Quickcheck 03: Solutions

Your friend used summations to make a model for the running time of one of their functions. Simplify their summations into a final (exact) closed form. Then determine what the big-$\mathcal{O}$ of the function is.

$$\sum_{i=0}^{n-1}(6i^2-3)+\sum_{i=0}^{n-1}\frac{2^i}{n}+\sum_{i=0}^{n/2-1}\sum_{j=0}^{i-1}1$$

Here is a list of identities that may be useful:

Manipulating Sums:

$$\sum_{i=a}^{b}(x+y)=\sum_{i=a}^{b}x+\sum_{i=a}^{b}y \qquad \sum_{i=a}^{b}f(i)=\sum_{i=0}^{b}f(i)-\sum_{i=0}^{a-1}f(i) \qquad \sum_{i=a}^{b}c\cdot f(i)=c\sum_{i=a}^{b}f(i)$$

Geometric Series Identities:

$$\sum_{i=0}^{n-1}x^i=\frac{x^n-1}{x-1} \qquad \sum_{i=0}^{\infty}x^i=\frac{1}{1-x}\text{ if }-1<x<1$$

Other Common Summations:

$$\sum_{i=0}^{n-1}i=\frac{n(n-1)}{2} \qquad \sum_{i=0}^{n-1}i^2=\frac{n(n-1)(2n-1)}{6} \qquad \sum_{i=0}^{n-1}c=cn$$

Solution:

A good first step is to separate out summations so each only has one term.

$$\sum_{i=0}^{n-1}6i^2-\sum_{i=0}^{n-1}3+\sum_{i=0}^{n-1}\frac{2^i}{n}+\sum_{i=0}^{n/2-1}\sum_{j=0}^{i-1}1$$

A good second step is to factor out anything that doesn't change as the summation variable (i or j) changes:

$$6\sum_{i=0}^{n-1}i^2-\sum_{i=0}^{n-1}3+\frac{1}{n}\sum_{i=0}^{n-1}2^i+\sum_{i=0}^{n/2-1}\sum_{j=0}^{i-1}1$$

Now we can start applying identites. The first three summations each fit nicely into one of the given formulas. Finally in the fourth summation, the inner one also exactly meets an identity. Just plug in for each of them:

$$6\cdot\frac{n(n-1)(2n-1)}{6}-3n+\frac{1}{n}\cdot\frac{2^n-1}{2-1}+\sum_{i=0}^{n/2-1}i$$

Let's look at that last summation. The "summand" (the thing you add up each time) matches with one of our summations. But the bounds don't match the forumla. To apply the formula, $n/2$ (in our equation) has to play the role of n in the formula, and we get:

$$6\cdot\frac{n(n-1)(2n-1)}{6}-3n+\frac{1}{n}\cdot\frac{2^n-1}{2-1}+\frac{n/2(n/2-1)}{2}$$

What we have now is a “closed form” – there are no more summations and no more recursion. For this class, we can stop simplifying now.

What’s the dominating term? The third term is exponential, and the others are all polynomial, so the third term dominates. The final big-$\mathcal{O}$ is $\mathcal{O}\left(2^n/n\right)$.