CSE 373: P vs NP

Michael Lee

Monday, Mar 5, 2018

Previously:

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- We spent a lot of time learning how to solve problems
- We spent a lot of time analyzing algorithms

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- Take a step back and look at the bigger picture

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- Take a step back and look at the bigger picture
- Discuss an important open question in computer science: does $\mathrm{P} = \mathrm{NP}$?

What is “efficiency”?

But first:

What does it mean for a problem to be “efficient”?

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What does it mean for a problem to be “efficient”?

What do we even mean by “problem”, anyways?

What is a “decision problem”?

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Which of these are decision problems?

- `IS-PRIME`: "Is X prime? (Where X is some input)"
- `FIND-PRIME`: "What is the *n*-th prime number?"
- `SORT`: "Sort this list of numbers."
- `IS-SORTED`: "Is this list of numbers sorted?"

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- `IS-PRIME`: "Is X prime? (Where X is some input)"
  Yes, it's a yes-or-no question.
- `FIND-PRIME`: "What is the *n*-th prime number?"
  No. The answer is a number, not a boolean.
- `SORT`: "Sort this list of numbers."
  No; not a question.
- `IS-SORTED`: "Is this list of numbers sorted?"
  Yes, it's a yes-or-no question.

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`SHORTEST-PATH`: "What is the shortest path between two given nodes?"

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Example:

SHORTEST-PATH: "What is the shortest path between two given nodes?"

...can be turned into:

PATH: "Does there exist a path between two given nodes that consists of k edges?"

$k=1$ $k=2$ $k=3$

What is a “solvable” problem?

Solvable

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Example: `IS-PRIME` is solvable. Here's an algorithm:

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boolean isPrimeSolver(n):
    for (int i = 2; i < n; i++)
        if (X % i == 0):
            return false
    return true
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We won’t go into that today; look up the “halting problem” if you’re curious.

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- What do we even mean by “problem”, anyways?

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- $\mathcal{O}\left(30000000000000n^3\right)$: Technically yes...

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- Once we find a polynomial algorithm to a problem, we've historically been able to improve it to something reasonable
- Finding a polynomial runtime is a *VERY* low bar. If we can't even get that...

Examples of problems

Pretty much all problems we've studied have efficient solutions!

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We've studied two main types of algorithms: sorting algorithms and graph algorithms, and every one we've looked at so far could run in polynomial time.

(e.g "How do I sort this list", "What is the shortest path", "What is the MST"...)

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lol

PATH vs LONGEST-PATH

PATH

inputs

Given a graph and two vertices, does there exist some path between those two vertices that visits exactly k edges?

another input

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- To solve, run BFS and see if we visit the dest in k edges.
- We can solve this efficiently!

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LONGEST-PATH

Given a graph, does there exist a path between **any** two vertices that visits exactly k edges?

There is no known efficient solution to this problem.

To solve, use brute force.

2-COLOR vs 3-COLOR

2-COLOR

Given a graph, is it possible to assign each node one of two colors such that no two adjacent nodes share the same color?

- To solve, run BFS or DFS, alternate colors...
- We can solve this efficiently!

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Given a graph, is it possible to assign each node one of **three** colors such that no two adjacent nodes share the same color?"

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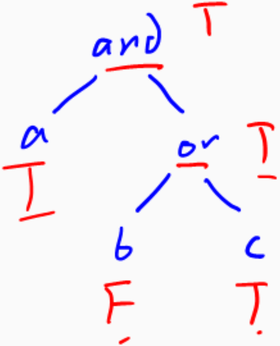
There is no known efficient solution to this problem.

To solve, use brute force: try all $\mathcal{O}\left(3^{|V|}\right)$ combinations.

CIRCUIT-VALUE

T F T

Given a boolean expression such as "a && (b || c)" and the truth values for every variable, is the final expression T?



CIRCUIT-VALUE vs CIRCUIT-SAT

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Given a boolean expression such as "`a && (b || c)`" and the truth values for every variable, is the final expression T?

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There is no known efficient solution to this problem.

To solve, use brute force: try every combination of variables.

Complexity classes

Observation: Some problems have polynomial solutions, some have worse.

Can we formalize this?

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Can we formalize this?

Complexity class

A **complexity class** is a set of problems limited by some resource constraint (time, space, etc)

Complexity class: P and EXP

The complexity class P

P is the set of all decision problems where there exists an algorithm that can solve all inputs in worst-case polynomial time.

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Examples: `IS-PRIME`, `IS-SORTED`, `PATH`, `2-COLOR`, `CIRCUIT-VALUE`, ...

Complexity class: P and EXP

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P is the set of all decision problems where there exists an algorithm that can solve all inputs in worst-case polynomial time.

Examples: `IS-PRIME`, `IS-SORTED`, `PATH`, `2-COLOR`, `CIRCUIT-VALUE`, ...

The complexity class EXP

EXP is the set of all decision problems where there exists an algorithm that can solve all inputs in worst-case exponential time.

Examples: `LONGEST-PATH`, `3-COLOR`, `CIRCUIT-SAT`...

Is P a subset of EXP?

Question: Suppose we have some random decision problem in P. Is that problem also in EXP?

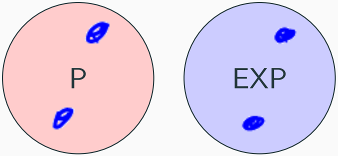
E.g. is `2-COLOR` in EXP?

Is P a subset of EXP?

There are three reasonable possibilities:

Answer 1: The sets are disjoint

E.g. if a problem is in P, it's not in EXP.



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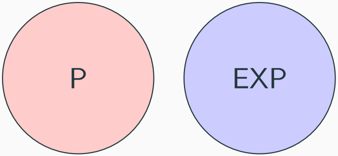
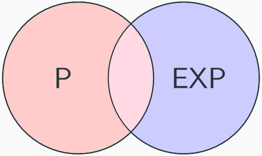


Answer 2: The sets overlap
E.g. some, but not all problems in P are in EXP



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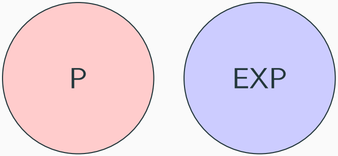

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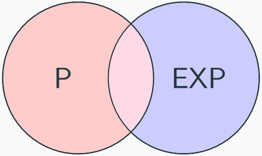
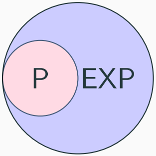
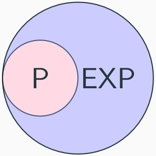


Answer 3: P is a subset of EXP
All problems in P are also in EXP



Is P a subset of EXP?

It turns out it's answer 3: P is a subset of EXP.



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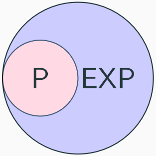

Reason: EXP is the set of decision problems where there exists an algorithm that solves the problem in *worst-case exponential time*.

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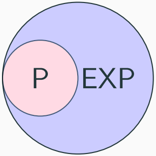

Reason: EXP is the set of decision problems where there exists an algorithm that solves the problem in *worst-case exponential time*.

So, if we can find a polynomial-time algorithm to a problem, we can definitely find an exponential one!

Is P a subset of EXP?

Example: We previously showed there exists an $\mathcal{O}(n)$ algorithm to check if a number n is prime:

```
boolean isPrimeSolver(n):
    for (int i = 2; i < n; i++):
        if (X % i == 0):
            return false
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So IS-PRIME $\in$ P.

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How do we show that IS-PRIME is in EXP?

```
boolean isPrimeSolver2(n):
    for (int i = 0; i < Math.pow(2, n); i++):
        print("lol")

    return isPrimeSolver(n)
```

This runs in exponential time and correctly solves all inputs.
So IS-PRIME is also in EXP.

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  - What does it mean to “solve” a decision problem?
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- What is a decision problem?
  - What does it mean to "solve" a decision problem?
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- What is a complexity class?
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  - P is a subset of EXP

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- What is a decision problem?
  - What does it mean to "solve" a decision problem?
  - What does it mean for an algorithm to be "efficient"?
- What is a complexity class?
  - P
  - EXP
  - P is a subset of EXP
- **Unfortunately, some problems we care about are in EXP**

A glimmer of hope...

Observation: Some problems in EXP have an interesting property:

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Big idea: NP is the set of decision problems that can be verified in polynomial time.

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Big idea: NP is the set of decision problems that can be verified in polynomial time.

If we can *verify* answers efficiently, can we *find* answers efficiently?

Solving vs verifying

Reminder: a solver is an algorithm that accepts an *instance* of a decision-problem and returns true or false.

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Another kind of algorithm – a *verifier*

Verifier

A **verifier** accepts as input:

1. Some instance of the decision problem
2. Some sort of "proof" or *certificate* of why the solver made whatever decision it made on that instance.

The complexity class NP

Suppose that we have some decision problem X where...

- There exists some solver for X

The complexity class NP

Suppose that we have some decision problem X where...

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- That solver says "yes" for some instance of X

The complexity class NP

Suppose that we have some decision problem X where...

- There exists some solver for X
- That solver says “yes” for some instance of X
- Whenever the solver says “yes”, it also returns some sort of “proof” or *certificate* of why they said “yes”.

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If there exists a verifier that...

- When given the instance and the certificate, always agrees the correct answer was "yes"
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...then X is in NP.

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Important note: The verifier only needs to exist when the solver says “yes”.

If the solver says “no”, we don’t care.

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A related complexity class: co-NP. Almost identical to NP, except for "NO" instances.

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Suppose that we have some decision problem X where...

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Suppose that we have some decision problem X where...

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- That solver says **"no"** for some instance of X

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...then X is in **co-NP**.

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boolean verify3Color(G, colorMap):
    for (v : G.vertices):
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            if (colorMap.get(v) == colorMap.get(w)):
                return false
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2-COLOR $\in$ P
$\in$ NP

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Is 2-color also in NP?

So, 3-COLOR $\in$ NP.

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