CSE 373: Minimum Spanning Trees: Prim and Kruskal

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Minimum spanning trees

Punchline: a MST of a graph connects all the vertices together while minimizing the number of edges used (and their weights).

Minimum spanning trees

Given a connected, undirected graph $G = (V, E)$, a **minimum spanning tree** is a *subgraph* $G' = (V', E')$ such that...

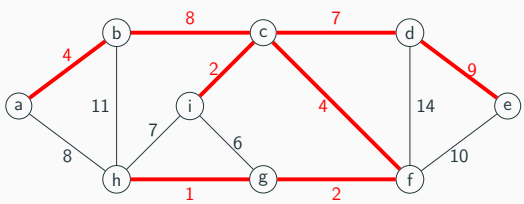
- $V = V'$ (G' is *spanning*)
- There exists a path from any vertex to any other one
- The sum of the edge weights in E' is *minimized*.

In order for a graph to have a MST, the graph must...

- ...be connected – there is a path from a vertex to any other vertex. (Note: this means $|V| \leq |E|$).
- ...be undirected.

Minimum spanning trees: example

An example of an minimum spanning tree (MST):



Minimum spanning trees: Applications

Example questions:

- We want to connect phone lines to houses, but laying down cable is expensive. How can we minimize the amount of wire we must install?
- We have items on a circuit we want to be "electrically equivalent". How can we connect them together using a minimum amount of wire?

Other applications:

- Implement efficient multiple constant multiplication
- Minimizing number of packets transmitted across a network
- Machine learning (e.g. real-time face verification)
- Graphics (e.g. image segmentation)

Minimum spanning trees: properties

Important properties:

- A valid MST cannot contain a cycle
- If we add or remove an edge from an MST, it's no longer a valid MST for that graph.
  Adding an edge introduces a cycle; removing an edge means vertices are no longer connected.
- If there are $|V|$ vertices, the MST contains exactly $|V| - 1$ edges.
- An MST is always a tree.
- If every edge has a unique weight, there exists a unique MST.

Minimum spanning trees: algorithm

Design question: how would you implement an algorithm to find the MST of some graph, assuming the edges *all have the same weight*?

Hints:

- Try modifying DFS or BFS.
- Try using an *incremental* approach: start with an empty graph, and steadily add nodes and edges.

Minimum spanning trees: approach 1, adding nodes

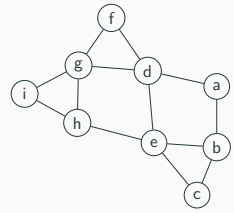
Intuition: We start with an "empty" MST, and steadily grow it.

Core algorithm:

1. Start with an arbitrary node.
2. Run either DFS or BFS, storing *edges* in our stack or queue.
3. As we visit nodes, add each edge we remove to our MST.

Minimum spanning trees: approach 1, adding nodes

An example using a modified version of DFS:



Stack:

Minimum spanning trees: approach 1, adding nodes

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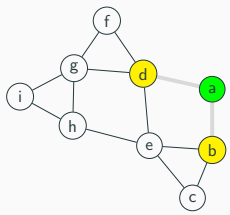

Stack: (a, b), (a, d),

Minimum spanning trees: approach 1, adding nodes

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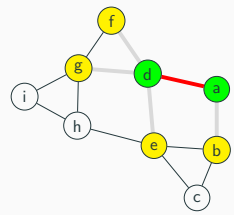
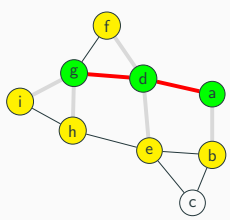


Stack: (a, b), (d, e), (d, f), (d, g),

Minimum spanning trees: approach 1, adding nodes

An example using a modified version of DFS:



Stack: (a, b), (d, e), (d, f), (g, h), (g, i),

Minimum spanning trees: approach 1, adding nodes

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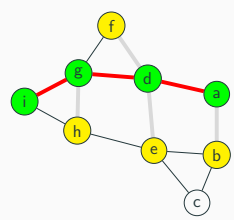

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Minimum spanning trees: approach 1, adding nodes

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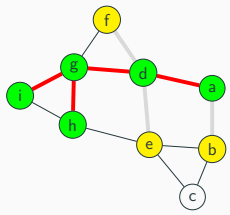

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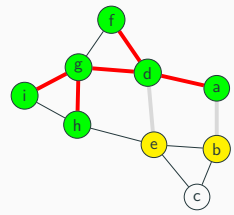

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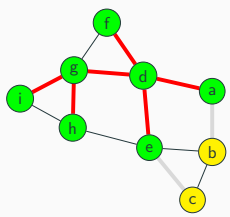

Stack: (a, b), (e, c),

Minimum spanning trees: approach 1, adding nodes

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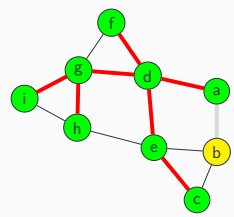

Stack: (a, b),

Minimum spanning trees: approach 1, adding nodes

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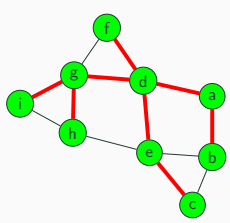


Stack:

Minimum spanning trees: approach 1, adding nodes

What if the edges have different weights?

Observation:

We solved a similar problem earlier this quarter, when studying shortest path algorithms!

Interlude: finding the shortest path

Review: How do we find the shortest path between two vertices?

- If the graph is unweighted: run BFS
- If the graph is weighted: run Dijkstra's

How does Dijkstra's algorithm work?

1. Give each vertex v a "cost": the cost of the shortest-known path so far between v and the start.
   (The cost of a path is the sum of the edge weights in that path)
2. Pick the node with the smallest cost, update adjacent node costs, repeat

Minimum spanning trees: approach 1, adding nodes

Intuition: We can use the same idea to find a MST!

Core idea: Use the exact same algorithm as Dijkstra's algorithm, but redefine the cost:

- **Previously, for Dijkstra's:**
  The cost of vertex v is the cost of the shortest-known path so far between v and the start
- **Now:**
  The cost of vertex v is the cost of the shortest-known path so far between v and *any node we've visited so far*

This algorithm is known as **Prim's algorithm**.

Compare and contrast: Dijkstra vs Prim

Pseudocode for Dijkstra's algorithm:

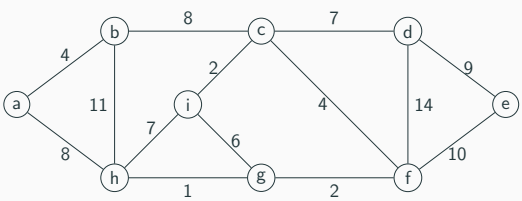
```
def dijkstra(start):
    backpointers = new SomeDictionary<Vertex, Vertex>()

    for (v : vertices):
        set cost(v) to infinity
    set cost(start) to 0

    while (we still have unvisited nodes):
        current = get next smallest node

        for (edge : current.getOutEdges()):
            newCost = min(cost(current) + edge.cost, cost(edge.dst))
            update cost(edge.dst) to newCost
            backpointers.put(edge.dst, edge.src)

    return backpointers
```


Compare and contrast: Dijkstra vs Prim

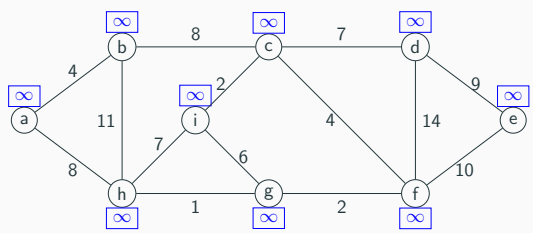
Pseudocode for Prim's algorithm:

```
def prim(start):
    backpointers = new SomeDictionary<Vertex, Vertex>()

    for (v : vertices):
        set cost(v) to infinity
    set cost(start) to 0

    while (we still have unvisited nodes):
        current = get next smallest node

        for (edge : current.getOutEdges()):
            newCost = min(edge.cost, cost(edge.dst))
            update cost(edge.dst) to newCost
            backpointers.put(edge.dst, edge.src)

    return backpointers
```


Prim's algorithm: an example

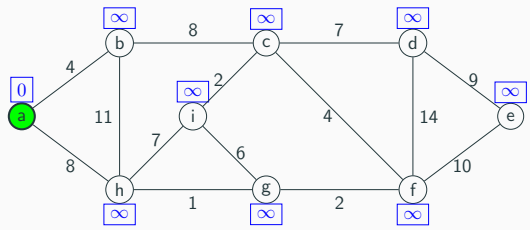


Prim's algorithm: an example



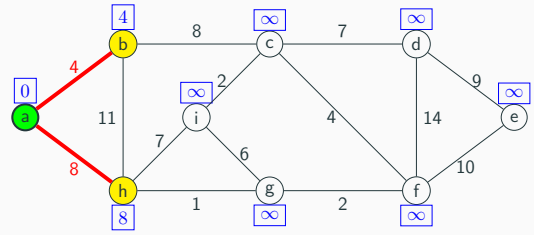
We initially set all costs to ∞, just like with Dijkstra.

Prim's algorithm: an example



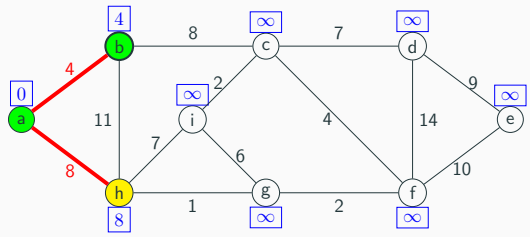
We pick an arbitrary node to start.

Prim's algorithm: an example



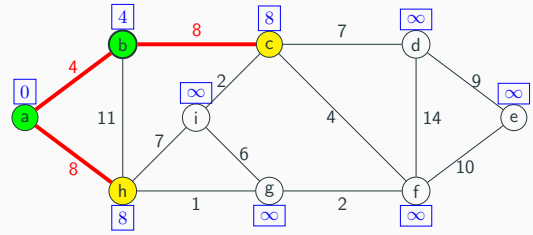
We update the adjacent nodes.

Prim's algorithm: an example



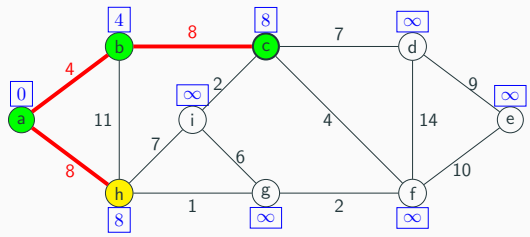
We select the one with the smallest cost.

Prim's algorithm: an example



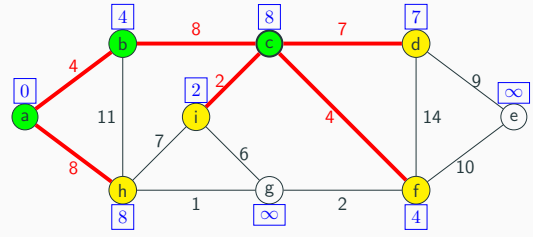
We potentially need to update h and c, but only c changes.

Prim's algorithm: an example



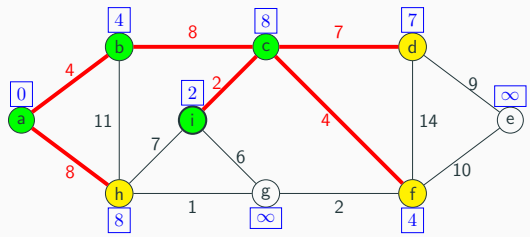
We (arbitrarily) pick c.

Prim's algorithm: an example



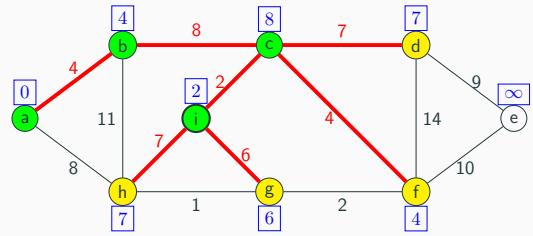
...and update the adjacent nodes. Note that we don't add the cumulative cost: the cost represents the shortest path to *any* green node, not to the start.

Prim's algorithm: an example



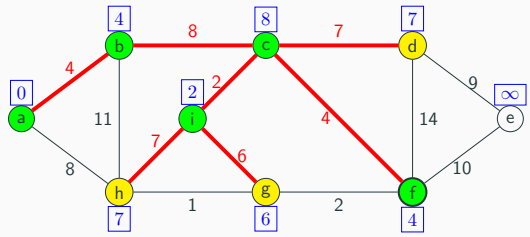
i has the smallest cost.

Prim's algorithm: an example



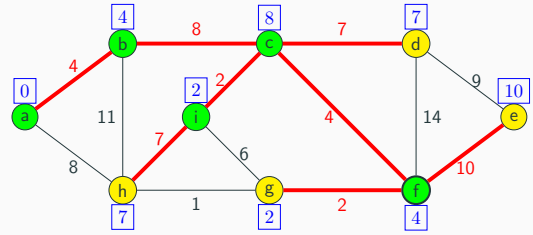
We update both unvisited nodes, and modify the edge to h since we now have a better option.

Prim's algorithm: an example



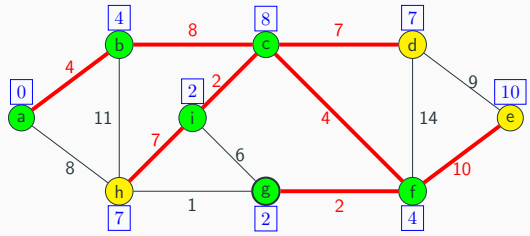
f has the smallest cost.

Prim's algorithm: an example



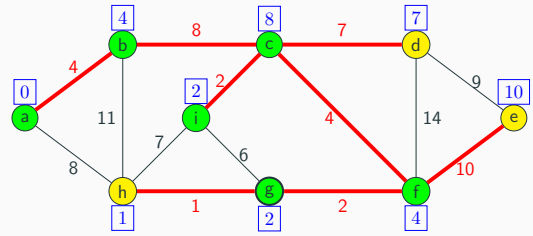
Again, we update the adjacent unvisited nodes.

Prim's algorithm: an example



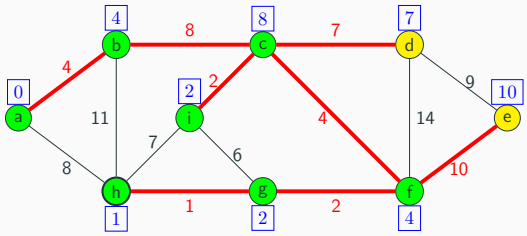
g has the smallest cost.

Prim's algorithm: an example



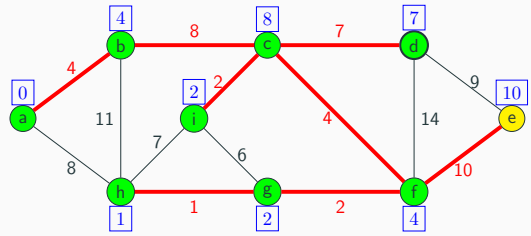
We update h again.

Prim's algorithm: an example



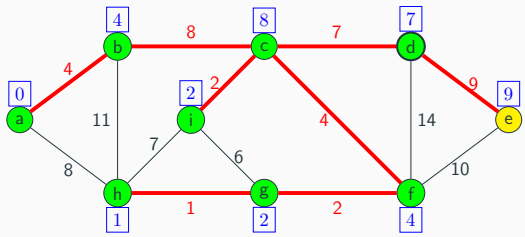
h has the smallest cost. Note that there nothing to update here.

Prim's algorithm: an example



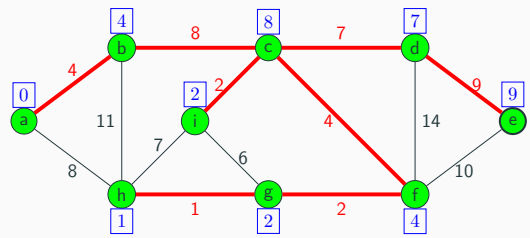
d has the smallest cost.

Prim's algorithm: an example



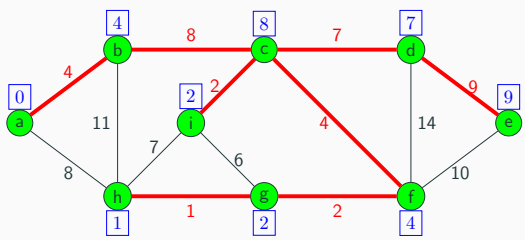
We can update e.

Prim's algorithm: an example



e has the smallest cost.

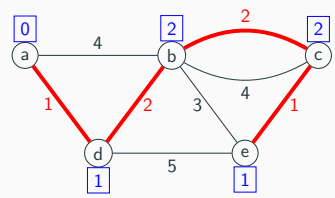
Prim's algorithm: an example



There are no more nodes left, so we're done.

Prim's algorithm: another example

Now you try. Start on node a.



Analyzing Prim's algorithm

Question: What is the worst-case asymptotic runtime of Prim's algorithm?

Answer: The same as Dijkstra's: $\mathcal{O}(|V|t_s + |E|t_u)$ where...

- t_s = time needed to get next smallest node
- t_u = time needed to update vertex costs

So, $\mathcal{O}(|V|\log(|V|) + |E|\log(|V|))$ if we stick to data structures we know how to implement; $\mathcal{O}(|V|\log(|V|) + |E|)$ if we use Fibonacci heaps.

Minimum spanning trees, approach 2

Recap: Prim's algorithm works similarly to Dijkstra's – we start with a single node, and "grow" our MST.

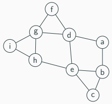
A second approach: instead of "growing" our MST, we...

- Initially place each node into its own MST of size 1 – so, we start with $|V|$ MSTs in total.
- Steadily combine together different MSTs until we have just one left
- How? Loop through every single edge, see if we can use it to join two different MSTs together.

This algorithm is called **Kruskal's algorithm**

Kruskal's algorithm

An example, for unweighted graphs. Note: each MST has a different color.



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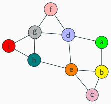

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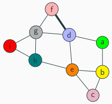

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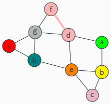

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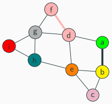

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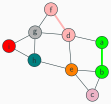

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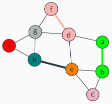

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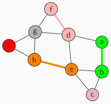

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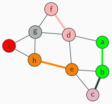

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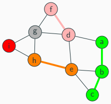

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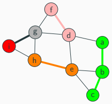

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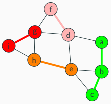

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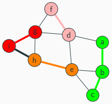

Kruskal's algorithm

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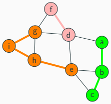

Kruskal's algorithm

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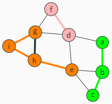

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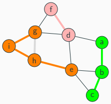

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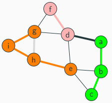

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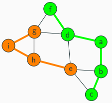

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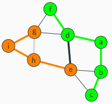

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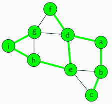

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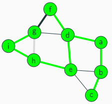

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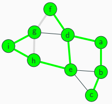

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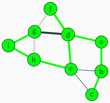

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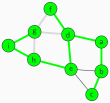

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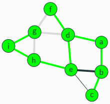

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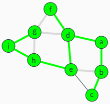

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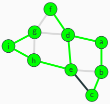

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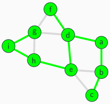


Kruskal's algorithm: weighted graphs

Question: How do we handle edge weights?

Answer: Consider edges sorted in ascending order by weight.

So, we look at the edge with the smallest weight first, the edge with the second smallest weight next, etc.

Kruskal's algorithm: pseudocode

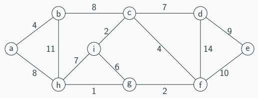
Pseudocode for Kruskal's algorithm:

```
def kruskal():
    mst = new SomeSet<Edge>()

    for (v : vertices):
        makeMST(v)

    sort edges in ascending order by their weight

    for (edge : edges):
        if findMST(edge.src) != findMST(edge.dst):
            union(edge.src, edge.dst)
            mst.add(edge)

    return mst
```

- `makeMST(v)`: stores v as a MST containing just one node
- `findMST(v)`: finds the MST that vertex is a part of
- `union(u, v)`: combines the two MSTs of the two given vertices, using the edge (u, v)

Kruskal's algorithm: example with a weighted graph

Now you try:



Kruskal's algorithm: example with a weighted graph

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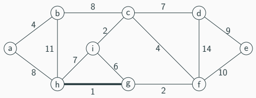

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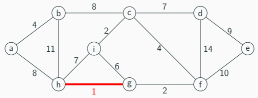

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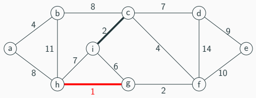

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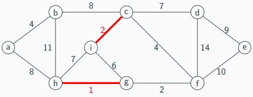

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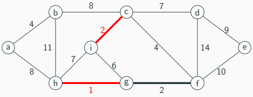

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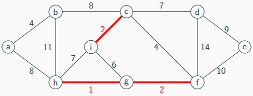

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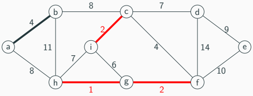

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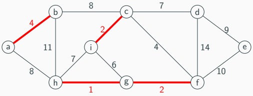

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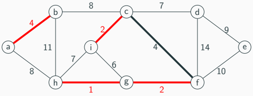

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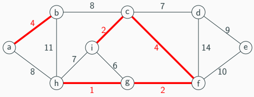

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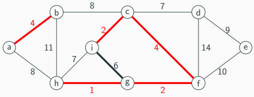

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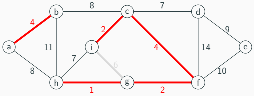

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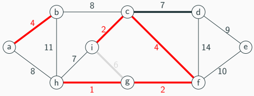

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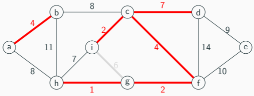

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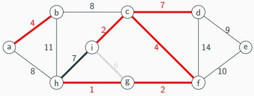

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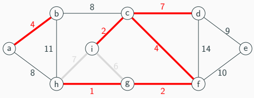

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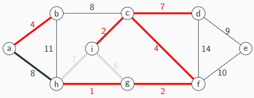

Kruskal's algorithm: example with a weighted graph

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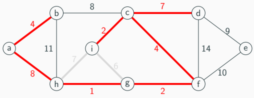

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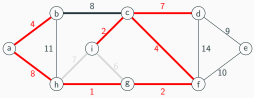

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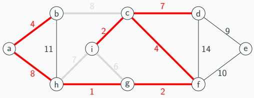

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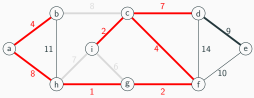

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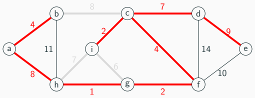

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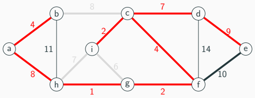

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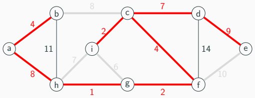

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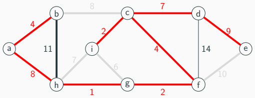

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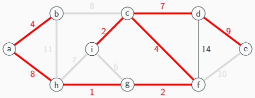

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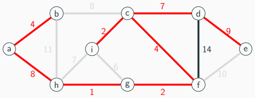

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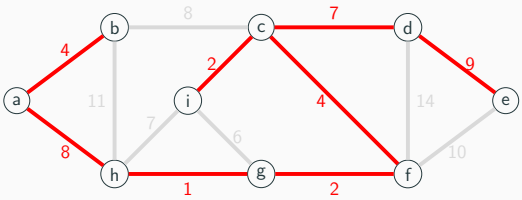

Kruskal's algorithm: analysis

What is the worst-case runtime?

```
def kruskal():
    mst = new SomeSet<Edge>()

    for (v : vertices):
        makeMST(v)

    sort edges in ascending order by their weight

    for (edge : edges):
        if findMST(edge.src) != findMST(edge.dst):
            union(edge.src, edge.dst)
            mst.add(edge)

    return mst
```

Note: assume that...

- `makeMST(v)` takes $\mathcal{O}(t_m)$ time
- `findMST(v)`: takes $\mathcal{O}(t_f)$ time
- `union(u, v)`: takes $\mathcal{O}(t_u)$ time

Kruskal's algorithm: analysis

- Making the $|V|$ MSTs takes $\mathcal{O}(|V| \cdot t_m)$ time
- Sorting the edges takes $\mathcal{O}(|E| \cdot \log(|E|))$ time, assuming we use a general-purpose comparison sort
- The final loop takes $\mathcal{O}(|E| \cdot t_f + |V| \cdot t_u)$ time

Putting it all together:

$$\mathcal{O}(|V| \cdot t_m + |E| \cdot \log(|E|) + |E| \cdot t_f + |V| \cdot t_u)$$

The DisjointSet ADT

But wait, what exactly is t_m, t_f, and t_u? How exactly do we implement `makeMST(v)`, `findMST(v)`, and `union(u, v)`?

We can do so using a new ADT called the `DisjointSet` ADT!