Previously:

- We spent a lot of time learning how to solve problems
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- We spent a lot of time analyzing algorithms
Today:

- Take a step back and look at the bigger picture
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▶ Take a step back and look at the bigger picture
▶ Discuss an important open question in computer science: does $P = NP$?
But first:

What does it mean for a problem to be “efficient”? 
What is “efficiency”?  

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What does it mean for a problem to be “efficient”?  

What do we even mean by “problem”, anyways?
What is a “decision problem”?

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Which of these are decision problems?

- **IS-PRIME**: “Is X prime? (Where X is some input)”

- **FIND-PRIME**: “What is the $n$-th prime number?”

- **SORT**: “Sort this list of numbers.”

- **IS-SORTED**: “Is this list of numbers sorted?”
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  Yes, it’s a yes-or-no question.

- **FIND-PRIME**: “What is the $n$-th prime number?”
  
  No. The answer is a number, not a boolean.

- **SORT**: “Sort this list of numbers.”
  
  No; not a question.

- **IS-SORTED**: “Is this list of numbers sorted?”
  
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**Question:** Why only talk about decision problems?

Answer: It simplifies things. Also, most problems can be turned into a decision problem with some tweaking, so not a big deal.

Example:

**SHORTEST-PATH:** “What is the shortest path between two given nodes?”

...can be turned into:

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A decision problem is **solvable** if there exists some algorithm that given any input, or *instance*, can correctly *decide* “yes” or “no”.

**Example:**

`IS-PRIME` is solvable. Here’s an algorithm:

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Surprisingly, yes.
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Surprisingly, yes.

We won’t go into that today; look up the “halting problem” if you’re curious.
What do we even mean by “problem”, anyways?
Questions:

- What do we even mean by “problem”, anyways?
- What does it mean for a problem to be “efficient”? 
What is an “efficient algorithm”? 

Efficient algorithm

An algorithm is **efficient** if the worst-case bound is a **polynomial**.
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**Question:** Are $n^{10000000}$ and $30000000000000n^3$ actually efficient in practice?

No, but...

- Once we find a polynomial algorithm to a problem, we’ve historically been able to improve it to something reasonable.
- Finding a polynomial runtime is a *VERY* low bar. If we can’t even get that...
Examples of problems

Pretty much all problems we’ve studied have efficient solutions!
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We’ve studied two main types of algorithms: sorting algorithms and graph algorithms, and every one we’ve looked at so far could run in polynomial time.

(e.g “How do I sort this list”, “What is the shortest path”, “What is the MST”...)
Great: do all solvable problems have efficient solutions?
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Haha, no.
Examples of problems

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Well, ok – do all *practical* problems we actually care about have efficient solutions?
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lol
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2-COLOR vs 3-COLOR

2-COLOR

Given a graph, is it possible to assign each node one of two colors such that no two adjacent nodes share the same color?

- To solve, run BFS or DFS, alternate colors...
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3-COLOR

Given a graph, is it possible to assign each node one of three colors such that no two adjacent nodes share the same color?

There is no known efficient solution to this problem.

To solve, use brute force: try all $O(3^V)$ combinations.
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Observation: Some problems have polynomial solutions, some have worse.

Can we formalize this?
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Can we formalize this?

**Complexity class**

A *complexity class* is a set of problems limited by some resource constraint (time, space, etc)
The complexity class P

P is the set of all decision problems where there exists an algorithm that can solve all inputs in worst-case polynomial time.
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Examples: IS-PRIME, IS-SORTED, PATH, 2-COLOR, CIRCUIT-VALUE, ...

Complexity class: P and EXP
The complexity class **P**

P is the set of all decision problems where there exists an algorithm that can solve all inputs in worst-case polynomial time.

Examples: IS-PRIME, IS-SORTED, PATH, 2-COLOR, CIRCUIT-VALUE, ...

The complexity class **EXP**

EXP is the set of all decision problems where there exists an algorithm that can solve all inputs in worst-case exponential time.

Examples: LONGEST-PATH, 3-COLOR, CIRCUIT-SAT...
Question: Suppose we have some random decision problem in P. Is that problem also in EXP?

E.g. is 2-COLOR in EXP?
Is P a subset of EXP?

There are three reasonable possibilities:

**Answer 1: The sets are disjoint**
E.g. if a problem is in P, it’s not in EXP.
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All problems in $P$ are also in $\text{EXP}$.
It turns out it’s answer 3: P is a subset of EXP.

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Reason: EXP is the set of decision problems where there exists an algorithm that solves the problem in *worst-case exponential time*. 
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**Answer 3: P is a subset of EXP**

All problems in P are also in EXP

Reason: EXP is the set of decision problems where there exists an algorithm that solves the problem in *worst-case exponential time*. So, if we can find a polynomial-time algorithm to a problem, we can definitely find an exponential one!
Example: We previously showed there exists an $O(n)$ algorithm to check if a number $n$ is prime:

```java
boolean isPrimeSolver(n):
    for (int i = 2; i < n; i++):
        if (X % i == 0):
            return false
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So IS-PRIME ∈ P.
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How do we show that $IS\text{-PRIME}$ is in $EXP$?
Is \( P \) a subset of \( \text{EXP} \)?

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So \( \text{IS-PRIME} \in P \).

How do we show that \( \text{IS-PRIME} \) is in \( \text{EXP} \)?

```java
boolean isPrimeSolver2(n):
    for (int i = 0; i < Math.pow(2, n); i++):
        print("lol")
    return isPrimeSolver(n)
```

This runs in exponential time and correctly solves all inputs. So \( \text{IS-PRIME} \) is also in \( \text{EXP} \).
Recap

To recap:

▶ What is a decision problem?
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- What is a decision problem?
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- What is a decision problem?
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Unfortunately, some problems we care about are in $\text{EXP}$.
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Big idea: NP is the set of decision problems that can be verified in polynomial time.
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Big idea: NP is the set of decision problems that can be verified in polynomial time.

If we can verify answers efficiently, can we find answers efficiently?
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A *verifier* accepts as input:

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Another kind of algorithm – a verifier

Verifier

A verifier accepts as input:

1. Some instance of the decision problem
2. Some sort of “proof” or certificate of why the solver made whatever decision it made on that instance.
The complexity class NP

Suppose that we have some decision problem $X$ where...

- There exists some solver for $X$
  - That solver says "yes" for some instance of $X$
  - Whenever the solver says "yes", it also returns some sort of "proof" or certificate of why they said "yes".

If there exists a verifier that...

- When given the instance and the certificate, always agrees the correct answer was "yes"
- Always runs in polynomial time

...then $X$ is in NP.
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The complexity class co-NP

**Important note:** The verifier only needs to exist when the solver says “yes”.

If the solver says “no”, we don’t care.
The complexity class co-NP

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A related complexity class: co-NP. Almost identical to NP, except for “NO” instances.
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The complexity class co-NP

Suppose that we have some decision problem $X$ where...

- There exists some solver for $X$
- That solver says “no” for some instance of $X$
- Whenever the solver says “no”, it also returns some sort of “proof” or certificate of why they said “no”.

If there exists a verifier that...

- When given the instance and the certificate, always agrees the correct answer was “no”
- Always runs in polynomial time

...then $X$ is in co-NP.
Example: showing 3-COLOR is in NP

I claim that 3-COLOR is in NP. How do we show this?

Step 1: Assume the preconditions are met. Suppose we have a magical solver for 3-COLOR, and it says "yes" for some graph $G$.

Step 2: Show that we can build a polynomial-time verifier, given $G$ and some certificate. Three things we must do:

1. How do we modify the solver so it returns a convincing certificate?
2. How do we check the certificate, whatever it is?
3. Does our verifier actually run in polynomial time?
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Part 2a: What would be a convincing certificate?

A map of vertices to colors! E.g. 
\{v_1 = red, v_2 = blue, v_3 = red, v_4 = green, \ldots \}.

Part 2b: How do we double-check this certificate?

Loop through all vertices, make sure neighbors have different colors!

```java
boolean verify3Color(G, colorMap):
    for (v : G.vertices):
        for (w : v.neighbors):
            if (colorMap.get(v) == colorMap.get(w)):
                return false
    return true
```

Part 2c: Does this verifier run in polynomial time?

Yes! It runs in $O(|V| + |E|)$ time!

So, 3-COLOR $\in$ NP.
**Part 2a:** What would be a convincing certificate?

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**Part 2b:** How do we double-check this certificate?

Loop through all vertices, make sure neighbors have different colors!

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Loop through all vertices, make sure neighbors have diff colors!
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So, \(3\text{-COLOR} \in \text{NP}\).
Example: showing $3$-COLOR is in NP

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Yes! It runs in $O(|V| + |E|)$ time!

So, 3-COLOR $\in$ NP.
Example: showing CIRCUIT-SAT is in NP

Question: is CIRCUIT-SAT in NP?
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**Question:** is CIRCUIT–SAT in NP?

**CIRCUIT–SAT**

Given a boolean expression such as “a && (b || c)” and the truth values for some of the variables, is there a way to set the remaining variables so that the output is T?
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