CSE 373: Disjoint sets

Michael Lee
Wednesday, Feb 28, 2018
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- **Kruskal’s algorithm:**
  Loop over edges, from smallest to largest. Use the edge only if it doesn’t introduce a cycle.
Kruskal’s algorithm: example with a weighted graph

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Kruskal’s algorithm: example with a weighted graph

Example of the algorithm:
Kruskal’s algorithm: analysis

Runtime analysis:

```python
def kruskal():
    for (v : vertices):
        makeMST(v)

    sort edges in ascending order by their weight

    mst = new SomeSet<Edge>()
    for (edge : edges):
        if findMST(edge.src) != findMST(edge.dst):
            union(edge.src, edge.dst)
            mst.add(edge)

    return mst
```

Note: assume that...

- `makeMST(v)` takes $O(t_m)$ time
- `findMST(v)`: takes $O(t_f)$ time
- `union(u, v)`: takes $O(t_u)$ time
Kruskal’s algorithm: analysis

- Making the $|V|$ MSTs takes $O(|V| \cdot t_m)$ time.
- Sorting the edges takes $O(|E| \cdot \log(|E|))$ time, assuming we use a general-purpose comparison sort.
- The final loop takes $O(|E| \cdot t_f + |V| \cdot t_u)$ time.
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Putting it all together:

$$O (|V| \cdot t_m + |E| \cdot \log(|E|) + |E| \cdot t_f + |V| \cdot t_u)$$
But wait, what exactly is $t_m$, $t_f$, and $t_u$? How exactly do we implement $\text{makeMST}(v)$, $\text{findMST}(v)$, and $\text{union}(u, v)$?
But wait, what exactly is $t_m$, $t_f$, and $t_u$? How exactly do we implement makeMST(v), findMST(v), and union(u, v)?

We can do so using a new ADT called the DisjointSet ADT!
Review: what is a set?

- A set is a “bag” of elements arranged in no particular order.
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We implemented a set in project 2: ChainedHashSet

Interesting note: sets come up all the time in math.
The DisjointSet ADT

Properties of a disjoint-set data structure:

- A disjoint-set data structure maintains a collection of many different sets.
The DisjointSet ADT

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What is a representative? Any sort of unique “identifier”.

Examples:
- We could pick some arbitrary element in the set to be the “representative”
- We could assign each set some unique integer id.
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print(findSet(d))
union(a, c)
union(b, d)
print(findSet(a) == findSet(c))
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union(c, b)
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What operations does a disjoint-set **NOT** support?

Answer: The ability to actually get the entire set. We can make a set, check if an item is in a set, and combine two sets, but we don’t have a built-in way of getting the entire set itself.

Insight: The few operations we need to support, the more creative our implementation can be. (If the client really wants the sets, they can get it themselves in $O(n)$ time – how?)
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(If the client really wants the sets, they can get it themselves in $O(n)$ time – how?)
So, how do we implement these?

Core idea:
▶ We represent each set as a tree
▶ The disjoint-set keeps track of a "forest" of trees

Intuitions:
▶ We want union-ing to be cheap. Combining two trees is cheap; we just manipulate pointers.
▶ We want a single "representative" per set. A tree has a single root!
DisjointSet: implementation

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High-level overview:

- **makeSet(x)**: Adds a new tree (of size 1) to our “forest”
- **findSet(x)**: Looks up the node, then finds root of tree
- **union(x, y)**: Combines two trees into one
Suppose we call makeSet(...) on 0 through 5.
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Each makeSet(...) adds a new tree to our “forest”.

Note that right now, each tree has only one element.
Suppose we call union(3, 5).

We combine those two trees into one.

Assumption: we have an $O(1)$ way of getting each node. (E.g. maintain a hashmap of numbers to node objects.)

Question: how do we implement findSet(...)?

Once we find a node, move upwards until we're looking at root.

Then, return the root's data field.
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Algorithm:
Find the roots of both trees and add one tree as a subchild of the other.
Which tree becomes the new root? For now, pick randomly.
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Now, suppose we call `union(2, 4)`. What happens?

![Diagram showing two trees after union operation]

- Step 3: We nest one tree inside the other.
Now, suppose we call union(2, 4). What happens?

Step 1: We look up 2 and 3
Now, suppose we call union(2, 4). What happens?

Step 2: We find the roots of 2 and 3
Now, suppose we call `union(2, 4)`. What happens?

```
    0
   /\  \
  /   \ /
1     2 3
    / \  /\  \
   /   \ /   \  \
  5     4
```

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What’s the worst-case runtime of our methods?
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Better question: are our trees guaranteed to be balanced?
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Hint: When union-ing, we pick which tree is nested randomly. Does that guarantee we’ll get a balanced tree?
The worst-case scenario:

0  1  2  3  4  5
DisjointSet: Analysis

The worst-case scenario:

Possible outcome of calling union(0, 1)
The worst-case scenario:

Possible outcome of calling `union(0, 2)`
DisjointSet: Analysis

The worst-case scenario:

Possible outcome of calling `union(0, 3)`
The worst-case scenario:

Possible outcome of calling `union(0, 4)`
The worst-case scenario:

Possible outcome of calling union(0, 5)
So, what are the worst-case runtimes?

- **makeSet(x):**

- **findSet(x):**

- **union(x, y):**
So, what are the worst-case runtimes?

- **makeSet(x):**
  \( \mathcal{O}(1) \) – creating the tree takes constant time

- **findSet(x):**
  \( \mathcal{O}(n) \) – if it’s a linked list, we need to traverse \( n \) elements!

- **union(x, y):**
  \( \mathcal{O}(n) \) – union calls findSet(...) on both elements

...where \( n \) is the total number of items added to the disjoint-set.
How can we improve disjoint sets?

1. **Union-by-rank:** Strategy to make sure trees are balanced.
2. **Path compression:** Hijack `findSet(x)` and make it do a little extra work to improve overall performance.
3. **Array representation:** Takes advantage of cache locality, simplifies implementation, etc.
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3. **Array representation:**
   Takes advantage of cache locality, simplifies implementation, etc.
Problem: Our trees could be unbalanced

Solution:

Let $\text{rank}(x)$ be a number representing the upper-bound of the height of $x$. So, $\text{rank}(x) \geq \text{height}(x)$.

We then...

1. Keep track of the rank of all trees.
2. When unioning, make the tree with the larger rank the root!
3. If it's a tie, pick one randomly and increase the rank by one.

(Why not keep track of the height? When we look at path compression, keeping track of the height becomes more challenging.)
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(Why not keep track of the height? When we look at path compression, keeping track of the height becomes more challenging.)
Example: Suppose we call union(1, 5)?

The tree with the root of “6” has the larger rank, so we make it the root.

Note: we’re not really “removing” the rank from node 0 – it’s just irrelevant, so we’re ignoring it and omitting it from the diagram to save space. We only care about the ranks at the roots.
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Example: Suppose we call union(5, 11)?

Here, there’s a tie. We break the tie arbitrarily, and increment the rank of the new tree by one.
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Here, there’s a tie. We break the tie arbitrarily, and increment the rank of the new tree by one.
Net effect? Our trees stay relatively balanced.

So, what are the worst-case runtimes now?

- **makeSet(x):**

- **findSet(x):**

- **union(x, y):**
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So, what are the worst-case runtimes now?

- **makeSet(x):**
  \( O(1) \) – still the same

- **findSet(x):**
  \( O(\log(n)) \) – since the tree is balanced

- **union(x, y):**
  \( O(\log(n)) \) – since union calls findSet
Path compression

Consider the following forest:

1
  2
  3
  11

7
  5
  6
  10
  13

4
  8
  9
  14
  0

Suppose we call findSet(3) a few hundred times. Why do we have to keep finding the root again and again?
Consider the following forest:

![Tree Diagram]

Suppose we call `findSet(3)` a few hundred times.
Consider the following forest:

Suppose we call `findSet(3)` a few hundred times.

Why do we have to keep finding the root again and again?
**Observation:** To find root, we must also traverse these nodes:

![Diagram showing nodes and arrows indicating traversal paths.]
**Path compression**

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What if, next time, we could just jump straight to the root?
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What if, next time, we could just jump straight to the root?

Same for the other nodes we visited
So, let’s do it!

Now what happens if we try calling `findSet(3)`?
Path compression

So, let’s do it!

Now what happens if we try calling $\text{findSet}(3)$?
So, let’s do it!

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```
Now what happens if we try calling findSet(3)?
```
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Now what happens if we try calling findSet(3)?
One additional note: path compression changes the heights of our trees.

This means it could be the case that rank $\neq$ height.

Is this a problem?
One additional note: path compression changes the heights of our trees.

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Is this a problem?

**Answer:** No; proof is beyond the scope of this class.
Now, what are the worst-case and best-case runtime of the following?

- **makeSet(x):**
  - $O(1)$ – still the same

- **findSet(x):**
  - In the best case, $O(1)$, in the worst case $O(\log(n))$

- **union(x, y):**
  - In the best case, $O(1)$, in the worst case $O(\log(n))$
Now, what are the worst-case and best-case runtime of the following?

- **makeSet(x):**
  \( \mathcal{O}(1) \) – still the same

- **findSet(x):**
  In the best case, \( \mathcal{O}(1) \), in the worst case \( \mathcal{O}(\log(n)) \)

- **union(x, y):**
  In the best case, \( \mathcal{O}(1) \), in the worst case \( \mathcal{O}(\log(n)) \)
Why are we doing this? To help us implement Kruskal’s algorithm:

```python
def kruskal():
    for (v : vertices):
        makeMST(v)

    sort edges in ascending order by their weight

    mst = new SomeSet<Edge>()
    for (edge : edges):
        if findMST(edge.src) != findMST(edge.dst):
            union(edge.src, edge.dst)
            mst.add(edge)

    return mst
```

- `makeMST(v)` takes $O(t_m)$ time
- `findMST(v)` takes $O(t_f)$ time
- `union(u, v)` takes $O(t_u)$ time
We concluded that the runtime is:

\[ O \left( |V| \cdot t_m + |E| \cdot \log(|E|) + |E| \cdot t_f + |V| \cdot t_u \right) \]

- Setup: \(|V| \cdot t_m\)
- Sorting edges: \(|E| \cdot \log(|E|)\)
- Core loop: \(|E| \cdot t_f + |V| \cdot t_u\)
We concluded that the runtime is:

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Well, we just said that in the worst case:

- \( t_m \in O(1) \)
- \( t_f \in O(\log(|V|)) \)
- \( t_u \in O(\log(|V|)) \)
We concluded that the runtime is:

\[ \mathcal{O} \left( \left| V \right| \cdot t_m + \left| E \right| \cdot \log(\left| E \right|) + \left| E \right| \cdot t_f + \left| V \right| \cdot t_u \right) \]

Well, we just said that in the worst case:

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- \( t_f \in \mathcal{O}(\log(|V|)) \)
- \( t_u \in \mathcal{O}(\log(|V|)) \)

So the worst-case overall runtime of Kruskal’s is:

\[ \mathcal{O} \left( \left| V \right| + \left| E \right| \cdot \log(\left| E \right|) + (\left| E \right| + \left| V \right|) \cdot \log(\left| V \right|) \right) \]
Our worst-case runtime:

\[ O(|V| + |E| \cdot \log(|E|) + (|E| + |V|) \cdot \log(|V|)) \]
Back to Kruskal’s

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One minor improvement: since our edge weights are numbers, we can likely use a linear sort and improve the runtime to:

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...and we’re left with something that’s basically the same as Prim’s algorithm.
Disjoint-sets, amortized analysis

...or are we?

Observation: each call to findSet(x) improves all future calls. How much of a difference does that make?

Interesting result: It turns out union and find are amortized $\log^* n$. 
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Iterated log

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Example:

- $\log^*(2) = \log(2) = 1$
- $\log^*(4) = \log(\log(4)) = 2$
- $\log^*(8) = \log(\log(\log(8))) = 3$
- $\log^*(65536) = \log^*(2^{2^{2^2}}) = 4$
- $\log^*(2^{65536}) = \ldots = 5$
What is $2^{65536}$?

$2^{65536}$ =

2003529930406846464979072351560255750447825475569751419
2650169737108940595563114530895061308809333481010382343429072
6318182294938211881266886950636476154702916504187191635158796
6347219442930927982084309104855990570159318959639524863372367
2030029169695921561087649488892540908059114570376752085002066
7156370236612635974714480711177481588091413574272096719015183
6282560618091458852699826141425030123391108273603843767876449
0432059603791244909057075603140350761625624760318637931264847
0374378295497561377098160461441330869211810248595915238019533
1030292162800160568670105651646750568038741529463842244845292
5373614425336143737290883037946012747249584148649159306472520
1515569392262818069165079638106413227530726714399815850881129
2628901134237782705567421080070065283963322155077831214288551
A big number

4376370598692891375715374000198639433246489005254310662966916
5243419174691389632476560289415199775477703138064781342309596
1909606545913008901888875880847336259560654448885014473357060
5881709016210849971452956834406197969056546981363116205357936
9791403236328496233046421066136200220175787851857409162050489
7117818204001872829399434461862243280098373237649318147898481
1945271300744022076568091037620399920349202390662626449190916
7985461515778839060397720759279378852241294301017458086862263
3692847258514030396155585643303854506886522131148136384083847
7826379045960718687672850976347127198889068047824323039471865
0525660978150729861141430305816927924971409161059417185352275
8875044775922183011587807019755357222414000195481020056617735
8978149953232520858975346354700778669040642901676380816174055
0405117670093673202804549339027992491867306539931640720492238
4748152806191669009338057321208163507076343516698696250209690
A big number

6340696503084422585596703927186946115851379338647569974856867
0079823960604393478850861649260304945061743412365828352144806
72667684180708375458622114082365798029612000274413244384324023
3125740354501935242877643088023285085588608996277445816468085
7875115807014743763867976955049991643998284357290415378143438
8473034842619033888414940313661398542576355771053355802066221
8557706008255128889333222643628198483861323957067619140963853
3832374343758830859233722284644287996245605476932428998432652
6773783731732880632107532112386806046747084280511664887090847
7029120816110491255559832236624486855665140268464120969498259
0565519216188104341226838996283071654868525536914850299539675
5039549383718534059000961874894739928804324963731657538036735
8671017578399481847179849824694806053208199606618343401247609
6639519778021441199752546704080608499344178256285092726523709
8986515394621930046073645079262129759176982938923670151709920
...I got tired of copying and pasting, but we’re not even a fourth of the way through.
...I got tired of copying and pasting, but we’re not even a fourth of the way through.

Punchline? $\log^*(n) \leq 5$, for basically any reasonable value of $n$. 
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Runtime of Kruskal? $O((|E| + |V|) \log^*(|V|)) \approx O(|E| + |V|)$
Inverse of the Ackerman function

But wait!

Somebody then came along and proved that find and union are amortized $\mathcal{O}(\alpha(n))$ – the inverse of the Ackermann function.

This grows even more slowly than $\log^*(n)$!