Section 03: Solutions

Review Problems

1. Code Analysis

For each of the following code blocks, what is the worst-case runtime? Give a big-Θ bound.

(a)
```
public IList<String> repeat(DoubleLinkedList<String> list, int n) {
    IList<String> result = new DoubleLinkedList<String>();
    for(String str : list) {
        for(int i = 0; i < n; i++) {
            result.add(str);
        }
    }
    return result;
}
```

Solution:

The runtime is $\Theta(nm)$, where m is the length of the input list and n is equal to the int `n` parameter.

One thing to note here is that unlike many of the methods we've analyzed before, we can't quite describe the runtime of this algorithm using just a single variable: we need two, one for each loop.

(b)
```
public void foo(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 5; j < i; j++) {
            System.out.println("Hello!");
        }

        for (int j = i; j >= 0; j -= 2) {
            System.out.println("Hello!");
        }
    }
}
```

Solution:

The inner loop executes about $i - 5 + i/2$ operations per loop. So we execute about

$$\sum_{i=0}^{n-1} i - 5 + i/2 = \frac{3}{2}\sum_{i=0}^{n-1} i - \sum_{i=0}^{n-1} 5 = \frac{3}{2} * \frac{(0+n-1)*n}{2} - 5n = \frac{3n(n-1)}{4} - 5n$$

which means the runtime is $\Theta\left(n^2\right)$.

(c)
```
public int num(int n){
    if (n < 10) {
        return n;
    } else if (n < 1000) {
        return num(n - 2);
    } else {
        return num(n / 2);
    }
}
```

Solution:

The answer is $\Theta(\log(n))$.

One thing to note is that the second case effectively has no impact on the runtime. That second case occurs only for $n < 1000$ – when discussing asymptotic analysis, we only care what happens with the runtime as n grows large.

(d)
```
public int foo(int n) {
    if (n <= 0) {
        return 3;
    }
    int x = foo(n - 1);
    System.out.println("hello");
    x += foo(n - 1);
    return x;
}
```

Solution:

The answer is $\Theta(2^n)$.

In order to determine that this is exponential, let's start by considering the following recurrence:

$$T(n) = \begin{cases} 1 & \text{If } n = 0 \\ 2T(n-1) + 1 & \text{Otherwise} \end{cases}$$

While we could unfold this to get an exact closed form, we can approximate the final asymptotic behavior by taking a step back and thinking on a higher level what this is doing.

Basically, what happens is we take the work done by $T(n-1)$ and multiply it by 2. If we ignore the $+1$ constant work done in the recursive case, the net effect is that we multiply 2 approximately n times. This simplifies to 2^n.

2. Binary Search Trees

(a) Write a method `validate` to validate a BST. Although the basic algorithm can be converted to any data structure and work in any format, if it helps, you may write this method for the `IntTree` class:

```
public class IntTree {
    private IntTreeNode overallRoot;

    // constructors and other methods omitted for clarity

    private class IntTreeNode {
        public int data;
        public IntTreeNode left;
        public IntTreeNode right;

        // constructors omitted for clarity
    }
}
```

Solution:

```
public boolean validate() {
    return validate(overallRoot, Integer.MIN_VALUE, Integer.MAX_VALUE);
}

private boolean validate(IntTreeNode root, int min, int max) {
    if (root == null) {
        return true;
    } else if (root.data > max || root.data < min) {
        return false;
    } else {
        return validate(root.left, min, root.data - 1) &&
            validate (root.right, root.data + 1, max);
    }
}
```

Section Problems

3. Modeling recursive functions

(a) Consider the following method.

```
public static int f(int n) {
    if (n == 0) {
        return 0;
    }

    int result = 0;
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < i; j++) {
            result++;
        }
    }
    return 5 * f(n / 2) + 3 * result + 2 * f(n / 2);
}
```

(i) Find a recurrence $T(n)$ modeling the worst-case runtime of `f(n)`. **Hint:** You may need to use Gauss's summation identity (see the last page).

Solution:

$$T(n) = \begin{cases} 1 & \text{When } n = 0 \\ \frac{n(n-1)}{2} + 2T(n/2) & \text{Otherwise} \end{cases}$$

The runtime for the two loops is $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1 = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$.

(ii) Find a recurrence $W(n)$ modeling the *returned integer output* of `f(n)`.

Solution:

$$W(n) = \begin{cases} 0 & \text{When } n = 0 \\ \frac{3n(n-1)}{2} + 7W(n/2) & \text{Otherwise} \end{cases}$$

After the loop, `result` is $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1 = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$. And we use `3 * result`.

(b) Consider the following method.

```
public static int g(n) {
    if (n <= 1) {
        return 1000;
    }
    if (g(n / 3) > 5) {
        for (int i = 0; i < n; i++) {
            System.out.println("Hello");
        }
        return 5 * g(n / 3);
    } else {
        for (int i = 0; i < n * n; i++) {
            System.out.println("World");
        }
        return 4 * g(n / 3);
    }
}
```

(i) Find a recurrence $S(n)$ modeling the worst-case runtime of `g(n)`.

Solution:

$$S(n) = \begin{cases} 1 & \text{When } n \leq 1 \\ 2S(n/3) + n & \text{Otherwise} \end{cases}$$

Important: note that the if statement contains a recursive call that must be evaluated for $n > 1$.

(ii) Find a recurrence $X(n)$ modeling the *returned integer output* of `g(n)`.

Solution:

$$X(n) = \begin{cases} 1000 & \text{When } n \leq 1 \\ 5T(n/3) & \text{Otherwise} \end{cases}$$

(iii) Find a recurrence $P(n)$ modeling the *printed output* of `g(n)`.

Solution:

$$P(n) = 2P(n/3) + n$$

(c) Consider the following set of recursive methods.

```
public int test(int n) {
    IDictionary<Integer, Integer> dict = new AvlDictionary<>();
    populate(n, dict);
    int counter = 0;
    for (int i = 0; i < n; i++) {
        counter += dict.get(i);
    }
    return counter;
}

private void populate(int k, IDictionary<Integer, Integer> dict) {
    if (k == 0) {
        dict.put(0, k);
    } else {
        for (int i = 0; i < k; i++) {
            dict.put(i, i);
        }
        populate(k / 2, dict);
    }
}
```

(i) Write a mathematical function representing the *worst-case runtime* of `test`.

You should write two functions, one for the runtime of `test` and one for the runtime of `populate`.

Solution:

The runtime of the `populate` method is:

$$P(k) = \begin{cases} \log(N) & \text{When } k = 0 \\ k\log(N) + P(k/2) & \text{Otherwise} \end{cases}$$

Here, N is the maximum possible value of n.

The runtime of the `test` method is then $R(n) = P(n) + n\log(n)$.

4. Master Theorem

For each of the recurrences below, use the Master Theorem to find the big-Θ of the closed form or explain why Master Theorem doesn't apply. (See the last page for the definition of Master Theorem.)

(a) $T(n) = \begin{cases} 18 & \text{if } n \leq 5 \\ 3T(n/4) + n^2 & \text{otherwise} \end{cases}$ **Solution:**

This is the correct form for Master Theorem. We want to compare $\log_4(3)$ to 2. $\log_4(3)$ is between 0 and 1 (since $4^0 < 3 < 4^1$), so $\log_4(3) < 2$. We're thus in the case where the answer is $\Theta\left(n^2\right)$.

(b) $T(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ 9T(n/3) + n^2 & \text{otherwise} \end{cases}$ **Solution:**

We want to compare $\log_3(9)$ to 2. $\log_3(9)$ is 2 (since $3^2 = 9$) since the two things we're comparing are equal, we have $\Theta\left(n^2 \log n\right)$ as our final answer.

(c) $T(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ \log(n)T(n/2) + n & \text{otherwise} \end{cases}$ **Solution:**

This recurrence is not in the right form to use the Master Theorem. The coefficient of $T(n/2)$ needs to be a constant, not a function of n.

(d) $T(n) = \begin{cases} 1 & \text{if } n \leq 19 \\ 4T(n/3) + n & \text{otherwise} \end{cases}$ **Solution:**

We want to compare $\log_3(4)$ to 1. $\log_3(4)$ is between 1 and 2 (since $3^1 < 4 < 3^2$), so $\log_3(4).1$. In this case, the Master Theorem says our result is $\Theta\left(n^{\log_3(4)}\right)$

(e) $T(n) = \begin{cases} 5 & \text{if } n \leq 24 \\ 2T(n-2) + 5n^3 & \text{otherwise} \end{cases}$ **Solution:**

This recurrence is not in the right form to use Master Theorem. It's only applicable if we are dividing the input size, not if we're subtracting from it.

5. Tree method walk-through

Consider the following recurrence: $A(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ 3A(n/6) + n & \text{otherwise} \end{cases}$

We want to find an *exact* closed form of this equation by using the tree method. Suppose we draw out the total work done by this method as a tree, as discussed in lecture. Let n be the initial input to A.

(a) What is the size of the input at level i (as in class, call the root level 0)?

Solution:

We divide by 6 at each level, so the input size is $n/6^i$.

(b) What is the number of nodes at level i?

Note: let $i = 0$ indicate the level corresponding to the root node. So, when $i = 0$, your expression should be equal to 1.

Solution:

Each (non-base-case) node produces 3 more nodes, so at level i we have 3^i nodes.

(c) What is the total work at the i^{th} **recursive** level?

Solution:

Combining our last two parts: $3^i \cdot \frac{n}{6^i} = \frac{n}{2^i}$

(d) What is the last level of the tree?

Solution:

We hit our base case when $n/6^i = 1$, which is at level $i = \log_6(n)$.

(e) What is the work done in the base case?

Solution:

From previous parts, there are $3^{\log_6(n)}$ nodes at that level, from the recurrence each does 1 unit of work, so we get $1 \cdot 3^{\log_6(n)}$ work.

(f) Combine your answers from previous parts to get an expression for the total work.

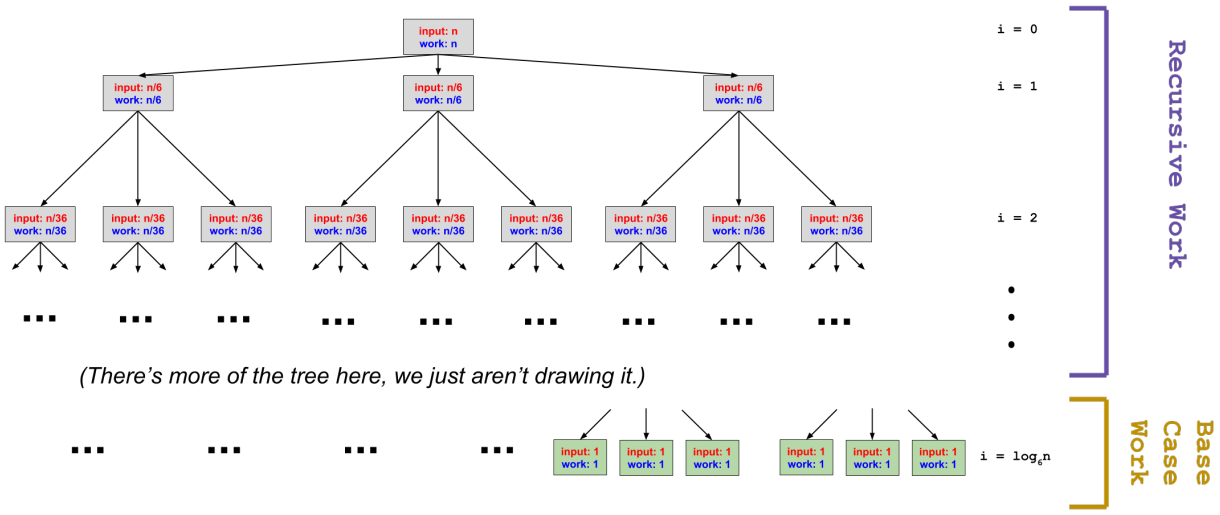
Solution:

We know the expression for the work done at each recursive level and the base case level. Combining these we have:

$$\sum_{i=0}^{\log_6(n)-1} \frac{n}{2^i} + 3^{\log_6(n)}$$

Side note: Yes, it says $\log_6(n) - 1$ and not $\log_6(n)$. This is because the last level ($\log_6(n)$) is the base case level and has a different formula.

Here's a drawing of of the tree. Note that we distinguish the **input** to a node (red) and the **work** done by a node (blue) since these can be different when we're applying the tree method.



(g) Simplify to a closed form.

Note: you do not need to simplify your answer, once you found the closed form. Hint: You should use the finite geometric series identity somewhere while finding a closed form.

Solution:

We combine all the pieces and simplify:

$$\begin{aligned} A(n) &= \sum_{i=0}^{\log_6(n)-1} \frac{n}{2^i} + 3^{\log_6(n)} \\ &= n \sum_{i=0}^{\log_6(n)-1} \left(\frac{1}{2}\right)^i + 3^{\log_6(n)} \end{aligned}$$

We'll apply two identities: to the summation we apply the finite geometric series identity; and to the base-case work, we apply the power of a log identity. We get:

$$A(n) = n \cdot \frac{\left(\frac{1}{2}\right)^{\log_6(n)} - 1}{\frac{1}{2} - 1} + n^{\log_6(3)}$$

You don't have to simplify further, but if you were to simplify, you would get:

$$\begin{aligned} A(n) &= n \cdot \frac{\left(\frac{1}{2}\right)^{\log_6(n)} - 1}{1/2 - 1} + n^{\log_6(3)} \\ &= -2n\left(n^{\log_6(1/2)} - 1\right) + n^{\log_6(3)} \\ &= -2n\left(n^{\log_6(3/6)} - 1\right) + n^{\log_6(3)} \\ &= -2n\left(n^{\log_6(3)-\log_6(6)} - 1\right) + n^{\log_6(3)} \\ &= -2n\left(\frac{n^{\log_6(3)}}{n} - 1\right) + n^{\log_6(3)} \\ &= -2n^{\log_6(3)} + 2n + n^{\log_6(3)} \\ &= 2n - n^{\log_6(3)} \end{aligned}$$

(h) Use the master theorem to find a big-Θ bound of $A(n)$. (See the last page for the definition of Master Theorem.)

Solution:

We check that $\log_6(3) < 1$, so we get $A(n) \in \Theta(n)$, which is consistent with our answer in the last part.

6. More tree method recurrences

For each of the following recurrences, find their closed form using the tree method. Then, check your answer using the master method (if applicable). It may be a useful guide to use the steps from section 4 of this handout to help you with all the parts of solving a recurrence problem fully.

(a) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 3 & \text{otherwise} \end{cases}$

Solution:

Given this recurrence, we know...

- Input size at level i: $n/2^i$
- Number of nodes on level i: $1^i = 1$
- Total work done per level: $1 \cdot 3 = 3$
- Last level of the tree: $\log_2(n)$
- Total work done in base case: $1 \cdot 1^{\log_3(n)} = 1$

So we get the expression:

$$\left(\sum_{i=0}^{\log_2(n)-1} 3 \right) + 1$$

Using the summation of a constant identity, we get:

$$3\log_2(n) + 1$$

We can apply the master thereom here. Note that $\log_b(a) = \log_2(1) = 0 = c$, which means that $T(n) \in \Theta\left(n^c \log(n)\right)$ which is $T(n) \in \Theta\left(n^0 \log(n)\right)$ which further simplifies to $T(n) \in \Theta\left(\log(n)\right)$.

This agrees with our simplified form.

(b) $S(q) = \begin{cases} 1 & \text{if } q = 1 \\ 2S(q-1)+1 & \text{otherwise} \end{cases}$

Solution:

Given this recurrence, we know...

- Size of input at level i is $q - i$
- Number of nodes on level i: 2^i
- Total work done at (recursive) level i: $2^i \cdot 1$
- Last level of the tree: $q - 1$
- Total work done in base case: $1 \cdot 2^{q-1}$

Note that these expressions look a little different from the ones we've seen up above. This is because we aren't *dividing* our terms by some constant factor – instead, we're *subtracting* them.

So we get the expression:

$$\left(\sum_{i=0}^{q-1-1} 2^i \right) + 2^{q-1}$$

We apply the finite geometric series to get:

$$\frac{2^{q-1} - 1}{2 - 1} + 2^{q-1}$$

If we wanted to simplify, we'd get:

$$2^q - 1$$

Note that we may NOT apply the master theorem here – our original recurrence doesn't match the form given in the theorem.

(c) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 8T(n/2) + 4n^2 & \text{otherwise} \end{cases}$

Solution:

Given this recurrence, we know...

- Size of input at level i is $n/2^i$
- Number of nodes on level i: 8^i
- Total work done per (recursive) level: $8^i \cdot 4\left(\frac{n}{2^i}\right)^2 = 8^i \cdot 4 \cdot \frac{n^2}{4^i}$
- Last level of the tree: When $n/2^i = 1$, i.e. $\log_2(n)$
- Total work done in base case: $1 \cdot 8^{\log_2(n)}$

So we get the expression:

$$\left(\sum_{i=0}^{\log_2(n)-1} 8^i \cdot 4 \cdot \frac{n^2}{4^i}\right) + 8^{\log_2(n)}$$

We can simplify by pulling the $4n^2$ out of the summation:

$$4n^2\left(\sum_{i=0}^{\log_2(n)-1} \frac{8^i}{4^i}\right) + 8^{\log_2(n)}$$

This further simplifies to:

$$4n^2\left(\sum_{i=0}^{\log_2(n)-1} 2^i\right) + 8^{\log_2(n)}$$

After applying the finite geometric series identity, we get:

$$4n^2 \cdot \frac{2^{\log_2(n)} - 1}{2 - 1} + 8^{\log_2(n)}$$

This is a closed form so we could stop, but if we want a tidy solution, we can continue simplifying:

$$\begin{aligned} T(n) &= 4n^2 \cdot \frac{2^{\log_2(n)} - 1}{2 - 1} + 8^{\log_2 2(n)} \\ &= 4n^2 \cdot \left(2^{\log_2(n)} - 1\right) + 8^{\log_2(n)} \\ &= 4n^2 \cdot \left(n^{\log_2(2)} - 1\right) + n^{\log_2(8)} \\ &= 4n^2 \cdot (n-1) + n^3 \\ &= 5n^3 - 4n^2 \end{aligned}$$

We can apply the master thereom here. Note that $log_b(a) = \log_2(8) = 3 > 2 = c$, which means that $T(n) \in \Theta\left(n^{\log_b(a)}\right)$ which is $T(n) \in \Theta\left(n^{\log_2 8}\right)$ which in turn simplifies to $T(n) \in \Theta\left(n^3\right)$.

This agrees with our simplified form.

Food for Thought

7. TreeMap implemented as a Binary Search Tree

Consider the following method, which is a part of a Binary Search Tree implementation of a TreeMap class.

```
public V find(K key) {
    return find(this.root, key);
}

private V find(Node<K, V> current, K key) {
    if (current == null) {
        return null;
    }
    if (current.key.equals(key)) {
        return current.value;
    }
    if (current.key.compareTo(key) > 0) {
        return find(current.left, key);
    } else {
        return find(current.right, key);
    }
}
```

(a) We want to analyze the runtime of our `find(x)` method in the best possible case and the worst possible case. What does our tree look like in the best possible case? In the worst possible case?

Solution:

In the best possible case, our tree will be completely balanced. In the worst possible case, it will be completely unbalanced, resembling a linked list.

(b) Write a recurrence to represent the worst-case runtime for `find(x)` in terms of n, the number of elements contained within our tree. Then, provide a Θ bound.

Solution:

The recurrence representing the worst-case runtime of `find(x)` is:

$$T_w(n) = \begin{cases} 1 & \text{when } n = 0 \\ 1 + T(n-1) & \text{otherwise} \end{cases}$$

That is, every time we recurse, we are able to eliminate only one node from the span of possibilities we must consider. This is possible in case the tree is absolutely unbalanced (think of a tree that looks like a linked list).

This recurrence is in $\Theta(n)$.

(c) Assuming we have an optimally structured tree, write a recurrence for the runtime of `find(x)` (again in terms of n). Then, provide a Θ bound.

Solution:

The recurrence representing the best-case runtime of `find(x)` is:

$$T_w(n) = \begin{cases} 1 & \text{when } n = 0 \\ 1 + T(n/2) & \text{otherwise} \end{cases}$$

That is, every time we recurse, we are able to eliminate about half of the nodes we must consider.

This recurrence is in $\Theta(\log(n))$.

Challenge Problems

8. Modeling recursive functions

Consider the following recursive function. You may assume that the input will be a multiple of 3.

```
public int test(int n) {
    if (n <= 6) {
        return 2;
    } else {
        int curr = 0;
        for (int i = 0; i < n * n; i++) {
            curr += 1;
        }
        return curr + test(n - 3);
    }
}
```

(a) Write a recurrence modeling the *worst-case runtime* of `test`.

Solution:

$$T(n) = \begin{cases} 1 & \text{When } n \leq 6 \\ n^2 + T(n-3) & \text{Otherwise} \end{cases}$$

(b) Unfold the recurrence into a summation (for $n \geq 6$).

Solution:

$$1 + \sum_{i=3}^{n/3} (3i)^2$$

Modeling this recurrence correctly is slightly challenging because we want to decrease n in increments of 3.

To do this, what we do is set the summation bounds to go up to $n/3$ instead of n, and multiply i on the inside by 3, simulating changing i in those increments.

We then also set the lower summation bound to be 3 instead of 0 or 1. That way, our summation will only consider numbers in the range 9 to n – if we set $i = 2$ or lower, our summation would double-cound $n \leq 6$, which should be handled by the base case.

Note: our model only works if n is a multiple of 3.

(c) Simplify the summation into a closed form (for $n \geq 6$).

Solution:

$$\begin{aligned}
1 + \sum_{i=3}^{n/3} (3i)^2 &= 1 + \sum_{i=0}^{n/3} (3i)^2 - \sum_{i=0}^{2} (3i)^2 && \text{Adjusting summation bounds} \\
&= 1 + 9 \sum_{i=0}^{n/3} i^2 - \sum_{i=0}^{2} (3i)^2 && \text{Pulling out a constant} \\
&= 1 + 9 \sum_{i=0}^{n/3} i^2 - (0 + 9 + 36) && \text{Evaluating the summation} \\
&= 9 \frac{\frac{n}{3}\left(\frac{n}{3} + 1\right)\left(\frac{2n}{3} + 1\right)}{6} - 44 && \text{Sum of squares}
\end{aligned}$$

A "closed form", within the context of this class, is just any expression that does not contain a summation or is recursive. This means we can stop here without needing to further simplify the expression.

That said, if you wanted to continue simplifying, we could:

$$\begin{aligned}
9 \frac{\frac{n}{3}\left(\frac{n}{3} + 1\right)\left(\frac{2n}{3} + 1\right)}{6} - 44 &= \frac{9}{6}\left(\frac{n}{3}\left(\frac{n}{3} + 1\right)\left(\frac{2n}{3} + 1\right)\right) - 44 \\
&= \frac{1}{2}\left(n\left(\frac{n}{3} + 1\right)\left(\frac{2n}{3} + 1\right)\right) - 44 \\
&= \frac{1}{2}\left(n\left(\frac{2}{9}n^2 + n + 1\right)\right) - 44 \\
&= \frac{1}{9}n^3 + \frac{1}{2}n^2 + \frac{1}{2}n - 44
\end{aligned}$$

9. Recurrences

For the following recurrence, use the unfolding method to first convert the recurrence into a summation. Then, find a big-Θ bound on the function in terms of n. Assume all division operations are integer division.

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2T(n/3) + n & \text{otherwise} \end{cases}$$

Solution:

In order to determine what this expression looks like as a summation, it helps to first partially unroll it. When unfolding a recurrence like this it helps to

(a) Distribute the 2 (coefficient of the recursive call) at each step, to avoid having too many parentheses within parentheses.

(b)

$$\begin{aligned} T(n) &= n + 2T\left(\frac{n}{3}\right) \\ &= n + 2\left(2T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) = n + \frac{2n}{3} 2^2 T\left(\frac{n}{3^2}\right) \\ &= n + \frac{2n}{3} + 2^2\left(2T\left(\frac{n}{3^3} + \frac{n}{3^2}\right)\right) = n + \frac{2n}{3} + \frac{2^2 n}{3^2} + 2^3 T\left(\frac{n}{3^3}\right) \\ &= n + \frac{2n}{3} + \frac{2^2 n}{3^2} + 2^3\left(\frac{n}{3^3} + T\left(\frac{n}{3^4}\right)\right) = n + \frac{2n}{3} + \frac{2^2 n}{3^2} + \frac{2^3 n}{3^3} + 2^4 T\left(\frac{n}{3^4}\right) \end{aligned}$$

We can start to see the pattern now: our summation is roughly of the form

$$n + \frac{2n}{3} + \frac{2^2 n}{3^2} + \cdots + \frac{2^{i-1} n}{3^{i-1}} + 2^i\left(\frac{n}{3^i}\right)$$

Choosing $i = \log_3(n)$ puts us in the base case, and we get:

$$T(n) = n + \frac{2n}{3} + \frac{2^2 n}{3^2} + \cdots + \frac{2^{\log_3(n)-1} n}{3^{\log_3(n)-1}} + 2^{\log_3(n)}$$

Notice that unlike with most of our previous recurrences, the term for the base case is not a constant. Because each of our recursive calls makes two other recursive calls, we hit the base case more than once (in fact $2^{\log_3(n)}$ times).

Rewriting to be in summation form:

$$\begin{aligned} T(n) &= \sum_{i=0}^{\log_3(n)-1} \frac{2^i n}{3^i} + 2^{\log_3 n} \\ &= n \sum_{i=0}^{\log_3(n)-1} \left(\frac{2}{3}\right)^i + 2^{\log_3 n} \end{aligned}$$

Let's handle the summation first. We could evaluate the summation exactly, but that would be a bit tedious. Instead, let's show that it evaluates to some constant (which we can supress in the $\Theta()$ at the end). Each term is positive, and the first is $2/3$, so it is at least some constant. On the other hand, if instead of stopping when $i = \log_3(n) - 1$, we kept going to $i = \infty$, we would get an infinite geometric series, whose closed form is

$\frac{1}{1-2/3} = 3$. Since this infinite series is larger than the finite one we care about, our series sums to at most 3. Thus it is between $2/3$ and 3, and definitely must be a constant.

Now let's figure out $2^{\log_3 n}$. Using logarithm identity 4 of the math review, we can change that number to $n^{\log_3(2)}$. So we have that

$$T(n) = \Theta(n) + n^{\log_3(2)}$$

Which of those is the dominating term? $\log_3(2)$ is the number which you raise 3 to to get 2. Since 2 is less than 3, $\log_3(2) < 1$. So the first term dominates and $\Theta(n)$ is our final answer.

Master Theorem

For recurrences in this form, where a, b, c, e are constants:

$$T(n) = \begin{cases} d & \text{if } n \leq \text{some constant} \\ aT(n/b) + e \cdot n^c & \text{otherwise} \end{cases}$$

$$T(n) \text{ is } \begin{cases} \Theta(n^c) & \text{if } \log_b(a) < c \\ \Theta(n^c \log n) & \text{if } \log_b(a) = c \\ \Theta\left(n^{\log_b(a)}\right) & \text{if } \log_b(a) > c \end{cases}$$

Useful summation identities

Splitting a sum

$$\sum_{i=a}^{b}(x+y) = \sum_{i=a}^{b} x + \sum_{i=a}^{b} y$$

Adjusting summation bounds

$$\sum_{i=a}^{b} f(x) = \sum_{i=0}^{b} f(x) - \sum_{i=0}^{a-1} f(x)$$

Factoring out a constant

$$\sum_{i=a}^{b} cf(i) = c\sum_{i=a}^{b} f(i)$$

Summation of a constant

$$\sum_{i=0}^{n-1} c = \underbrace{c + c + \ldots + c}_{n \text{ times}} = cn$$

Note: this rule is a special case of the rule on the left

Gauss's identity

$$\sum_{i=0}^{n-1} i = 0 + 1 + \ldots + n - 1 = \frac{n(n-1)}{2}$$

Sum of squares

$$\sum_{i=0}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}$$

Finite geometric series

$$\sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}$$

Infinite geometric series

$$\sum_{i=0}^{\infty} x_i = \frac{1}{1-x}$$

Note: applicable only when $-1 < x < 1$