CSE 373: Disjoint sets

Michael Lee

Wednesday, Feb 28, 2018

Review

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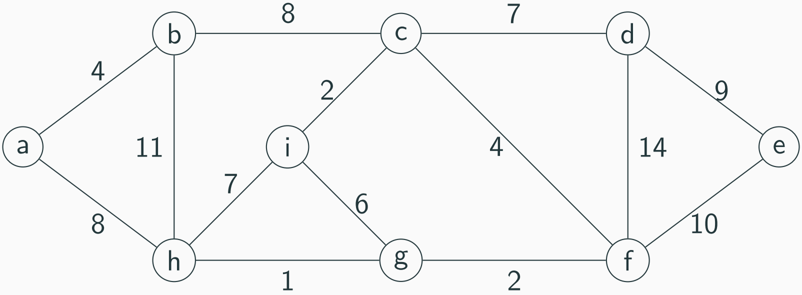
- **Prim's algorithm:**
  Nearly identical to Dijkstra's, except we use the distance to any already-visited node as the cost.
- **Kruskal's algorithm:**
  Loop over edges, from smallest to largest. Use the edge only if it doesn't introduce a cycle.

Kruskal's algorithm: example with a weighted graph

Example of the algorithm:



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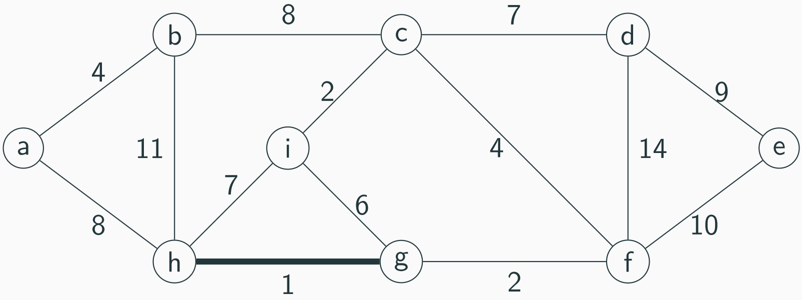

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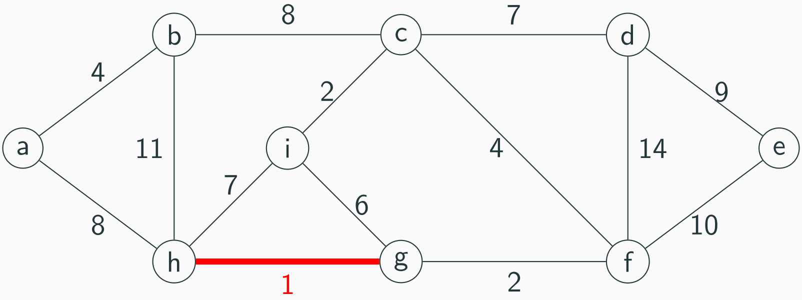

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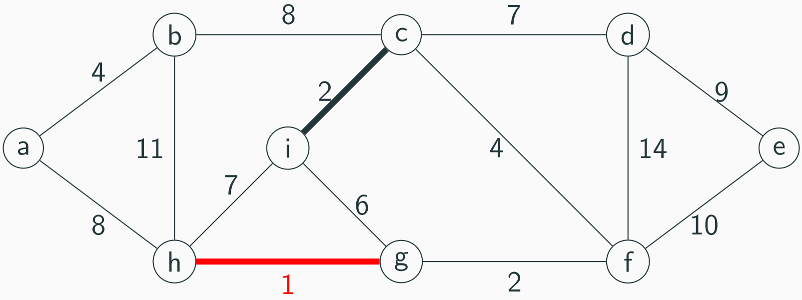

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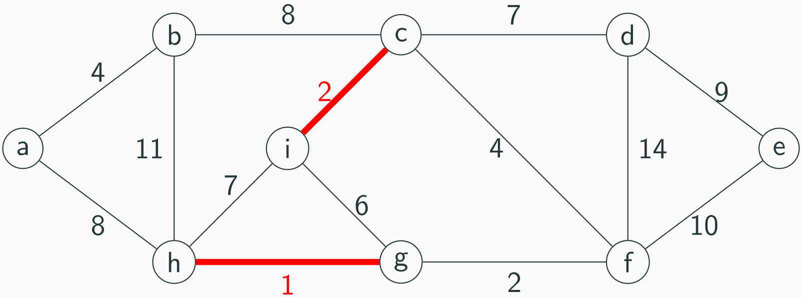

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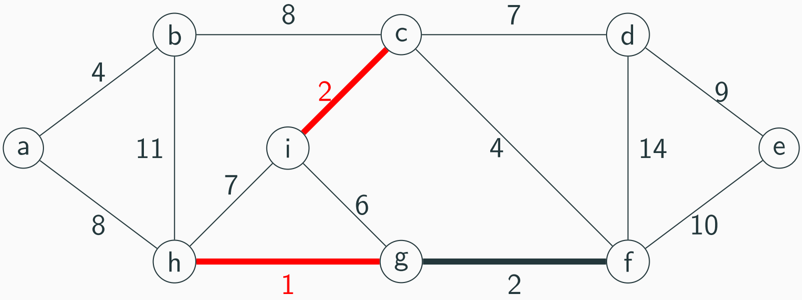

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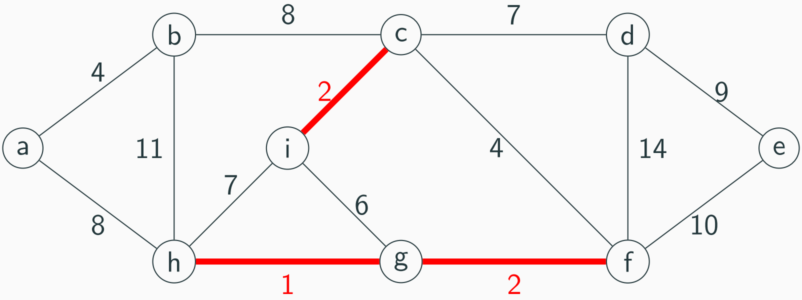

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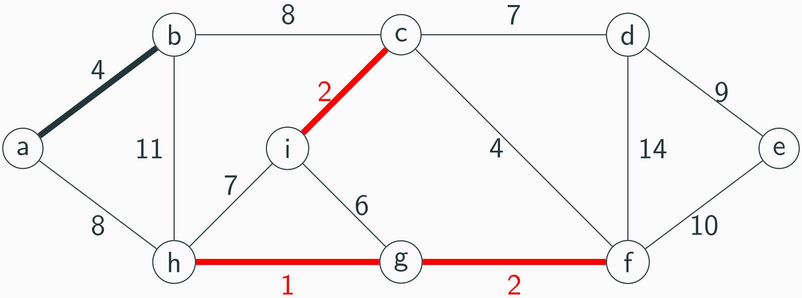

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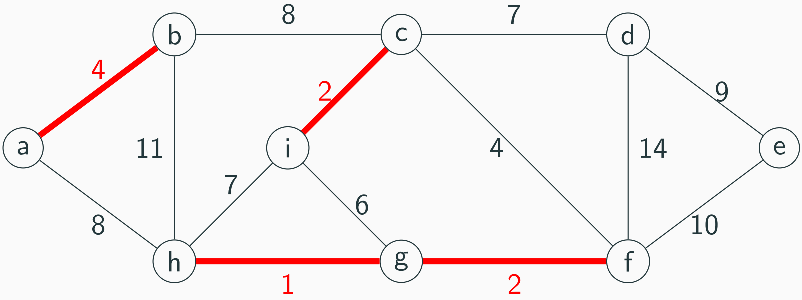

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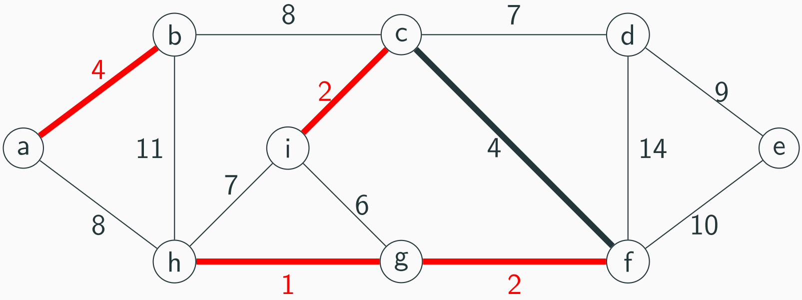

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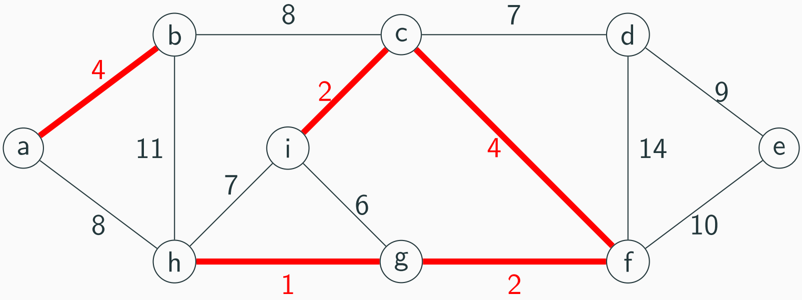

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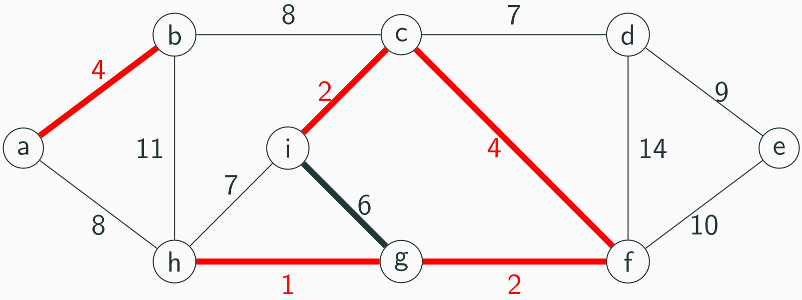

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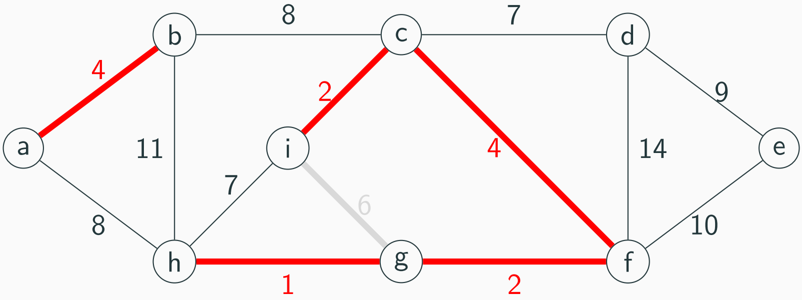

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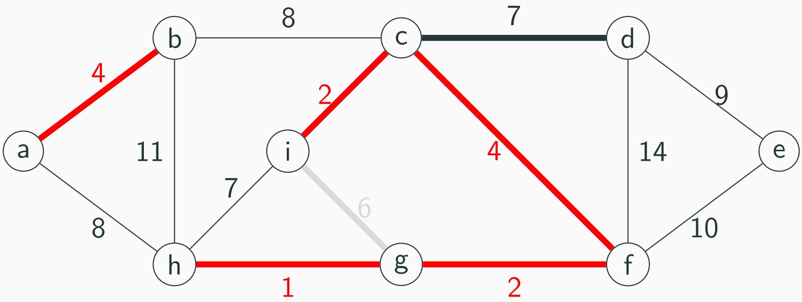

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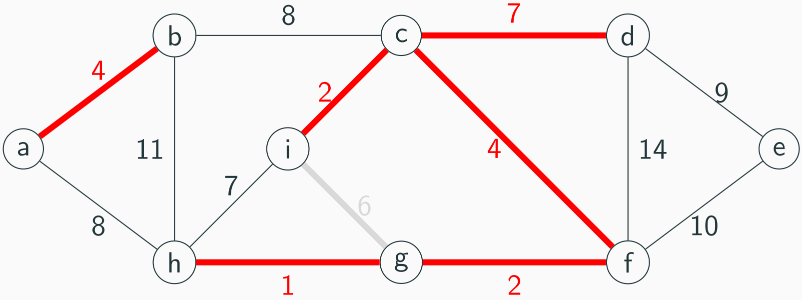

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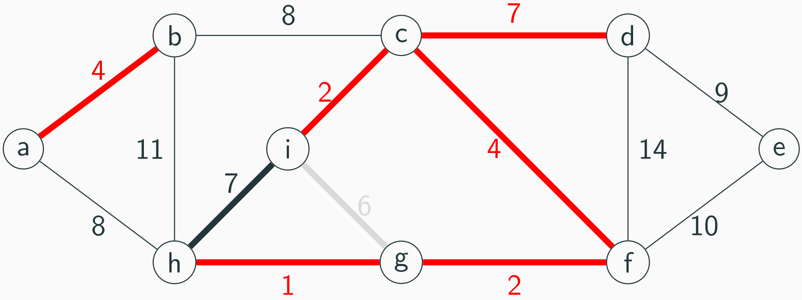

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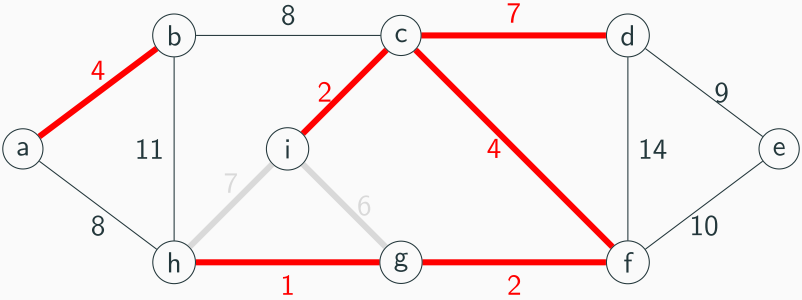

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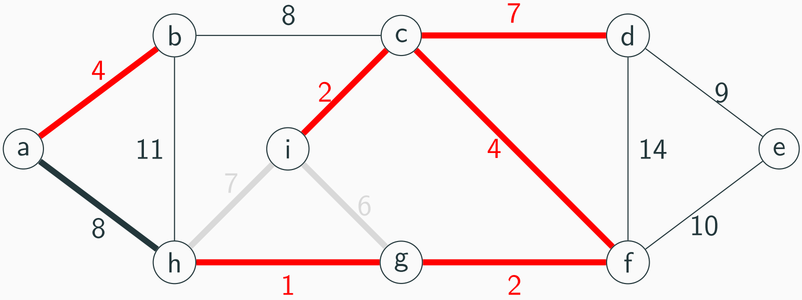

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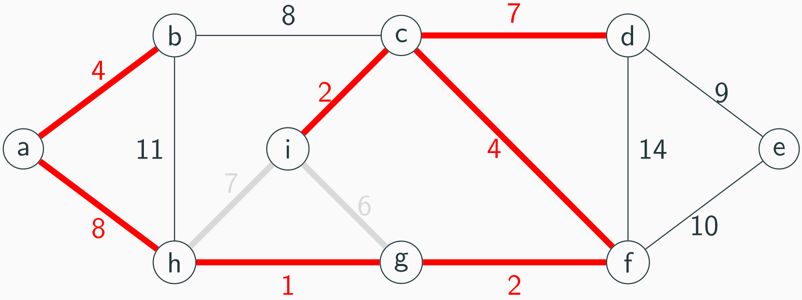

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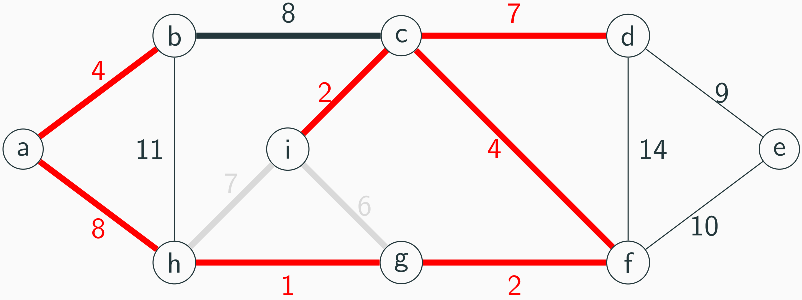

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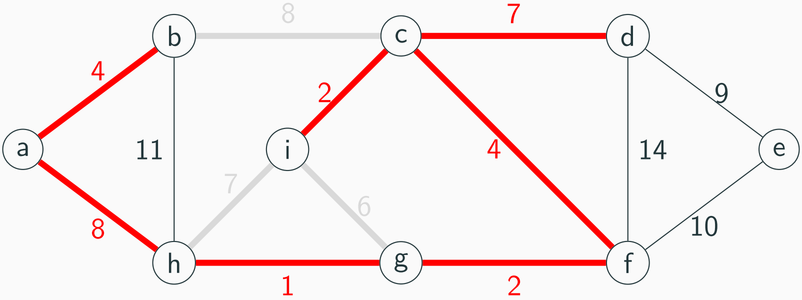

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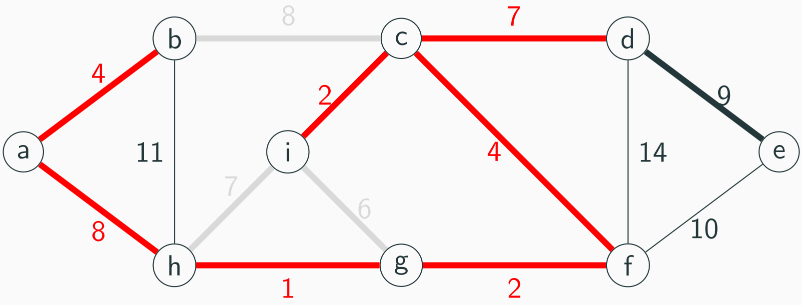

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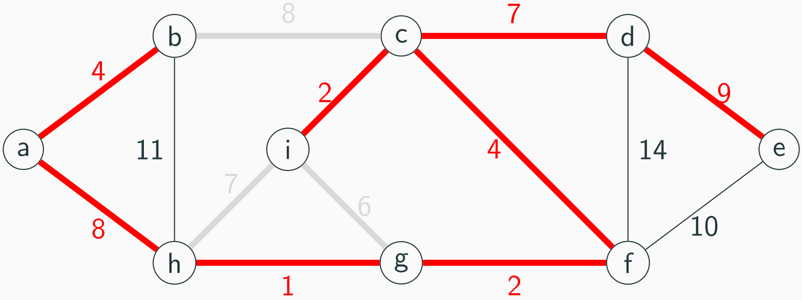

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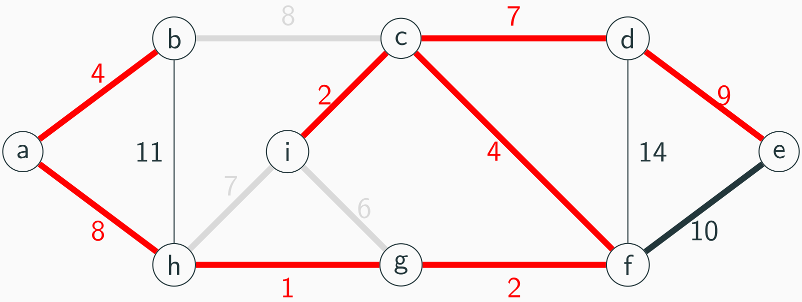

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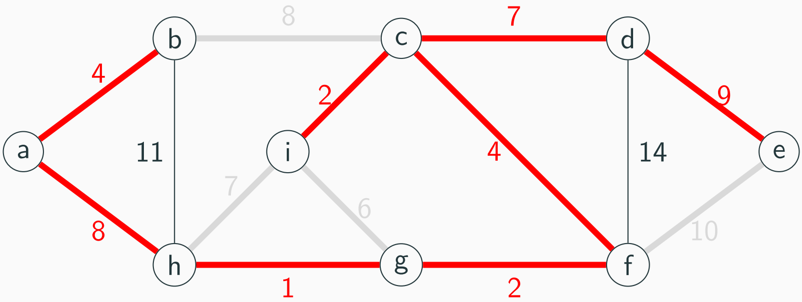

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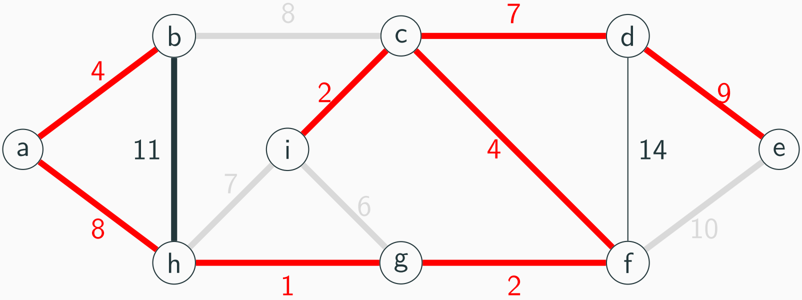

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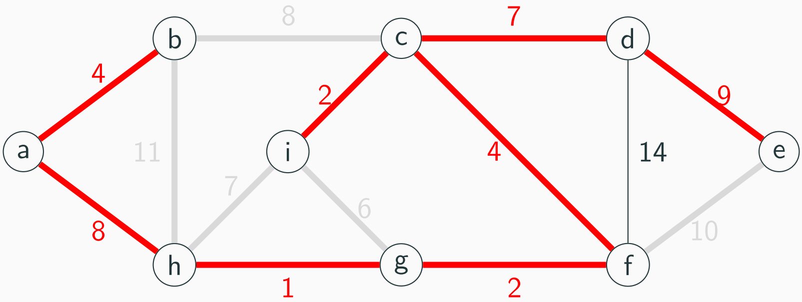

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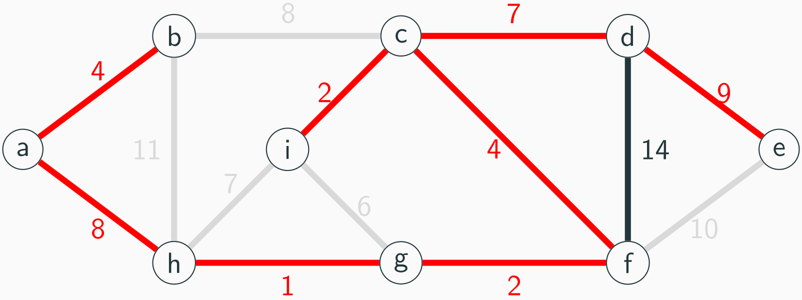

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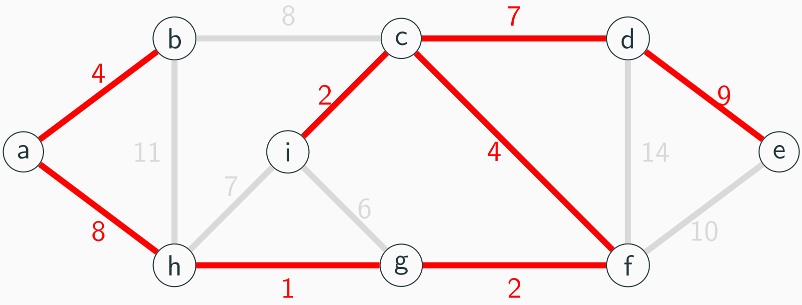

Kruskal's algorithm: analysis

Runtime analysis:

```
def kruskal():
    for (v : vertices):
        makeMST(v)

    sort edges in ascending order by their weight

    mst = new SomeSet<Edge>()
    for (edge : edges):
        if findMST(edge.src) != findMST(edge.dst):
            union(edge.src, edge.dst)
            mst.add(edge)

    return mst
```

Note: assume that...

- `makeMST(v)` takes $\mathcal{O}(t_m)$ time
- `findMST(v)`: takes $\mathcal{O}(t_f)$ time
- `union(u, v)`: takes $\mathcal{O}(t_u)$ time

Kruskal's algorithm: analysis

- Making the $|V|$ MSTs takes $\mathcal{O}(|V| \cdot t_m)$ time
- Sorting the edges takes $\mathcal{O}(|E| \cdot \log(|E|))$ time, assuming we use a general-purpose comparison sort
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Putting it all together:

$$\mathcal{O}(|V| \cdot t_m + |E| \cdot \log(|E|) + |E| \cdot t_f + |V| \cdot t_u)$$

The DisjointSet ADT

But wait, what exactly is t_m, t_f, and t_u? How exactly do we implement `makeMST(v)`, `findMST(v)`, and `union(u, v)`?

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We can do so using a new ADT called the `DisjointSet` ADT!

Interlude: What is a set?

Review: what is a set?

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Interesting note: sets come up all the time in math.

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  Examples:
  - We could pick some arbitrary element in the set to be the "representative"
  - We could assign each set some unique integer id.

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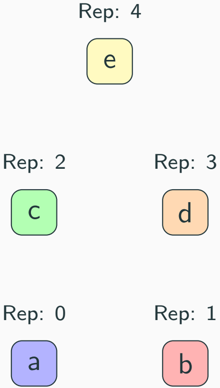

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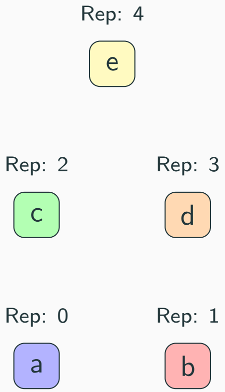

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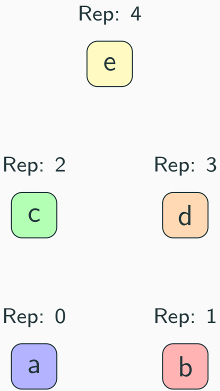

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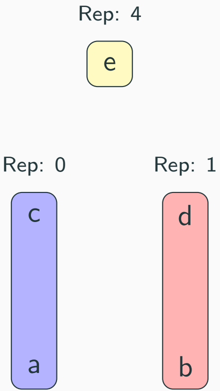

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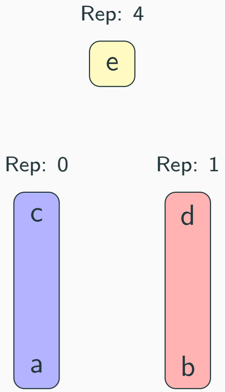

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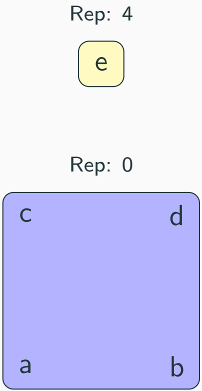

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Answer: The ability to actually get the entire set.

We can *make* a set, *check* if an item is in a set, and *combine* two sets, but we don't have a built-in way of *getting* the entire set itself.

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(If the client really wants the sets, they can get it themselves in $\mathcal{O}(n)$ time – how?)

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- We want union-ing to be cheap.
  Combining two trees is cheap; we just manipulate pointers.
- We want a single "representative" per set.
  A tree has a single root!

DisjointSet: implementation

High-level overview:

- **`makeSet(x)`**: Adds a new tree (of size 1) to our "forest"
- **`findSet(x)`**: Looks up the node, then finds root of tree
- **`union(x, y)`**: Combines two trees into one

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Suppose we call `makeSet(...)` on 0 through 5.

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Each `makeSet(...)` adds a new tree to our "forest".

Note that right now, each tree has only one element.

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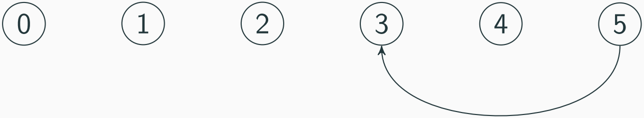

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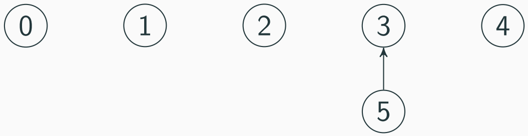

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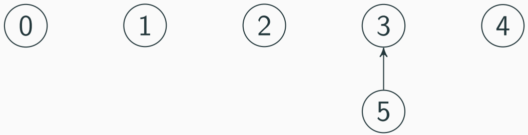

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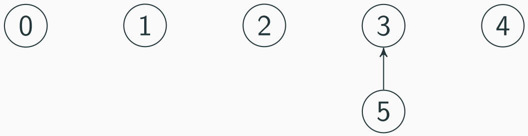
Assumption: we have an $\mathcal{O}(1)$ way of getting each node.
(E.g. maintain a hashmap of numbers to node objects.)

DisjointSet: implementation

Suppose we call `union(3, 5)`.



We combine those two trees into one.

Assumption: we have an $\mathcal{O}(1)$ way of getting each node.
(E.g. maintain a hashmap of numbers to node objects.)

Question: how do we implement `findSet(...)`?

DisjointSet: implementation

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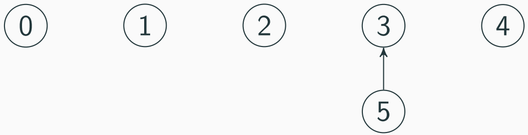

We combine those two trees into one.

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(E.g. maintain a hashmap of numbers to node objects.)

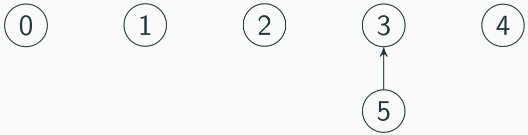
Question: how do we implement `findSet(...)`?

Once we find a node, move upwards until we're looking at root.

Then, return the root's data field.

DisjointSet: implementation

Suppose we call `union(5, 4)`.



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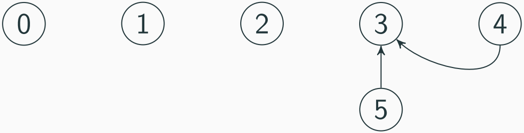

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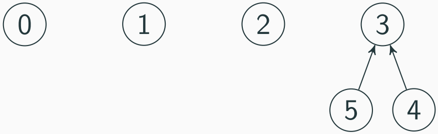
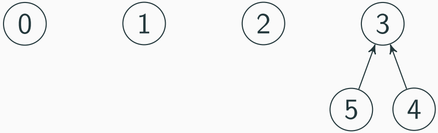


Algorithm: Find the roots of both trees and add one tree as a subchild of the other.

Which tree becomes the new root? For now, pick randomly.

DisjointSet: implementation

Suppose we call `union(0, 1)`, then `union(2, 0)`.



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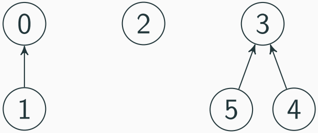

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DisjointSet: implementation

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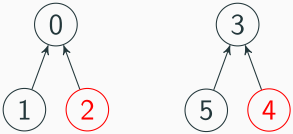
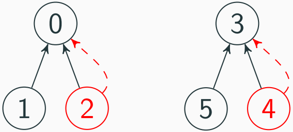


Step 1: We look up 2 and 3

DisjointSet: implementation

Now, suppose we call `union(2, 4)`. What happens?



Step 2: We find the roots of 2 and 3

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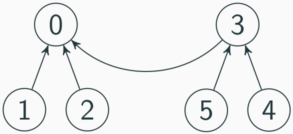
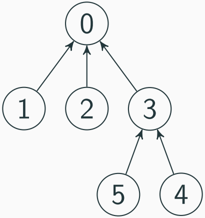


Step 3: We nest one tree inside the other

DisjointSet: implementation

Now, suppose we call `union(2, 4)`. What happens?



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DisjointSet: Analysis

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Hint: When union-ing, we pick which tree is nested randomly.
Does that guarantee we'll get a balanced tree?

DisjointSet: Analysis

The worst-case scenario:

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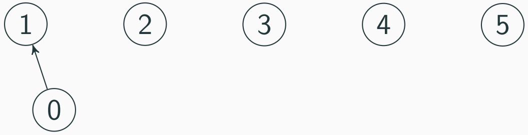

Possible outcome of calling `union(0, 1)`

DisjointSet: Analysis

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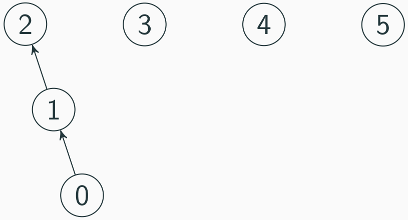
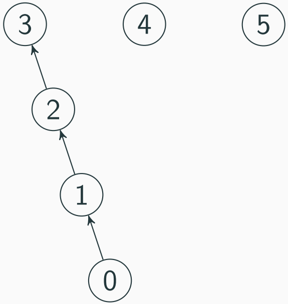


Possible outcome of calling `union(0, 2)`

DisjointSet: Analysis

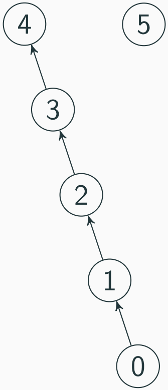
The worst-case scenario:



Possible outcome of calling `union(0, 3)`

DisjointSet: Analysis

The worst-case scenario:



Possible outcome of calling `union(0, 4)`

DisjointSet: Analysis

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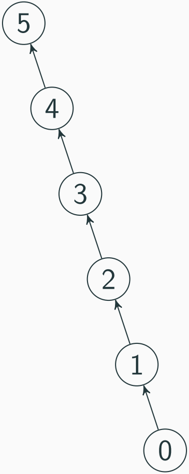

Possible outcome of calling `union(0, 5)`

DisjointSet: implementation

So, what are the worst-case runtimes?

- **`makeSet(x)`**:
- **`findSet(x)`**:
- **`union(x, y)`**:

DisjointSet: implementation

So, what are the worst-case runtimes?

- **`makeSet(x)`**:
  $\mathcal{O}(1)$ – creating the tree takes constant time
- **`findSet(x)`**:
  $\mathcal{O}(n)$ – if it's a linked list, we need to traverse n elements!
- **`union(x, y)`**:
  $\mathcal{O}(n)$ – union calls `findSet(...)` on both elements

...where n is the total number of items added to the disjoint-set.

Improving DisjointSet

How can we improve disjoint sets?

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1. **Union-by-rank:**
   Strategy to make sure trees are balanced
2. **Path compression:**
   Hijack `findSet(x)` and make it do a little extra work to improve overall performance.
3. **Array representation:**
   Takes advantage of cache locality, simplifies implementation, etc.

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Let rank(x) be a number representing the upper-bound of the height of x. So, rank(x) $\geq$ height(x).

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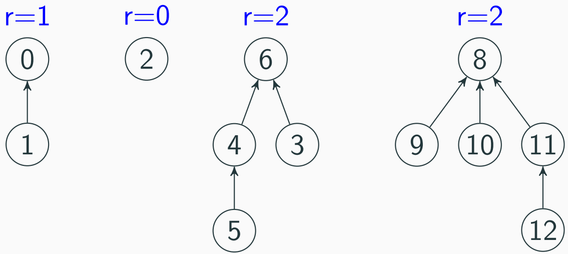
(Why not keep track of the height? When we look at path compression, keeping track of the height becomes more challenging.)

Union-by-rank

Example: Suppose we call `union(1, 5)`?



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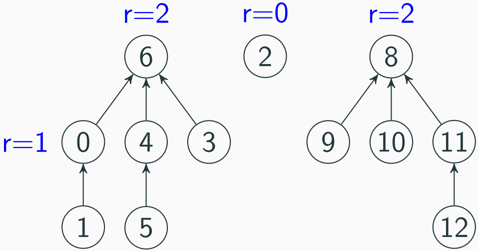

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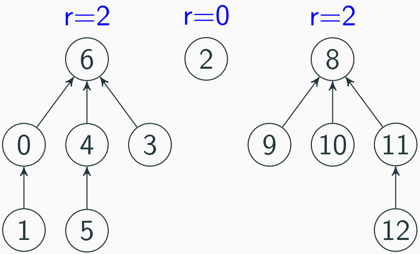
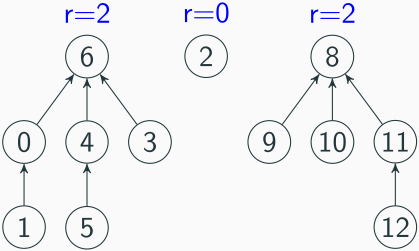


The tree with the root of “6” has the larger rank, so we make it the root.

Note: we’re not really “removing” the rank from node 0 – it’s just irrelevant, so we’re ignoring it and omitting it from the diagram to save space. We only care about the ranks at the roots.

Union-by-rank

Example: Suppose we call `union(5, 11)`?



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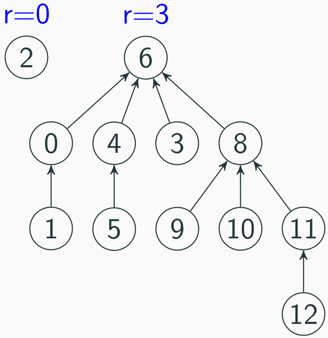

Here, there's a tie. We break the tie arbitrarily, and increment the rank of the new tree by one.

Union-by-rank

Net effect? Our trees stay relatively balanced.

So, what are the worst-case runtimes now?

- **`makeSet(x)`**:
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Union-by-rank

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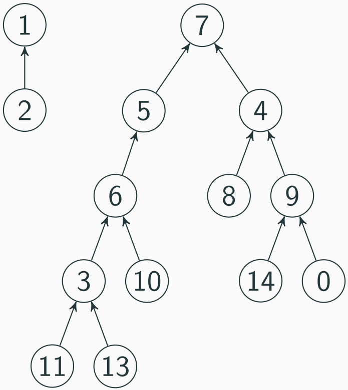
So, what are the worst-case runtimes now?

- **`makeSet(x)`**:
  $\mathcal{O}(1)$ – still the same
- **`findSet(x)`**:
  $\mathcal{O}(\log(n))$ – since the tree is balanced
- **`union(x, y)`**:
  $\mathcal{O}(\log(n))$ – since union calls `findSet`

Path compression

Consider the following forest:



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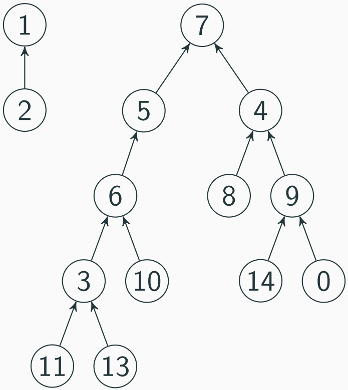

Suppose we call `findSet(3)` a few hundred times.

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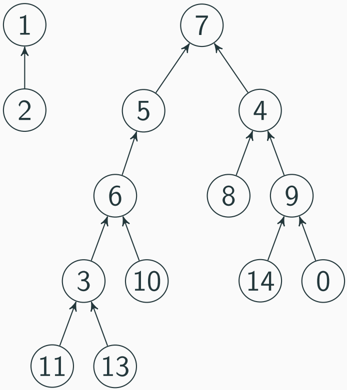
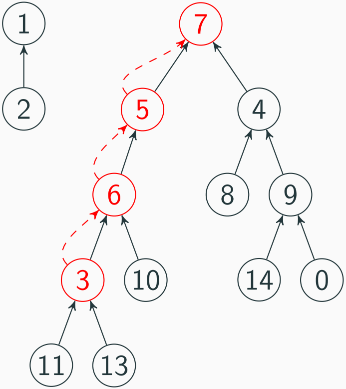


Suppose we call `findSet(3)` a few hundred times.

Why do we have to keep finding the root again and again?

Path compression

Observation: To find root, we must also traverse these nodes:



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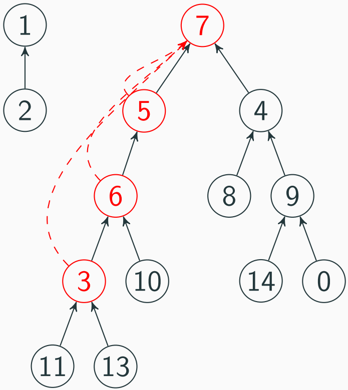

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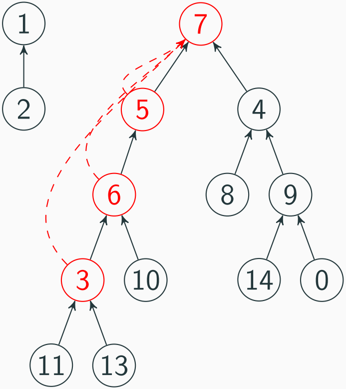
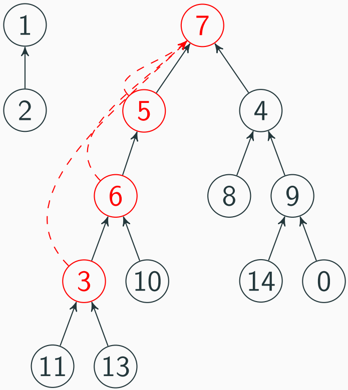


What if, next time, we could just jump straight to the root?

Same for the other nodes we visited

Path compression

So, let's do it!



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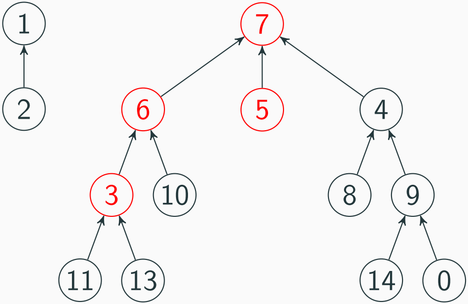

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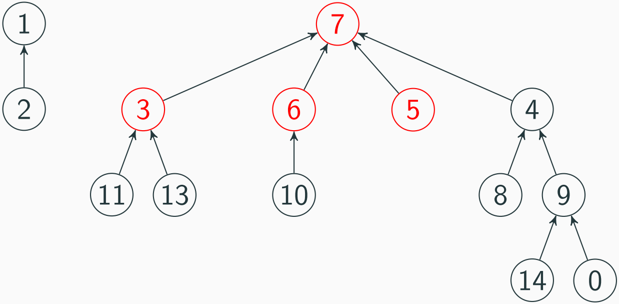

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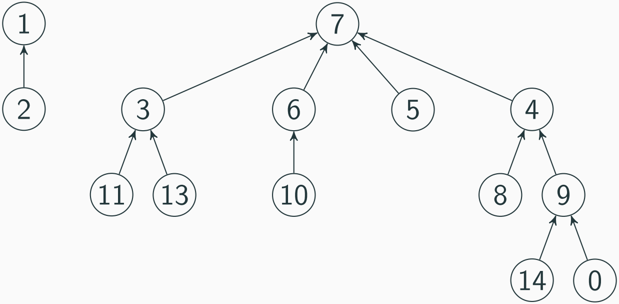

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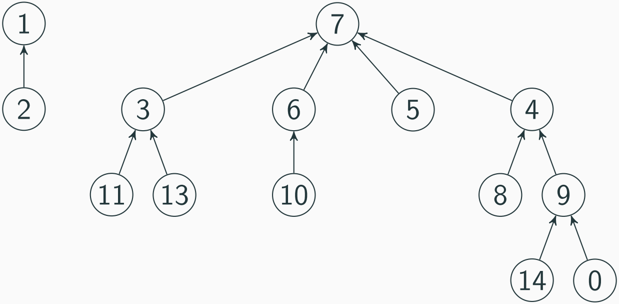

Now what happens if we try calling `findSet(3)`?

Path compression

One additional note: path compression changes the heights of our trees.

This means it could be the case that rank $\neq$ height.

Is this a problem?

Path compression

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This means it could be the case that rank $\neq$ height.

Is this a problem?

Answer: No; proof is beyond the scope of this class

Path compression: runtime

Now, what are the worst-case and best-case runtime of the following?

- **`makeSet(x)`**:
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  In the best case, $\mathcal{O}(1)$, in the worst case $\mathcal{O}(\log(n))$
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Back to Kruskal's

Why are we doing this? To help us implement Kruskal's algorithm:

```
def kruskal():
    for (v : vertices):
        makeMST(v)

    sort edges in ascending order by their weight

    mst = new SomeSet<Edge>()
    for (edge : edges):
        if findMST(edge.src) != findMST(edge.dst):
            union(edge.src, edge.dst)
            mst.add(edge)

    return mst
```

- `makeMST(v)` takes $\mathcal{O}(t_m)$ time
- `findMST(v)`: takes $\mathcal{O}(t_f)$ time
- `union(u, v)`: takes $\mathcal{O}(t_u)$ time

Back to Kruskal's

We concluded that the runtime is:

$$\mathcal{O}\left(\underbrace{|V| \cdot t_m}_{\text{setup}} + \underbrace{|E| \cdot \log(|E|)}_{\text{sorting edges}} + \underbrace{|E| \cdot t_f + |V| \cdot t_u}_{\text{core loop}}\right)$$

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Well, we just said that in the worst case:

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So the worst-case overall runtime of Kruskal's is:

$$\mathcal{O}(|V| + |E| \cdot \log(|E|) + (|E| + |V|) \cdot \log(|V|))$$

Back to Kruskal's

Our worst-case runtime:

$$\mathcal{O}\left(|V| + |E| \cdot \log(|E|) + (|E| + |V|) \cdot \log(|V|)\right)$$

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One minor improvement: since our edge weights are numbers, we can likely use a *linear sort* and improve the runtime to:

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We can drop the $|V| + |E|$, since they're dominated by the last term:

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...and we're left with something that's basically the same as Prim's algorithm.

Disjoint-sets, amortized analysis

…or are we?

Disjoint-sets, amortized analysis

...or are we?

Observation: each call to `findSet(x)` improves all future calls.

How much of a difference does that make?

Disjoint-sets, amortized analysis

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Observation: each call to `findSet(x)` improves all future calls.
How much of a difference does that make?

Interesting result:

It turns out `union` and `find` are *amortized* $\log^*(n)$.

Disjoint-sets, amortized analysis

Iterated log

The expression $\log^*(n)$ is equivalent to the number of times you need to compute $\log(x)$ to bring the value down to at most 1

Disjoint-sets, amortized analysis

Iterated log

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Example:

- $\log^*(2) = \log(2) = 1$
- $\log^*(4) = \log(\log(4)) = 2$
- $\log^*(8) = \log(\log(\log(8))) = 3$
- $\log^*(65536) = \log^*(2^{2^{2^{2}}}) = 4$
- $\log^*(2^{65536}) = \ldots = 5$

A big number

What is 2^{65536}?

$2^{65536} =$

2003529930406846464979072351560255750447825475569751419
2650169737108940595563114530895061308809333481010382343429072
6318182294938211881266886950636476154702916504187191635158796
6347219442930927982084309104855990570159318959639524863372367
2030029169695921561087649488892540908059114570376752085002066
7156370236612635974714480711177481588091413574272096719015183
6282560618091458852699826141425030123391108273603843767876449
0432059603791244909057075603140350761625624760318637931264847
0374378295497561377098160461441330869211810248595915238019533
1030292162800160568670105651646750568038741529463842244845292
5373614425336143737290883037946012747249584148649159306472520
1515569392262818069165079638106413227530726714399815850881129
2628901134237782705567421080070065283963322155077831214288551

A big number

4376370598692891375715374000198639433246489005254310662966916
5243419174691389632476560289415199775477703138064781342309596
1909606545913008901888875880847336259560654448885014473357060
5881709016210849971452956834406197969056546981363116205357936
9791403236328496233046421066136200220175787851857409162050489
7117818204001872829399434461862243280098373237649318147898481
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8978149953232520858975346354700778669040642901676380816174055
0405117670093673202804549339027992491867306539931640720492238
4748152806191669009338057321208163507076343516698696250209690

A big number

6340696503084422585596703927186946115851379338647569974856867
00798239606043934788850861649260304945061743412365828352144806
7266768418070837548622114082365798029612000274413244384324023
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6639519778021441199752546704080608499344178256285092726523709
8986515394621930046073645079262129759176982938923670151709920

A big number

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Runtime of Kruskal? $\mathcal{O}\left((|E| + |V|)\log^*(|V|)\right) \approx \mathcal{O}\left(|E| + |V|\right)$

Inverse of the Ackerman function

But wait!

Somebody then came along and proved that find and union are amortized $\mathcal{O}(\alpha(n))$ – the inverse of the Ackermann function.

This grows even more slowly then $\log^*(n)$!