CSE 373: Disjoint sets continued

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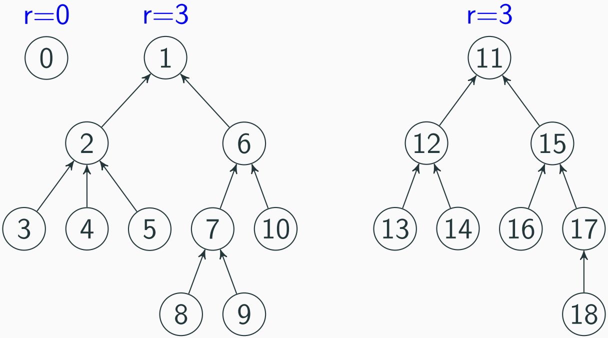
Friday, Mar 2, 2018

Warmup

Consider the following disjoint set.

What happens if we run `findSet(8)` then `union(4, 17)`?

Note: the `union(...)` method internally calls `findSet(...)`.



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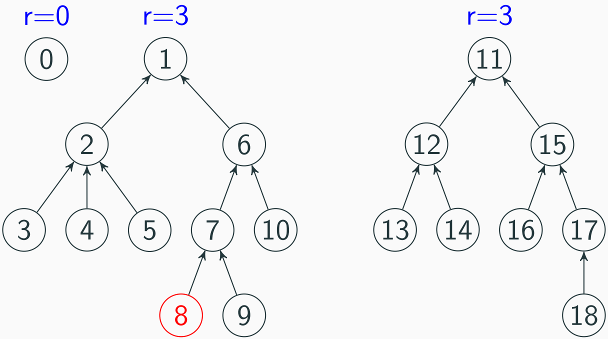

Step 1: We find the node corresponding to 8 in $\mathcal{O}(1)$ time

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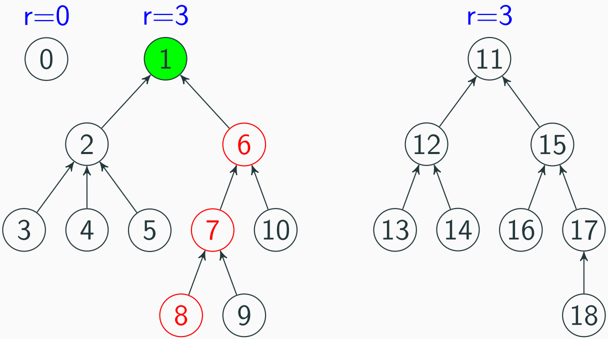

Step 2: We travel up the tree until we find the root

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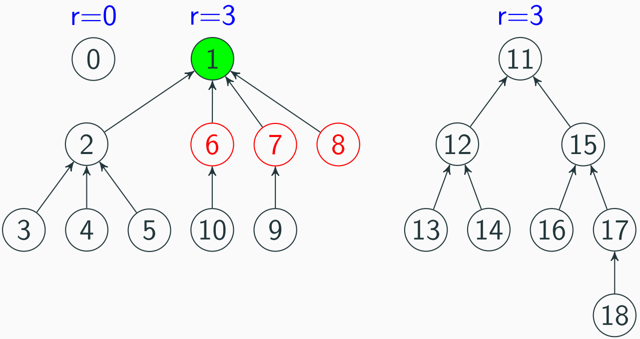
What happens when we run `findSet(8)`?



Step 3: We move each node we passed by (every red node) to point directly at the root.

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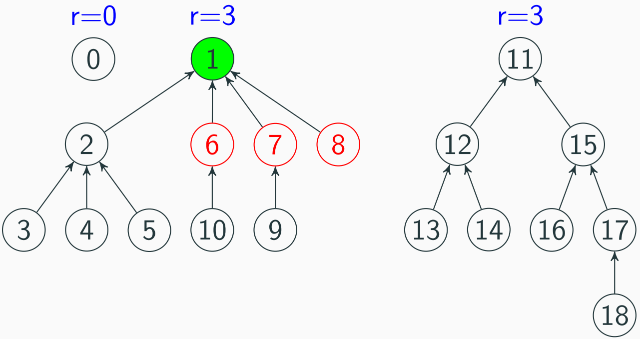
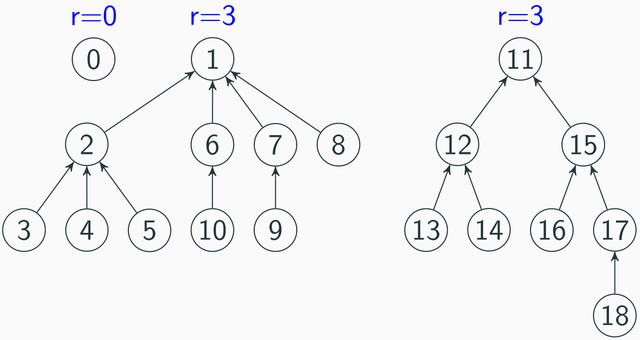


Step 3: We move each node we passed by (every red node) to point directly at the root.

Note: we do not update the rank (too expensive)

Warmup

What happens if we run `union(4, 17)`?



link(findSet(4), findSet(17))

Warmup

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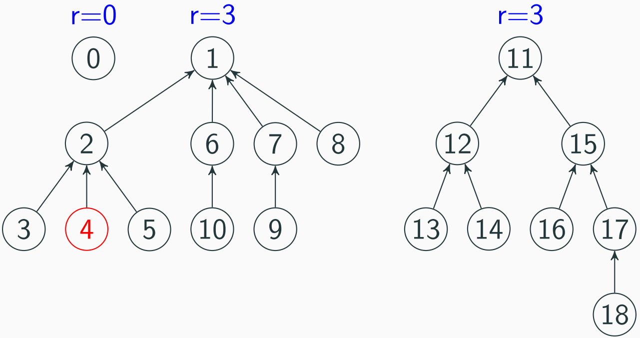

Step 1: We first run `findSet(4)`

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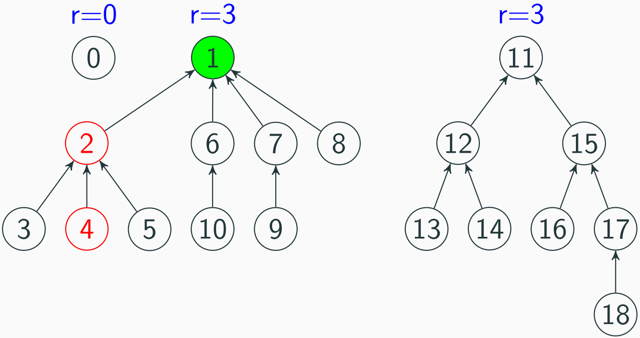

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So we need to crawl up and find the parent...

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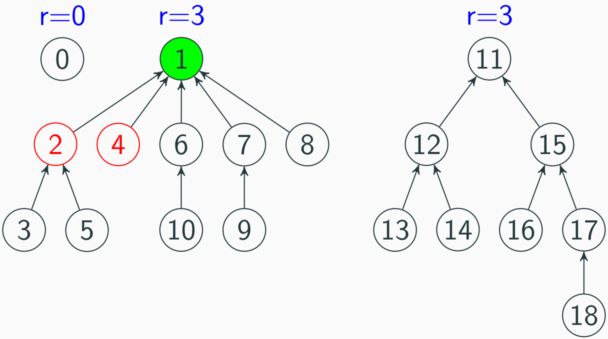


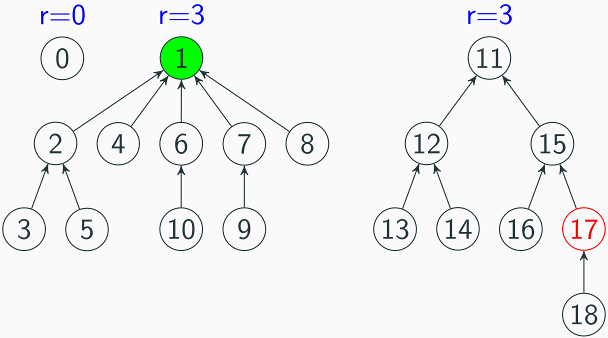
Step 1: We first run `findSet(4)`.

So we need to crawl up and find the parent...

...and make node "4" point directly at the root.

Warmup

What happens if we run `union(4, 17)`?



Step 2: We next run `findSet(17)` and repeat the process.

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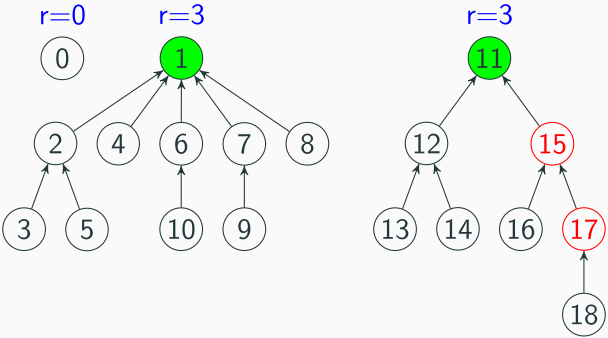

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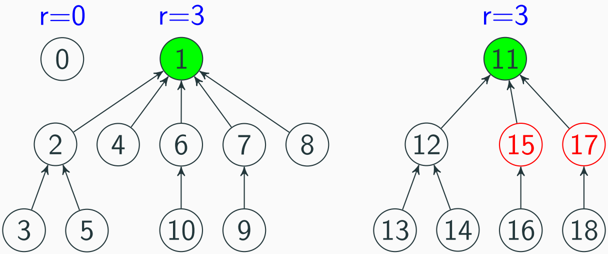

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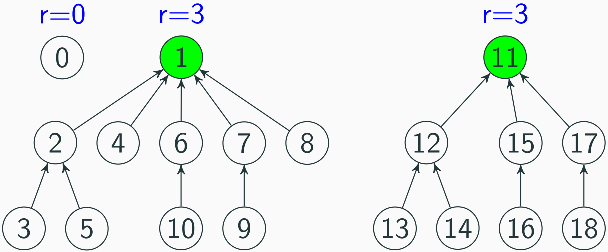
We've finished `findSet(4)` and `findSet(17)`, so now we need to finish the rest of `union(4, 17)` by linking the two trees together.



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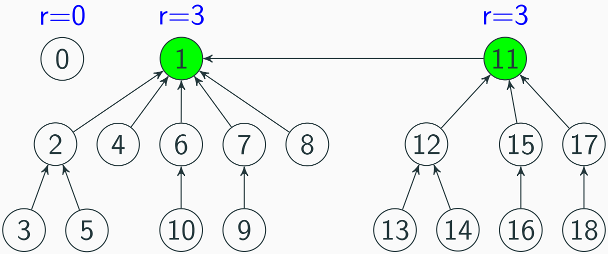

The ranks are the same, so we arbitrarily make set 1 the root and make set 11 the child.

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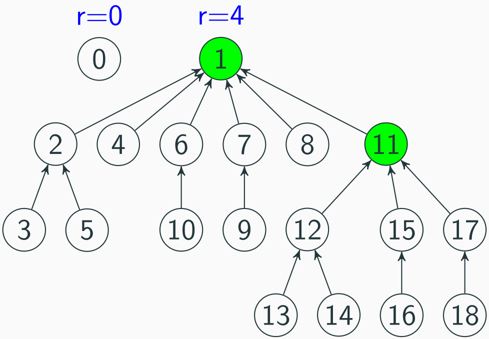

We then update the rank of set 1 and "forget" the rank of set 11.

Path compression: runtime

Now, what are the worst-case and best-case runtime of the following?

- **`makeSet(x)`**:
- **`findSet(x)`**:
- **`union(x, y)`**:

Path compression: runtime

Now, what are the worst-case and best-case runtime of the following?

- **`makeSet(x)`**:
  $\mathcal{O}(1)$ – still the same
- **`findSet(x)`**:
  In the best case, $\mathcal{O}(1)$, in the worst case $\mathcal{O}(\log(n))$
- **`union(x, y)`**:
  In the best case, $\mathcal{O}(1)$, in the worst case $\mathcal{O}(\log(n))$

Back to Kruskal's

Why are we doing this? To help us implement Kruskal's algorithm:

```
def kruskal():
    for (v : vertices):
        makeMST(v)

    sort edges in ascending order by their weight

    mst = new SomeSet<Edge>()
    for (edge : edges):
        if findMST(edge.src) != findMST(edge.dst):
            union(edge.src, edge.dst)
            mst.add(edge)

    return mst
```

- `makeMST(v)` takes $\mathcal{O}(t_m)$ time
- `findMST(v)`: takes $\mathcal{O}(t_f)$ time
- `union(u, v)`: takes $\mathcal{O}(t_u)$ time

Back to Kruskal's

We concluded that the runtime is:

$$\mathcal{O}\left(\underbrace{|V| \cdot t_m}_{\text{setup}} + \underbrace{|E| \cdot \log(|E|)}_{\text{sorting edges}} + \underbrace{|E| \cdot t_f + |V| \cdot t_u}_{\text{core loop}}\right)$$

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Well, we just said that in the worst case:

- $t_m \in \mathcal{O}(1)$
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So the worst-case overall runtime of Kruskal's is:

$$\mathcal{O}(|V| + |E| \cdot \log(|E|) + (|E| + |V|) \cdot \log(|V|))$$

Back to Kruskal's

Our worst-case runtime:

$$\mathcal{O}\left(|V| + |E| \cdot \log(|E|) + (|E| + |V|) \cdot \log(|V|)\right)$$

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One minor improvement: since our edge weights are numbers, we can likely use a *linear sort* and improve the runtime to:

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We can drop the $|V| + |E|$ (they're dominated by the last term):

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...and we're left with something that's basically the same as Prim.

Disjoint-sets, amortized analysis

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How much of a difference does that make?

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Observation: each call to `findSet(x)` improves all future calls.
How much of a difference does that make?

Interesting result:

It turns out `union` and `find` are *amortized* $\log^*(n)$.

Iterated log

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What does this mean?

Interlude: repeated exponentiation

Observation:

- Multiplication is a shorthand for repeated addition*

$$2 \times 5 = 2 + 2 + 2 + 2$$

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$$2^5 = 2 \times 2 \times 2 \times 2 \times 2$$

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$$2 \,??\, 5 = 2^{2^{2^{2^2}}}$$

- Why stop there – is there a way of expressing repeated whatever-it-is-we-did up above?

$$2 \,??!!???\, 5 = 2 \,??\, 2 \,??\, 2 \,??\, 2 \,??\, 2$$

*assuming we use only integers

Interlude: Knuth's up-arrow notation

Yes – it's called **Knuth's up-arrow notation**

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$$2 \uparrow 5 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 16$$

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- Two arrows means *repeated exponentiation* – tetration

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- Three arrows means *repeated tetration*

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- etc...

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- $\log(...)$ is the inverse of $\uparrow$ (exponentiation)

$$\log_2(2 \uparrow 5) = \log_2(2^5) = 5$$

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$$\log_2(2 \uparrow 5) = \log_2(2^5) = 5$$

- $\log^*(...)$ is the inverse of $\uparrow\uparrow$ (tetration)

$$\log^*_2(2 \uparrow\uparrow 5) = \log^*_2(2^{2^{2^{2^2}}}) = 5$$

Up-arrows and iterated log

A slightly modified definition:

Iterated log

The expression $\log_b^*(n)$ is equivalent to the number of times we repeatedly compute $\log_b(x)$ to bring x down to at most 1.

This is equivalent to the inverse of $b \uparrow\uparrow x$.

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What does this look like?

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- $\log^*(2 \uparrow\uparrow 4) = \log^*(2^{2^{2^2}}) = \log(\log(\log(\log(65536)))) = 4$

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- $\log^*(2 \uparrow\uparrow 4) = \log^*(2^{2^{2^2}}) = \log(\log(\log(\log(65536)))) = 4$
- $\log^*(2 \uparrow\uparrow 5) = \log^*(2^{2^{2^{2^2}}}) =$ $\log(\log(\log(\log(\log(2^{65536}))))) = 5$

A big number

And what exactly is 2^{65536}?

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$= 2003529930406846464979072351560255750447825475569751419$
$2650169737108940595563114530895061308809333481010382343429072$
$6318182294938211881266886950636476154702916504187191635158796$
$6347219442930927982084309104855990570159318959639524863372367$
$2030029169695921561087649488892540908059114570376752085002066$
$7156370236612635974714480711177481588091413574272096719015183$
$6282560618091458852699826141425030123391108273603843767876449$
$0432059603791244909057075603140350761625624760318637931264847$
$0374378295497561377098160461441330869211810248595915238019533$
$1030292162800160568670105651646750568038741529463842244845292$
$5373614425336143737290883037946012747249584148649159306472520$
$1515569392262818069165079638106413227530726714399815850881129$
$2628901134237782705567421080070065283963322155077831214288551$

A big number

6755540733451072131124273995629827197691500548839052238043570
4584819795639315785351001899200002414196370681355984046403947
2194016069517690156119726982337890017641517190051133466306898
1402193834814354263873065395529696913880241581618595611006403
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7117818204001872829399434461862243280098373237649318147898481
1945271300744022076568091037620399920349202390662626449190916

A big number

7985461515778839060397720759279378852241294301017458086862263
3692847258514030396155585643303854506886522131148136384083847
7826379045960718687672850976347127198889068047824323039471865
0525660978150729861141430305816927924971409161059417185352275
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A big number

6340696503084422585596703927186946115851379338647569974856867
0079823960604393478850861649260304945061743412365828352144806
7266768418070837548622114082365798029612000274413244384324023
3125740354501935242877643088023285085588608996277445816468085
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5039549383718534059000961874894739928804324963731657538036735
8671017578399481847179849824694806053208199606618343401247609

A big number

6639519778021441199752546704080608499344178256285092726523709
8986515394621930046073645079262129759176982938923670151709920
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7783905630048248379983969202922221548614590237347822268252163
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A big number

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A big number

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A big number

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A big number

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A big number

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A big number

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A big number

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A big number

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A big number

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A big number

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A big number

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A big number

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A big number

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A big number

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A big number

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Runtime of Kruskal?

$$\mathcal{O}\left((|E| + |V|) \log^*(|V|)\right) \leq \mathcal{O}\left((|E| + |V|)5\right) \approx \mathcal{O}\left(|E| + |V|\right)$$

Inverse of the Ackermann function

But wait!

Somebody then came along and proved an even tighter bound. It turns out `findSet(...)` and `union(...)` are amortized $\mathcal{O}\left(\alpha(n)\right)$ – the inverse of the Ackermann function.

The Ackermann function

The Ackermann function is a recursive function designed to grow extremely quickly:

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0 \\ A(m - 1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m - 1, A(m, n - 1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

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So, the runtime of Kruskal's is even better! It's

$$\mathcal{O}\left((|E| + |V|)\alpha(|V|)\right) \le \mathcal{O}\left((|E| + |V|)4\right)$$

...for any practical size of $|V|$.

Are we done yet?

But wait, there's more!

Recap

To recap, we found that the runtimes of `findSet(...)` and `union(...)` were...

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- After applying **path compression**, $\mathcal{O}(\alpha(n)) \approx \mathcal{O}(1)$
- One final optimization: **array representation**.
  It doesn't lead to an asymptotic improvement, but it does lead to a constant factor speedup (and simplifies implementation).

Array representation

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For example:

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Observation: It seems wasteful to have allocate an entire object just to store two fields

Array representation

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- The int field takes up 4 bytes
- The pointer to the parent takes up 8 bytes (assuming 64-bit)
- The object itself also uses up an additional 16 bytes
- This adds up to 28, but in a 64 bit computer, we always "pad" or round up to the nearest multiple of 8. So, this object will use up 32 bytes of memory.

Array representation

Idea: Just use an array of ints instead!

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Core idea:

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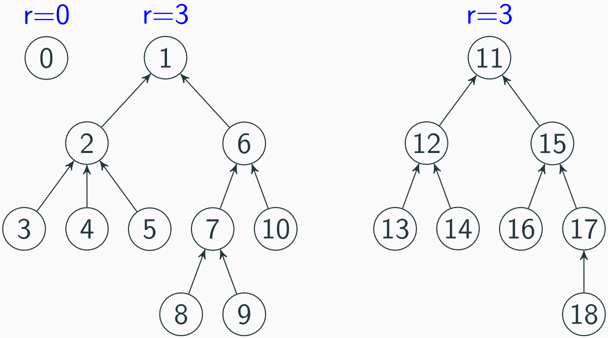
- Make the element in the array be the index of the parent

Array representation

Example:



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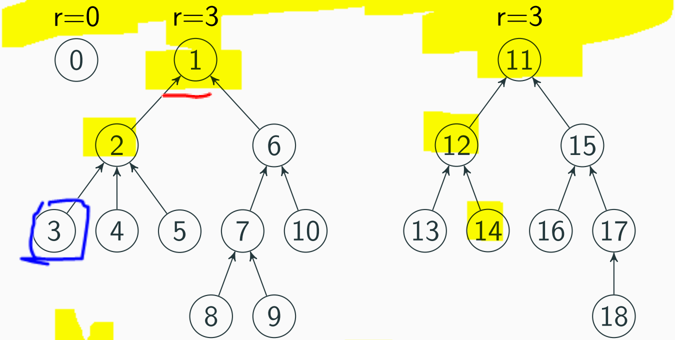
Example:



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
-	-	1	2	2	2	1	6	7	7	7	-	11	12	12	11	15	15	17

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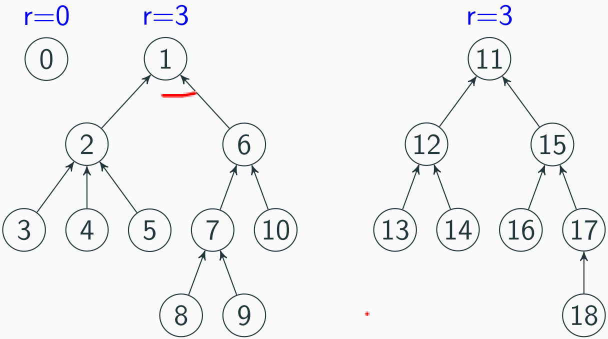
Idea 1: Rather then leaving the root cells empty, just stick the ranks there.

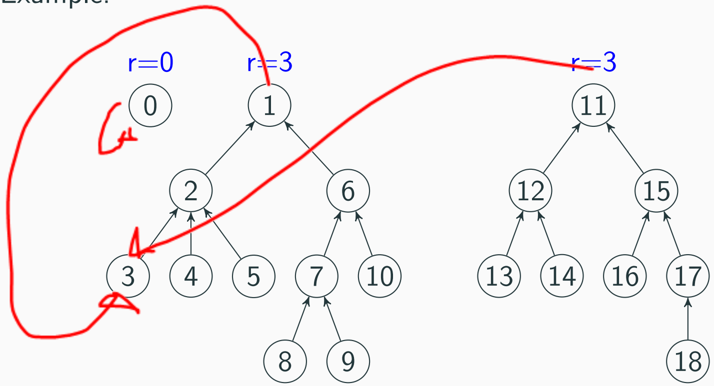
Array representation

Example:



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	3	1	2	2	2	1	6	7	7	7	3	11	12	12	11	15	15	17

Example:



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	3	1	2	2	2	1	6	7	7	7	3	11	12	12	11	15	15	17

What's wrong with this idea?

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Problem: How do we tell whether a number is supposed to be a rank or an index to the parent?

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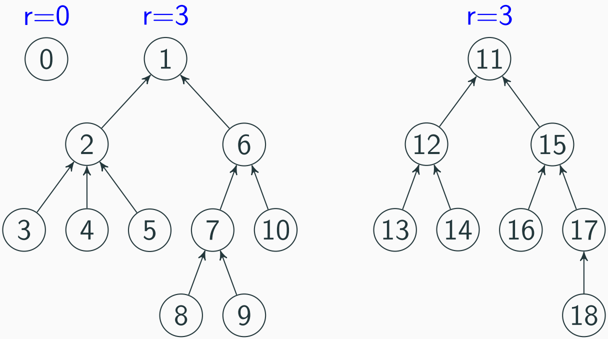
A trick: Rather then storing just the rank, let's store the negative of the rank!

So, if a number is positive, it's an index. If the number is negative, it's a rank (and that node is a root).

Array representation

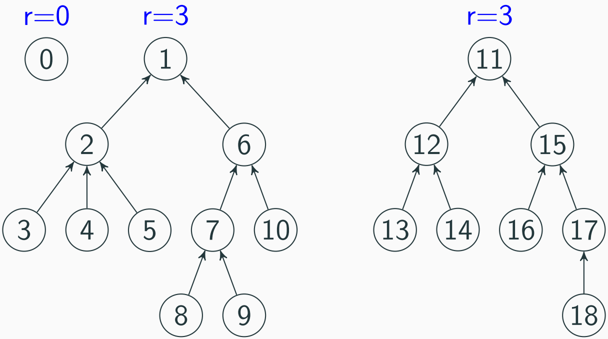
Example:



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
-0	-3	1	2	2	2	1	6	7	7	7	-3	11	12	12	11	15	15	17

Array representation

Example:



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
-0	-3	1	2	2	2	1	6	7	7	7	-3	11	12	12	11	15	15	17

What's wrong with this idea?

Problem: What's the difference between 0 and -0?

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Solution: Instead of just storing $-\text{rank}$, store $-\text{rank} - 1$.

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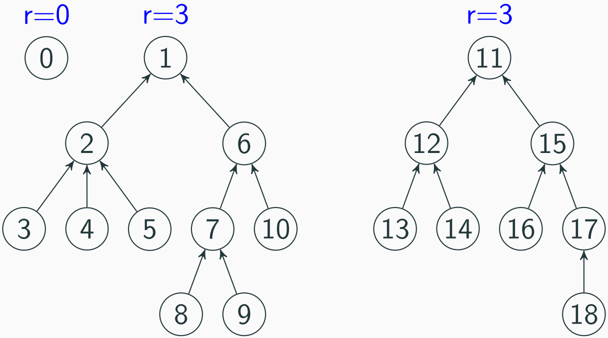
(Alternatively, redefine the rank to be the upper bound of the number of *levels* in the tree, rather then the *height*.)

Array representation

Example:



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
-1	-4	1	2	2	2	1	6	7	7	7	-4	11	12	12	11	15	15	17

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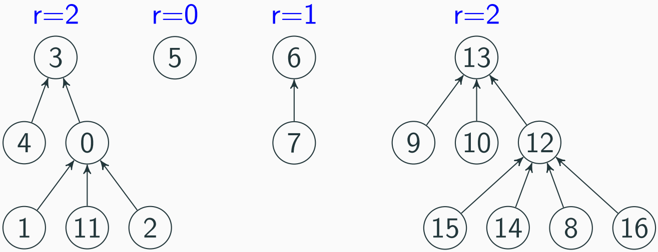
Now you try – what is the array representation of this disjoint set?



Array representation

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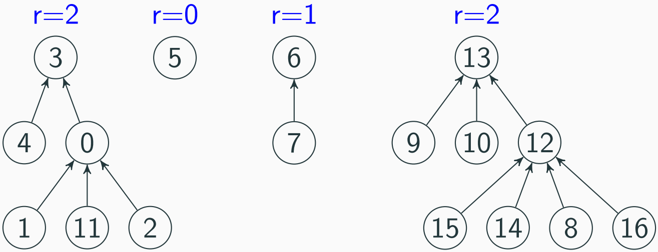

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	0	0	-3	3	-1	-2	6	12	13	13	0	13	-3	12	12	12

Recap

And that's it for graphs. Topics covered:

- Graph definitions, graph representations

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- Finding the shortest path: Dijkstra's algorithm
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- Minimum spanning trees: Prim's and Kruskal's
- Disjoint sets

Recap

Next time: What does it mean for a problem to be "hard"?