] != null && !this.array[index].key.equals(key)) {
            i += 1;
            index = (index + i * i) % this.array.length;
        }

        if (this.array[index] == null) {
            throw new NoSuchKeyException(”Key-value pair not in dictionary”);
        }

        this.array[index] = null;

        return this.array[index].value;
    }
}
```

(b) Let's suppose the Pair array has the following elements (pretend the array fit on one line):

["lily", V_2]	["castle", V_6]	["resource", V_1]	["hard", V_9]	["bathtub", V_0]
["wage", V_4]	["refund", V_7]	["satisfied", V_6]	["spring", V_8]	["spill", V_3]

And, that the following keys have the following hash codes:

Key	Hash Code
"bathtub"	9744
"resource"	4452
"lily"	7410
"spill"	2269
"wage"	8714
"castle"	2900
"satisfied"	9251
"refund"	8105
"spring"	6494
"hard"	9821

What happens when we call delete with the following inputs? Be sure write out the resultant array, and to do these method calls *in order*. (**Note**: If a call results in an infinite loop or an error, explain what happened, but don't change the array contents for the next question.)

(i) `delete("lily")`

(ii) `delete("spring")`

(iii) `delete("castle")`

(iv) `delete("bananas")`

(v) `delete(null)`

(c) List four different test cases you would write to test this method. For each test case, be sure to either describe or draw out what the table's internal fields look like, as well as the expected outcome (assuming the delete method was implemented correctly). **Hint**: You may use the inputs previously given to help you identify tests, but it's up to you to describe what kind of input they are testing generally.

Identities

Splitting a sum

$$\sum_{i=a}^{b}(x+y)=\sum_{i=a}^{b}x+\sum_{i=a}^{b}y$$

Factoring out a constant

$$\sum_{i=a}^{b}cf(i)=c\sum_{i=a}^{b}f(i)$$

Change-of-base identity

$$\log_a(n)=\frac{\log_b(n)}{\log_a(b)}$$

Gauss's identity

$$\sum_{i=0}^{n-1}i=0+1+\ldots+n-1=\frac{n(n-1)}{2}$$

Finite geometric series

$$\sum_{i=0}^{n-1}r^i=\frac{r^n-1}{r-1}$$

Master theorem

Given a recurrence of the form:

$$T(n)=\begin{cases}d & \text{if } n=1\\ aT\left(\frac{n}{b}\right)+n^c & \text{otherwise}\end{cases}$$

We know that:

- If $\log_b(a)<c$ then $T(n)\in\Theta\left(n^c\right)$
- If $\log_b(a)=c$ then $T(n)\in\Theta\left(n^c\log(n)\right)$
- If $\log_b(a)>c$ then $T(n)\in\Theta\left(n^{\log_b(a)}\right)$