CSE 373: P vs NP

Michael Lee
Monday, Mar 5, 2018
Previously:

- We spent a lot of time learning how to solve problems
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- We spent a lot of time analyzing algorithms
Today:

- Take a step back and look at the bigger picture
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- Discuss an important open question in computer science: does $P = NP$?
What is “efficiency”?  

But first:  

What does it mean for a problem to be “efficient”?  

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What does it mean for a problem to be “efficient”?  

What do we even mean by “problem”, anyways?
What is a “decision problem”? 

Decision problem

A decision problem is any arbitrary yes-or-no question on an infinite set of inputs.
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Which of these are decision problems?

- **IS-PRIME**: “Is X prime? (Where X is some input)”
  - Yes, it's a yes-or-no question.

- **FIND-PRIME**: “What is the \( n \)-th prime number?”
  - No. The answer is a number, not a boolean.

- **SORT**: “Sort this list of numbers.”
  - No; not a question.

- **IS-SORTED**: “Is this list of numbers sorted?”
  - Yes, it’s a yes-or-no question.
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What is a “decision problem”?

**Question:** Why only talk about decision problems?

Answer: It simplifies things. Also, most problems can be turned into a decision problem with some tweaking, so not a big deal.

Example: SHORTEST-PATH: “What is the shortest path between two given nodes?”

...can be turned into: PATH: “Does there exist a path between two given nodes that consists of \( k \) edges?”
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SHORTEST-PATH: “What is the shortest path between two given nodes?” 

...can be turned into: 

PATH: “Does there exist a path between two given nodes that consists of $k$ edges?” 

$k = 1$  
$k = 2$  
$k = 3$
What is a “solvable” problem?

A decision problem is **solvable** if there exists some algorithm that given any input, or *instance*, can correctly *decide* “yes” or “no”.

**Example:** 

*IS-PRIME* is solvable. Here’s an algorithm:

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boolean isPrimeSolver(n):
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Surprisingly, yes.
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Surprisingly, yes.

We won’t go into that today; look up the “halting problem” if you’re curious.
Questions:

- What do we even mean by “problem”, anyways?
Definitions

Questions:

- What do we even mean by “problem”, anyways?
- What does it mean for a problem to be “efficient”? 
**What is an “efficient algorithm”?**

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- $O(n \log(n))$: Yes, $n \log(n) \in O(n^2)$, which is a polynomial
- $O(30000000000000000000000n^3)$: Technically yes...
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- Once we find a polynomial algorithm to a problem, we’ve historically been able to improve it to something reasonable.
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► Once we find a polynomial algorithm to a problem, we’ve historically been able to improve it to something reasonable.

► Finding a polynomial runtime is a *VERY* low bar. If we can’t even get that...
Examples of problems

Pretty much all problems we’ve studied have efficient solutions!
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We’ve studied two main types of algorithms: sorting algorithms and graph algorithms, and every one we’ve looked at so far could run in polynomial time.

(e.g “How do I sort this list”, “What is the shortest path”, “What is the MST”...)
Great: do all solvable problems have efficient solutions?
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Haha, no.
Examples of problems

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Haha, no.

Well, ok – do all *practical* problems we actually care about have efficient solutions?
Great: do all solvable problems have efficient solutions?

Haha, no.

Well, ok – do all *practical* problems we actually care about have efficient solutions?

lol
PATH vs LONGEST-PATH

PATH
Given a graph and two vertices $u$ and $v$, does there exist some path from $u$ to $v$ that visits exactly $k$ edges?

▶ To solve, run BFS and see if we visit $v$ in $k$ steps.

▶ We can solve this efficiently!

LONGEST-PATH
Given a graph, does there exist a path between any two vertices that visits exactly $k$ edges?

There is no known efficient solution to this problem.

To solve, use brute force.
### PATH vs LONGEST-PATH

**PATH**

Given a graph and two vertices, does there exist some path between those two vertices that visits exactly $k$ edges?

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2-COLOR vs 3-COLOR

2-COLOR
Given a graph, is it possible to assign each node one of two colors such that no two adjacent nodes share the same color?

- To solve, run BFS or DFS, alternate colors...
- We can solve this efficiently!

3-COLOR
Given a graph, is it possible to assign each node one of three colors such that no two adjacent nodes share the same color?

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To solve, use brute force: try all $O(3^{|V|})$ combinations.
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CIRCUIT-VALUE vs CIRCUIT-SAT

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Given a boolean expression such as “a && (b || c)” and the truth values for every variable, is the final expression T?

To solve, convert into an abstract syntax tree and evaluate.

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CIRCUIT-SAT

Given a boolean expression such as “a && (b || c)” and the truth values for some of the variables, is there a way to set the remaining variables so that the output is T?

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Observation: Some problems have polynomial solutions, some have worse.

Can we formalize this?
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Can we formalize this?

**Complexity class**

A *complexity class* is a set of problems limited by some resource constraint (time, space, etc)
The complexity class P

P is the set of all decision problems where there exists an algorithm that can solve all inputs in worst-case polynomial time.
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Examples: IS-PRIME, IS-SORTED, PATH, 2-COLOR, CIRCUIT-VALUE, ...
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Examples: IS-PRIME, IS-SORTED, PATH, 2-COLOR, CIRCUIT-VALUE, ...

The complexity class EXP

EXP is the set of all decision problems where there exists an algorithm that can solve all inputs in worst-case exponential time.

Examples: LONGEST-PATH, 3-COLOR, CIRCUIT-SAT...
Question: Suppose we have some random decision problem in P. Is that problem also in EXP?

E.g. is 2-COLOR in EXP?
Is $P$ a subset of $EXP$?

There are three reasonable possibilities:

**Answer 1: The sets are disjoint**
E.g. if a problem is in $P$, it’s not in $EXP$. 

**Answer 2: The sets overlap**
E.g. some, but not all problems in $P$ are in $EXP$.

**Answer 3: $P$ is a subset of $EXP$**
All problems in $P$ are also in $EXP$. 

Is $P$ a subset of $\text{EXP}$?

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Reason: EXP is the set of decision problems where there exists an algorithm that solves the problem in *worst-case exponential time*. 
Is $P$ a subset of $\text{EXP}$?

It turns out it’s answer 3: $P$ is a subset of $\text{EXP}$.

**Answer 3: $P$ is a subset of $\text{EXP}$**

All problems in $P$ are also in $\text{EXP}$

Reason: $\text{EXP}$ is the set of decision problems where there exists an algorithm that solves the problem in *worst-case exponential time*. So, if we can find a polynomial-time algorithm to a problem, we can definitely find an exponential one!
Example: We previously showed there exists an $\mathcal{O}(n)$ algorithm to check if a number $n$ is prime:

```java
boolean isPrimeSolver(n):
    for (int i = 2; i < n; i++):
        if (X % i == 0):
            return false
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So IS-PRIME $\in$ P.
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How do we show that IS-PRIME is in EXP?

```java
boolean isPrimeSolver2(n):
    for (int i = 0; i < Math.pow(2, n); i++):
        print("lol")
    return isPrimeSolver(n)
```

This runs in exponential time and correctly solves all inputs. So IS-PRIME is also in EXP.
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- *Checking or verifying* if a solution is correct always takes polynomial time!
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Big idea: NP is the set of decision problems that can be verified in polynomial time.
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- *Checking or verifying* if a solution is correct always takes polynomial time!

**Big idea:** NP is the set of decision problems that can be verified in polynomial time.

If we can *verify* answers efficiently, can we *find* answers efficiently?
Reminder: a solver is an algorithm that accepts an *instance* of a decision-problem and returns true or false.
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Another kind of algorithm – a verifier
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The complexity class NP

Suppose that we have some decision problem X where...

- There exists some solver for X

- That solver says "yes" for some instance of X

- Whenever the solver says "yes", it also returns some sort of "proof" or certificate of why they said "yes".

- If there exists a verifier that...
  - When given the instance and the certificate, always agrees the correct answer was "yes"
  - Always runs in polynomial time

...then X is in NP.
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Important note: The verifier only needs to exist when the solver says “yes”.
If the solver says “no”, we don’t care.
The complexity class co-NP

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A related complexity class: co-NP. Almost identical to NP, except for “NO” instances.
The complexity class co-NP

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- There exists some solver for $X$

...then $X$ is in co-NP.
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Suppose that we have some decision problem \( X \) where...

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...then X is in co-NP.
Example: showing 3-COLOR is in NP

I claim that 3-COLOR is in NP. How do we show this?

Step 1: Assume the preconditions are met. Suppose we have a magical solver for 3-COLOR, and it says "yes" for some graph $G$.

Step 2: Show that we can build a polynomial-time verifier, given $G$ and some certificate. Three things we must do:

1. How do we modify the solver so it returns a convincing certificate?
2. How do we check the certificate, whatever it is?
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**Part 2a:** What would be a convincing certificate?

A map of vertices to colors! E.g.

```
{v_1 = red, v_2 = blue, v_3 = red, v_4 = green, ...}
```

**Part 2b:** How do we double-check this certificate?

Loop through all vertices, make sure neighbors have different colors!

```
boolean verify3Color(G, colorMap):
    for (v : G.vertices):
        for (w : v.neighbors):
            if (colorMap.get(v) == colorMap.get(w)):
                return false
    return true
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**Part 2c:** Does this verifier run in polynomial time?

Yes! It runs in $O(|V| + |E|)$ time!

So, 3-COLOR ∈ NP.
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So, \(3\text{-COLOR} \in \text{NP}\).
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Given a boolean expression such as “a && (b || c)” and the truth values for some of the variables, is there a way to set the remaining variables so that the output is T?

As before, assume you have a magical solver, and it said “yes” for some boolean expression $B$. 
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