CSE 373: P vs NP

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Overview

Previously:

- We spent a lot of time learning how to solve problems
- We spent a lot of time analyzing algorithms

Overview

Today:

- Take a step back and look at the bigger picture
- Discuss an important open question in computer science: does P = NP?

What is "efficiency"?

But first:

What does it mean for a problem to be "efficient"?

What do we even mean by "problem", anyways?

What is a "decision problem"?

Decision problem

A **decision problem** is any arbitrary yes-or-no question on an infinite set of inputs.

Which of these are decision problems?

- `IS-PRIME`: "Is X prime? (Where X is some input)"
  Yes, it's a yes-or-no question.
- `FIND-PRIME`: "What is the n-th prime number?"
  No. The answer is a number, not a boolean.
- `SORT`: "Sort this list of numbers."
  No; not a question.
- `IS-SORTED`: "Is this list of numbers sorted?"
  Yes, it's a yes-or-no question.

What is a "decision problem"?

Question: Why only talk about decision problems?

Answer: It simplifies things. Also, most problems can be turned into a decision problem with some tweaking, so not a big deal.

Example:

`SHORTEST-PATH`: "What is the shortest path between two given nodes?"

...can be turned into:

`PATH`: "Does there exist a path between two given nodes that consists of k edges?"

What is a "solvable" problem?

Solvable

A decision problem is **solvable** if there exists some algorithm that given any input, or *instance*, can correctly *decide* "yes" or "no".

Example: IS-PRIME is solvable. Here's an algorithm:

```
boolean isPrimeSolver(n):
    for (int i = 2; i < n; i++):
        if (X % i == 0):
            return false
    return true
```


What is a "solvable" problem?

Question: Are there problems that are unsolvable – problems that are impossible to solve?

Surprisingly, yes.

We won't go into that today; look up the "halting problem" if you're curious.

Definitions

Questions:

- What do we even mean by "problem", anyways?
- What does it mean for a problem to be "efficient"?

What is an "efficient algorithm"?

Efficient algorithm

An algorithm is **efficient** if the worst-case bound is a **polynomial**.

Examples: which of these runtime bounds are "efficient"?

- $\mathcal{O}\left(n^2\right)$: Yes, it's a polynomial
- $\mathcal{O}\left(2^n\right)$: No, it's an exponential
- $\mathcal{O}\left(n\log(n)\right)$: Yes, $n\log(n) \in \mathcal{O}\left(n^2\right)$, which is a polynomial
- $\mathcal{O}\left(n^{10000000}\right)$: Technically yes...
- $\mathcal{O}\left(30000000000000n^3\right)$: Technically yes...

What is an "efficient algorithm"?

Question: Are $n^{10000000}$ and $30000000000000n^3$ *actually* efficient in practice?

No, but...

- Once we find a polynomial algorithm to a problem, we've historically been able to improve it to something reasonable
- Finding a polynomial runtime is a *VERY* low bar. If we can't even get that...

Examples of problems

Pretty much all problems we've studied have efficient solutions!

We've studied two main types of algorithms: sorting algorithms and graph algorithms, and every one we've looked at so far could run in polynomial time.

(e.g "How do I sort this list", "What is the shortest path", "What is the MST"...)

Examples of problems

Great: do all solvable problems have efficient solutions?

Haha, no.

Well, ok – do all *practical* problems we actually care about have efficient solutions?

lol

PATH vs LONGEST-PATH

PATH

Given a graph and two vertices, does there exist some path between those two vertices that visits exactly k edges?

- To solve, run BFS and see if we visit the dest in k edges.
- We can solve this efficiently!

What if we tweak the problem a little?

LONGEST-PATH

Given a graph, does there exist a path between **any** two vertices that visits exactly k edges?

There is no known efficient solution to this problem.

To solve, use brute force.

2-COLOR vs 3-COLOR

2-COLOR

Given a graph, is it possible to assign each node one of two colors such that no two adjacent nodes share the same color?

- To solve, run BFS or DFS, alternate colors...
- We can solve this efficiently!

What if we tweak the problem a little?

3-COLOR

Given a graph, is it possible to assign each node one of **three** colors such that no two adjacent nodes share the same color?"

There is no known efficient solution to this problem.

To solve, use brute force: try all $\mathcal{O}\left(3^{|V|}\right)$ combinations.

CIRCUIT-VALUE vs CIRCUIT-SAT

CIRCUIT-VALUE

Given a boolean expression such as "`a && (b || c)`" and the truth values for every variable, is the final expression T?

- To solve, convert into an abstract syntax tree and evaluate.
- We can solve this efficiently!

CIRCUIT-SAT

Given a boolean expression such as "`a && (b || c)`" and the truth values for **some** of the variables, is there a way to set the remaining variables so that the output is T?

There is no known efficient solution to this problem.

To solve, use brute force: try every combination of variables.

Complexity classes

Observation: Some problems have polynomial solutions, some have worse.

Can we formalize this?

Complexity class

A **complexity class** is a set of problems limited by some resource constraint (time, space, etc)

Complexity class: P and EXP

The complexity class P

P is the set of all decision problems where there exists an algorithm that can solve all inputs in worst-case polynomial time.

Examples: IS-PRIME, IS-SORTED, PATH, 2-COLOR, CIRCUIT-VALUE, ...

The complexity class EXP

EXP is the set of all decision problems where there exists an algorithm that can solve all inputs in worst-case exponential time.

Examples: LONGEST-PATH, 3-COLOR, CIRCUIT-SAT...

Is P a subset of EXP?

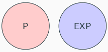
Question: Suppose we have some random decision problem in P. Is that problem also in EXP?

E.g. is 2-COLOR in EXP?

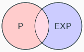
Is P a subset of EXP?

There are three reasonable possibilities:

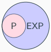
Answer 1: The sets are disjoint
E.g. if a problem is in P, it's not in EXP.



Answer 2: The sets overlap
E.g. some, but not all problems in P are in EXP



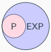
Answer 3: P is a subset of EXP
All problems in P are also in EXP



Is P a subset of EXP?

It turns out it's answer 3: P is a subset of EXP.

Answer 3: P is a subset of EXP
All problems in P are also in EXP



Reason: EXP is the set of decision problems where there exists an algorithm that solves the problem in *worst-case exponential time*.

So, if we can find a polynomial-time algorithm to a problem, we can definitely find an exponential one!

Is P a subset of EXP?

Example: We previously showed there exists an $\mathcal{O}(n)$ algorithm to check if a number n is prime:

```
boolean isPrimeSolver(n):
    for (int i = 2; i < n; i++):
        if (X % i == 0):
            return false
    return true
```

So IS-PRIME $\in$ P.

How do we show that IS-PRIME is in EXP?

```
boolean isPrimeSolver2(n):
    for (int i = 0; i < Math.pow(2, n); i++):
        print("lol")

    return isPrimeSolver(n)
```

This runs in exponential time and correctly solves all inputs.
So IS-PRIME is also in EXP.

Recap

To recap:

- What is a decision problem?
  - What does it mean to "solve" a decision problem?
  - What does it mean for an algorithm to be "efficient"?
- What is a complexity class?
  - P
  - EXP
  - P is a subset of EXP
- **Unfortunately, some problems we care about are in EXP**

A glimmer of hope...

Observation: Some problems in EXP have an interesting property:

- They may take either polynomial or exponential time to *solve*, but either way...
- *Checking or verifying* if a solution is correct always takes polynomial time!

Big idea: NP is the set of decision problems that can be verified in polynomial time.

If we can *verify* answers efficiently, can we *find* answers efficiently?

Solving vs verifying

Reminder: a solver is an algorithm that accepts an *instance* of a decision-problem and returns true or false.

Another kind of algorithm – a *verifier*

Verifier

A **verifier** accepts as input:

1. Some instance of the decision problem
2. Some sort of “proof” or *certificate* of why the solver made whatever decision it made on that instance.

The complexity class NP

The complexity class NP

Suppose that we have some decision problem X where...

- There exists some solver for X
- That solver says “yes” for some instance of X
- Whenever the solver says “yes”, it also returns some sort of “proof” or *certificate* of why they said “yes”.

If there exists a verifier that...

- When given the instance and the certificate, always agrees the correct answer was “yes”
- Always runs in polynomial time

...then X is in NP.

The complexity class co-NP

Important note: The verifier only needs to exist when the solver says “yes”.

If the solver says “no”, we don’t care.

A related complexity class: co-NP. Almost identical to NP, except for “NO” instances.

The complexity class co-NP

The complexity class co-NP

Suppose that we have some decision problem X where...

- There exists some solver for X
- That solver says **“no”** for some instance of X
- Whenever the solver says “no”, it also returns some sort of “proof” or *certificate* of why they said **“no”**.

If there exists a verifier that...

- When given the instance and the certificate, always agrees the correct answer was **“no”**
- Always runs in polynomial time

...then X is in **co-NP**.

Example: showing 3-COLOR is in NP

I claim that 3-COLOR is in NP. How do we show this?

Step 1: Assume the preconditions are met.

Suppose we have a magical solver for 3-COLOR, and it says “yes” for some graph G.

Step 2: Show that we can build a polynomial-time verifier, given G and some certificate.

Three things we must do:

1. How do we modify the solver so it returns a convincing certificate?
2. How do we check the certificate, whatever it is?
3. Does our verifier actually run in polynomial time?

Example: showing 3-COLOR is in NP

Part 2a: What would be a convincing certificate?

A map of vertices to colors! E.g. $\{v_1 = \text{red}, v_2 = \text{blue}, v_3 = \text{red}, v_4 = \text{green}, \ldots\}$.

Part 2b: How do we double-check this certificate?

Loop through all vertices, make sure neighbors have diff colors!

```
boolean verify3Color(G, colorMap):
    for (v : G.vertices):
        for (w : v.neighbors):
            if (colorMap.get(v) == colorMap.get(w)):
                return false
    return true
```

Part 2c: Does this verifier run in polynomial time?

Yes! It runs in $\mathcal{O}(|V| + |E|)$ time!

So, 3-COLOR $\in$ NP.

Example: showing CIRCUIT-SAT is in NP

Question: is `CIRCUIT-SAT` in NP?

`CIRCUIT-SAT`

Given a boolean expression such as "`a && (b || c)`" and the truth values for **some** of the variables, is there a way to set the remaining variables so that the output is T?

As before, assume you have a magical solver, and it said "yes" for some boolean expression B.

Three questions to answer:

1. How do we modify the solver so it returns a convincing certificate?
2. How do we check the certificate, whatever it is?
3. Does our verifier actually run in polynomial time?