CSE 373: P vs NP; reductions

Michael Lee
Wednesday, Mar 7, 2018
Warmup

Remind your neighbor:

- What is a decision-problem?
- What is P, EXP, and NP?
Warmup

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- What is a decision-problem?
  A yes-or-no question
- What is P, EXP, and NP?
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- What is P, EXP, and NP?

  1. P is the set of all decision problems that can be solved in worst-case *polynomial* time
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- What is a decision-problem?
  A yes-or-no question

- What is P, EXP, and NP?
  1. P is the set of all decision problems that can be solved in worst-case polynomial time
  2. EXP is the set of all decision problems that can be solved in worst-case exponential time
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- What is a decision-problem?
  A yes-or-no question
- What is P, EXP, and NP?
  1. P is the set of all decision problems that can be solved in worst-case polynomial time
  2. EXP is the set of all decision problems that can be solved in worst-case exponential time
  3. NP is the set of all decision problems where we can verify all “yes” answers in worst-case polynomial time
Final logistics:

- Thursday, March 15
- 2:30 to 4:20
- Gowen 301
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Review sessions:

- Monday, Mar 12: EEB 125, 4:30 to 6:30
- Tuesday, Mar 13: EEB 105, 4:30 to 6:30
The final will be cumulative, but skewed towards new material.

Post-midterm topics:

1. Heaps
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2. Sorting, basic divide-and-conquer
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   ▶ Definitions
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5. P and NP
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Pre-midterm topics:
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1. Asymptotic analysis, modeling code as equations
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1. Asymptotic analysis, modeling code as equations
2. Anything related to dictionaries
3. Caching and locality
General study tips for mechanical problems:

1. Drill until you can complete them very quickly
2. Invent your own problems and check them using online tools

General study tips for non-mechanical problems:

1. Do tons of practice
2. Minor differences matter; make sure you ask about them
3. Definitions are important; make sure you know them
4. For each data structure and algorithm we've studied, try writing a document summarizing (a) the high-level idea of how to implement them and (b) the best, average, and worst-case runtimes.
5. Think about what would happen if you were to tweak some aspect of a data structure or algorithm
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General tips when asked to analyze algorithms or code:

1. Don't make assumptions about what the code is doing, actually read it.
2. Try mentally running the code on specific examples.

General tips when asked to write pseudocode:

1. Keep a mental list of every data structure and algorithm we've studied. When stuck, go through that list one-by-one and try and find one that seems applicable.
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Syllabus change:
Final

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Previously:

- Midterm was 20% of grade
- Final was 20% of grade
Syllabus change:

Previously:

- Midterm was 20% of grade
- Final was 20% of grade

Now:

- Your lowest-scoring exam will be 15% of grade
- Your highest-scoring exam will be 25% of grade
Recap

Last time:

- Introduced the idea of decision problems and complexity classes
- Introduced the complexity classes $P$ and $\mathsf{EXP}$
- Found some (useful!) problems are, unfortunately, in $\mathsf{EXP}$
- But many of those problems are also in $\mathsf{NP}$!

Question: if there are problems where we can verify answers efficiently, does that mean we can also find answers efficiently?
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CIRCUIT-SAT

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Given a boolean expression such as “a && (b || c)” and the truth values for some of the variables, is there a way to set the remaining variables so that the output is T?
Is CIRCUIT-SAT in NP?

Question: is CIRCUIT-SAT in NP?

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**Step 1:** Assume you have a magical solver, and it said “yes” for some boolean expression $B$. 
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Question: is CIRCUIT-SAT in NP?

CIRCUIT-SAT

Given a boolean expression such as “a && (b || c)” and the truth values for some of the variables, is there a way to set the remaining variables so that the output is T?

Step 1: Assume you have a magical solver, and it said “yes” for some boolean expression $B$.

Step 2: Three questions to answer.

1. How do we modify the solver so it returns a convincing certificate for $B$?
2. How do we check the certificate, whatever it is?
3. Does our verifier actually run in polynomial time?
Is CIRCUIT-SAT in NP?

Step 2a: How do we modify the solver so it returns a convincing certificate?

Idea: return a map of the variable assignments!

{a=true, b=false, c=true, d=false, ...}

2b: How do we check the certificate, whatever it is?

Idea: try evaluating the expression!

```java
boolean verifyCircuitSat(BooleanAst B, Dictionary<String, Boolean> certificate) {
    return evaluateExpr(B, certificate);
}

private boolean evaluateExpr(B, certificate) {
    // Do something similar to toDoubleHelper, back from project 1
}
```

2c: Does our verifier actually run in polynomial time?

Yes: we visit each node and edge in the tree a constant number of times.
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- 2-COLOR easier or harder than 3-COLOR?
- 3-COLOR easier or harder than CIRCUIT-SAT?
Yes, using *reductions*.
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### Reductions

Given two decision problems $A$ and $B$, we can show that $A$ is "harder than or the same difficulty as" $B$ by...

\[
a \geq b
\]
Yes, using *reductions*.

**Reductions**

Given two decision problems \( A \) and \( B \), we can show that \( A \) is “harder than or the same difficulty as” \( B \) by...

1. Assuming we have some magical solver for \( A \)
Yes, using *reductions*.

**Reductions**

Given two decision problems $A$ and $B$, we can show that $A$ is “harder then or the same difficulty as” $B$ by...

1. Assuming we have some magical solver for $A$
2. Create an algorithm which calls the magical solver to solve $B$
Yes, using *reductions*.

### Reductions

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Core ideas: If solving $A$ lets us also solve $B$, then...

- $A$ was "harder then" (or the same as) $B$
Yes, using *reductions*.

**Reductions**

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Core ideas: If solving $A$ lets us also solve $B$, then...

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- The $B$ was really a special case of $A$ all along!
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Core ideas: If solving $A$ lets us also solve $B$, then...

- $A$ was “harder then” (or the same as) $B$
- The $B$ was really a special case of $A$ all along!
- We’ve *reduced* the number of distinct problems in the world by one.
Showing 2-COLOR reduces to 3-COLOR

We want to show that 2-COLOR reduces to 3-COLOR: that 3-COLOR is “harder then” 2-COLOR.
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**Step 1:** Assume we have a magical solver for 2-COLOR
Showing 2-COLOR reduces to 3-COLOR

We want to show that 2-COLOR reduces to 3-COLOR: that 3-COLOR is “harder than” 2-COLOR.

**Step 1:** Assume we have a magical solver for 3-COLOR.

**Step 2:** Using this magical solver, how do we solve an instance of 2-COLOR?
Showing 2-COLOR reduces to 3-COLOR

We want to show that 2-COLOR reduces to 3-COLOR: that 3-COLOR is “harder than” 2-COLOR.

**Step 1:** Assume we have a magical solver for 2-COLOR

**Step 2:** Using this magical solver, how do we solve an instance of 2-COLOR?

**Answer:**

1. Start by adding a new vertex to the graph
2. Connect this vertex to all other nodes
3. Give this vertex some color. This forces all other vertices to have a only one of two colors!
4. Run the solver for 3-COLOR, return the result
New question: How do we show two problems are the same?
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Intuition:

- To show two numbers $a$ and $b$ are the same, we can show $a \geq b$ and $a \leq b$. 
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Intuition:

- To show two numbers \( a \) and \( b \) are the same, we can show \( a \geq b \) and \( a \leq b \).
- To show two functions \( f(n) \) and \( g(n) \) are asymptotically the same, we can show that \( f(n) \) both dominates and is dominated by \( g(n) \).
New question: How do we show two problems are the same?

Intuition:

- To show two numbers $a$ and $b$ are the same, we can show $a \geq b$ and $a \leq b$.
- To show two functions $f(n)$ and $g(n)$ are asymptotically the same, we can show that $f(n)$ both dominates and is dominated by $g(n)$.
- To show two decision problems $A$ and $B$ are the same, we can show that $A$ reduces to $B$ and $B$ reduces to $A$. 

## LONG-PATH and HAM-PATH

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## LONG-PATH and HAM-PATH

### LONG-PATH
Given a graph $G$ and some integer $k$, does $G$ contain some path that uses $k$ edges?

### HAM-PATH
Given a graph $G$, does $G$ have a path that visits every vertex?
LONG-PATH and HAM-PATH

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**Goal:** Show that LONG-PATH and HAM-PATH are the same
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**Goal:** Show that LONG-PATH and HAM-PATH are the same

**Step 1:**
Reduce HAM-PATH to LONG-PATH

---

Note: paths can't revisit the same node
# LONG-PATH and HAM-PATH

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Given a graph $G$, does $G$ have a path that visits every vertex?

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**Step 1:**
Reduce HAM-PATH to LONG-PATH

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boolean hamPathSolver(G) {
    return longPathSolver(G, |V| - 1)
}
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LONG-PATH and HAM-PATH

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Step 2: Reduce LONG-PATH to HAM-PATH

```java
boolean longPathSolver(G, k) {
    for (G2=(v1, v2, ..., vk) : G):
        if (hamPathSolver(G2)):
            return true;
    return false;
}
```
LONG-PATH and HAM-PATH

LONG-PATH
Given a graph $G$ and some integer $k$, does $G$ contain some path that uses $k$ edges?

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}
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Punchline: HAM-PATH and LONG-PATH are actually the same problem in disguise!
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**Question:** Are there other problems that are secretly the same problem in disguise?
Equivalent problems

**Punchline:** HAM-PATH and LONG-PATH are actually the same problem in disguise!

**Question:** Are there other problems that are secretly the same problem in disguise?

Yes! It turns out that...

- CIRCUIT-SAT
- 3-COLOR
- HAM-PATH
- LONG-PATH

...are all the same problem.
Is there some problem that’s “harder than or same as” all of the problems we’ve seen so far?
Is there some problem that’s “harder then or same as” all of the problems we’ve seen so far?

Yes! For example, CIRCUIT-SAT (and therefore HAM-PATH and LONG-PATH and 3-COLOR).
Is there some problem that’s “harder than or same as” all of the problems we’ve seen so far?

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**NP-HARD**

A decision problem is NP-HARD if that decision problem is “harder than or as hard as” any other problem in NP.
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**NP-HARD**

A decision problem is NP-HARD if that decision problem is “harder then or as hard as” any other problem in NP.

Alternative phrasing: if every single decision problem in NP reduces to $X$, then $X$ is NP-HARD.

**NP-COMPLETE**

A decision problem is NP-COMPLETE if it is both in NP and in NP-HARD.
Punchline: If we have a way of solving any NP-HARD problem, we have a way of solving every problem we’ve looked at so far.
NP-HARD and NP-COMPLETE

How do these relate?
How do these relate?

How do all relate to P?
Is $P$ a subset of $\text{EXP}$?

Last time, we asked if $P$ is a subset of $\text{EXP}$. 
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**Answer 1: The sets are disjoint**
E.g. if a problem is in $P$, it’s not in $\text{EXP}$.
Is $P$ a subset of $\text{EXP}$?

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**Answer 1: The sets are disjoint**
E.g. if a problem is in $P$, it’s not in $\text{EXP}$.

**Answer 2: The sets overlap**
E.g. some, but not all problems in $P$ are in $\text{EXP}$.
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**Answer 2: The sets overlap**
E.g. some, but not all problems in $P$ are in $\text{EXP}$

**Answer 3: $P$ is a subset of $\text{EXP}$**
All problems in $P$ are also in $\text{EXP}$
Is P a subset of EXP?

Last time, we asked if P is a subset of EXP.

It turns out, yes, P is indeed a subset of EXP:

**Answer 3: P is a subset of EXP**

All problems in P are also in EXP.
Is $P$ a subset of $\text{EXP}$?

Last time, we asked if $P$ is a subset of $\text{EXP}$.

It turns out, yes, $P$ is indeed a subset of $\text{EXP}$:

**Answer 3: $P$ is a subset of $\text{EXP}$**

All problems in $P$ are also in $\text{EXP}$

Reason: $\text{EXP}$ is the set of decision problems where there exists an algorithm that solves the problem in *worst-case exponential time*. 
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It turns out, yes, P is indeed a subset of EXP:

**Answer 3: P is a subset of EXP**

All problems in P are also in EXP

Reason: EXP is the set of decision problems where there exists an algorithm that solves the problem in *worst-case exponential time*. So, if we can find a polynomial-time algorithm to a problem, we can definitely find an exponential one!
Is $P$ a subset of $NP$?

New question: is a $P$ a subset of $NP$?
New question: is a \( P \) a subset of \( \text{NP} \)?

**Answer 1: The sets are disjoint**
E.g. if a problem is in \( P \), it’s not in \( \text{NP} \).
New question: is a P a subset of NP?

**Answer 1: The sets are disjoint**
E.g. if a problem is in P, it’s not in NP.

**Answer 2: The sets overlap**
E.g. some, but not all problems in P are in NP
New question: is a P a subset of NP?

**Answer 1: The sets are disjoint**
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E.g. some, but not all problems in P are in NP

**Answer 3: P is a subset of NP**
All problems in P are also in NP
New question: is a P a subset of NP?

It turns out, yes.

**Answer 3: P is a subset of NP**

All problems in P are also in NP.
Is P a subset of NP?

Reason: Let’s say we have some decision problem \( X \).

**Step 1:** Assume we have a magical solver for \( X \), and it said “yes” for some input.
Is P a subset of NP?

Reason: Let’s say we have some decision problem X.

Step 1: Assume we have a magical solver for X, and it said “yes” for some input.

Step 2: Three questions to answer.
Is P a subset of NP?

Reason: Let’s say we have some decision problem X.

**Step 1:** Assume we have a magical solver for X, and it said “yes” for some input.

**Step 2:** Three questions to answer.

1. How do make the solver so it returns a convincing certificate?
Is P a subset of NP?

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**Step 1:** Assume we have a magical solver for X, and it said “yes” for some input.

**Step 2:** Three questions to answer.

  1. How do make the solver so it returns a convincing certificate? One possible certificate: return the string “\_(\^_^)/\_”.
Is P a subset of NP?

Reason: Let’s say we have some decision problem X.

**Step 1:** Assume we have a magical solver for X, and it said “yes” for some input.

**Step 2:** Three questions to answer.

1. How do make the solver so it returns a convincing certificate?
   One possible certificate: return the string “\_\_(\_\_)/\_”.
2. How do we check the certificate, whatever it is?
Is P a subset of NP?

Reason: Let’s say we have some decision problem X.

Step 1: Assume we have a magical solver for X, and it said “yes” for some input.

Step 2: Three questions to answer.

1. How do make the solver so it returns a convincing certificate?
   One possible certificate: return the string “¬\(_\{\_\(\_\)\}_\)/¬”.

2. How do we check the certificate, whatever it is?
   Idea: just *ignore* the certificate
   ```java
   boolean verifyX(input, certificate) {
       return solverX(input);
   }
   ```
Is P a subset of NP?

Reason: Let’s say we have some decision problem X.

**Step 1:** Assume we have a magical solver for X, and it said “yes” for some input.

**Step 2:** Three questions to answer.

1. **How do make the solver so it returns a convincing certificate?**
   One possible certificate: return the string "__(\(\_\_\_\))__".

2. **How do we check the certificate, whatever it is?**
   
   Idea: just *ignore* the certificate
   
   ```java
   boolean verifyX(input, certificate) {
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   ```

3. **Does our verifier actually run in polynomial time?**
   
   Yep. If X was originally in P, then we know by definition solverX runs in polynomial time.
Punchline: For any problem in $P$, we can build a verifier by just re-using the solver!
Third question: is $P = NP$?
Third question: is P = NP?

**Answer 1: No**
P is a subset of NP, but that’s it.
Is $P = NP$?

Third question: is $P = NP$?

**Answer 1: No**
P is a subset of NP, but that’s it.

**Answer 2: Yes**
Not only is $P$ a subset of NP, they’re exactly the same.
Third question: is P = NP?

**Answer 1: No**
P is a subset of NP, but that’s it.

**Answer 2: Yes**
Not only is a P a subset of NP, they’re exactly the same

**Answer: We don’t know.**
What if $P \neq NP$?

**Answer 1:** No

$P$ is a subset of $NP$, but that's it.
What if $P \neq NP$?

**Answer 1: No**
P is a subset of NP, but that’s it.

- Have your name be immortalized in CS textbooks forever
What if $P \neq NP$?

**Answer 1: No**

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- Have your name be immortalized in CS textbooks forever
- Win 1 million dollars for solving a Millenium Prize problem
What if $P \neq NP$?

Answer 1: No

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- The world otherwise looks the same
What if $P \neq NP$?

If $P \neq NP$, and we have an NP problem, what do we do?

- Try and find approximate solutions
- Use probabilistic algorithms
- Use solvers that work efficiently on many (but not all!) instances of NP-COMPLETE problems. (E.g. programs like z3, which solve CIRCUIT-SAT)
- Find a way of reducing your problem into some famous NP-HARD problem and use a solver
- Crowdsource. Observation: lots of games are actually NP (e.g. sudoku).
- Actual example: Foldit, a protein folding “game”
- Something something quantum computing? (Lots of caveats, not practical right now, doesn’t solve everything, even if they work.)
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What if $P = NP$?

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What if this is reality?
What if $P = NP$?

What if this is reality?

AND what if we have an efficient way of solving any NP-COMPLETE problem?
What if $P = NP$?

- Have your name be immortalized in CS textbooks forever

- Win 1 million dollars for solving a Millenium Prize problem

- Finding a way of generating a proof of anything (assuming the proof is a reasonable length)

- Win 5 million more dollars for solving the remaining Millenium Prize problems

- Crack all of modern encryption, and have all the dollars

- Crack all of modern encryption, and have access to all information, public or private

- Literally cure cancer
What if P = NP?

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