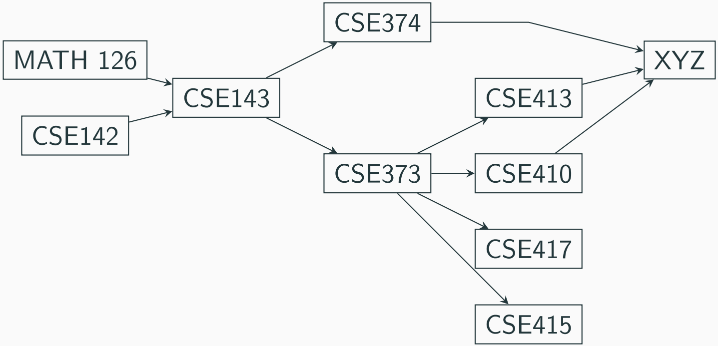
CSE 373: Topological Sort and Minimum Spanning Trees

Michael Lee

Friday, Feb 23, 2018

Topological sort

Design question: suppose we have a bunch of classes with pre-requisites.



Topological sort

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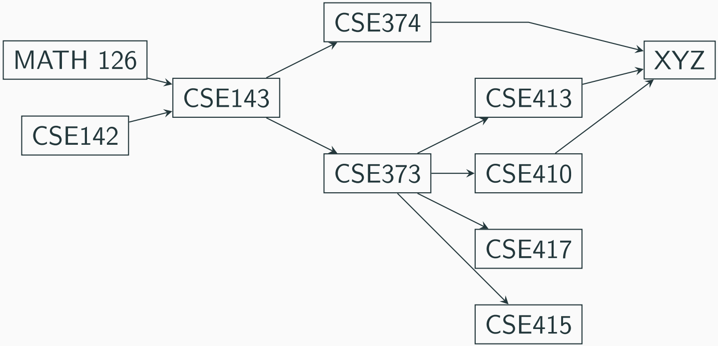

Goal: list out classes in a "valid" order

For example: 126, 142, 143, 374, 373, 417, 410, 413, XYZ, 415

Topological sort

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Given a *directed, acyclic graph* (DAG), running **topological sort** on that graph will produce a list of all the vertices in an order such that no vertex appears before another vertex that has an edge to it.

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Example applications:

- Any scheduling problem (scheduling courses, scheduling threads)
- Computing order to recompute cells in spreadsheet
- Determining order to compile files using a MAkefile

In general: taking a dependency graph and coming up with order of execution.

Topological sort

Questions

- Can we perform topo-sort on graphs containing cycles?
- Is there always one unique output per graph?

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- Can we perform topo-sort on graphs containing cycles?
  No: how do we decide which node comes first?
- Is there always one unique output per graph?
  No: see example on inked slides

Topological sort: algorithm

Intuition:

- The only nodes we can start with are also nodes that have in-degree 0

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- The only nodes we can start with are also nodes that have in-degree 0
- So, start by adding those to the list
- Is there some way of “repeating” this process?

Topological sort

Setup

- Look at each vertex and record its in-degree somewhere

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Core loop

- Choose an arbitrary vertex a with in-degree 0

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- For each vertex b adjacent to a, decrement the in-degree of b

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- Choose an arbitrary vertex a with in-degree 0
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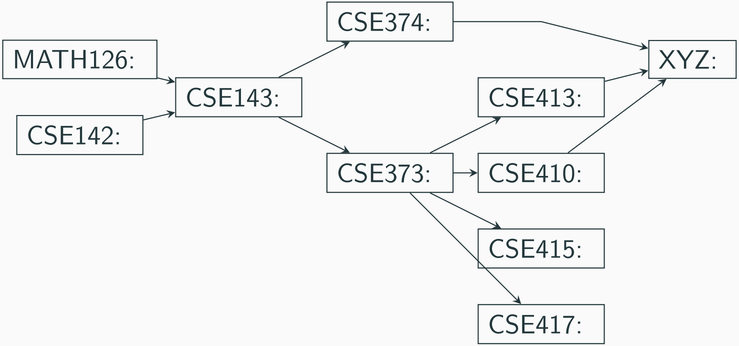
- Repeat

Topological sort: Example 1

Example again:



;

Output:

Topological sort: Example 1

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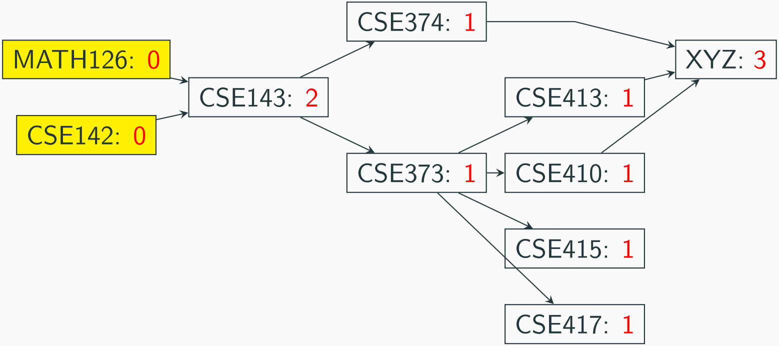

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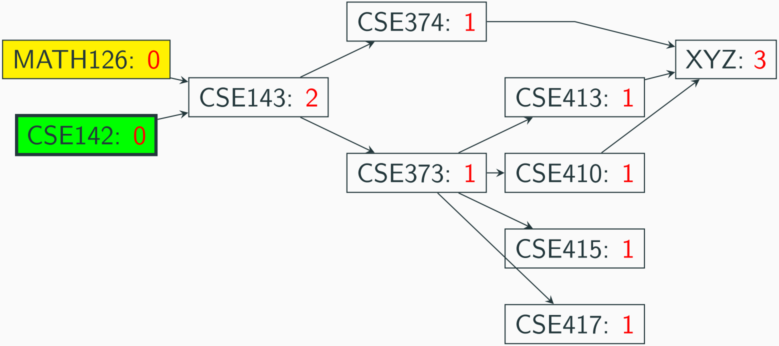
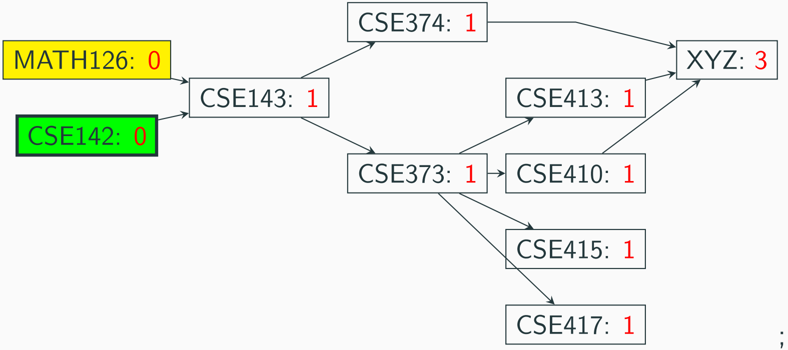


;

Output: CSE142,

Topological sort: Example 1

Example again:



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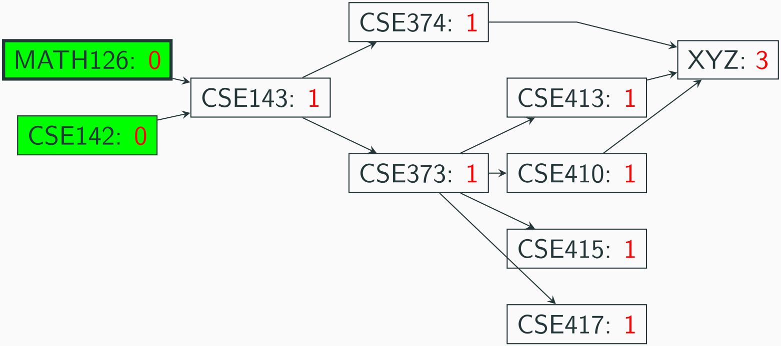

;

Output: CSE142, MATH126,

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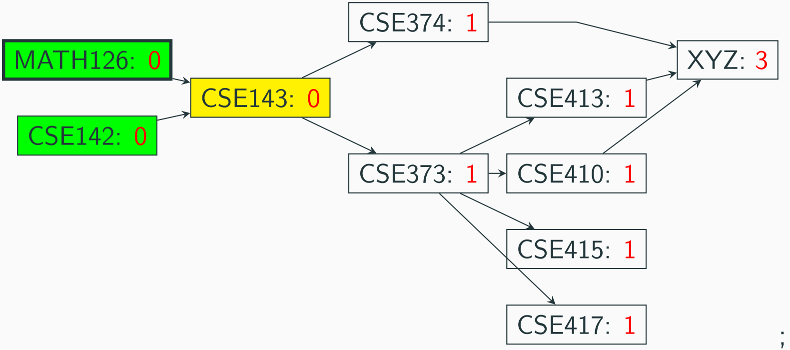

Output: CSE142, MATH126,

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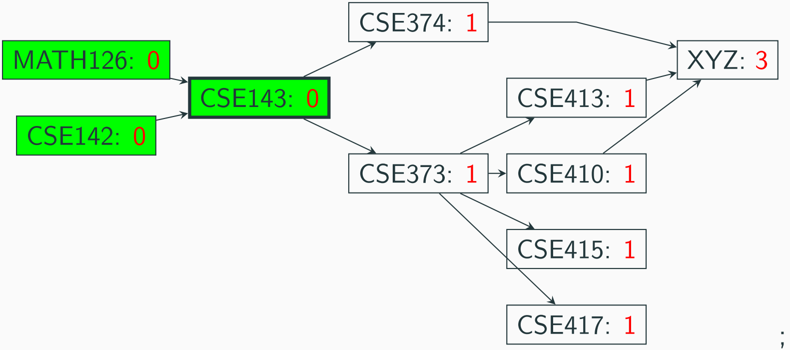

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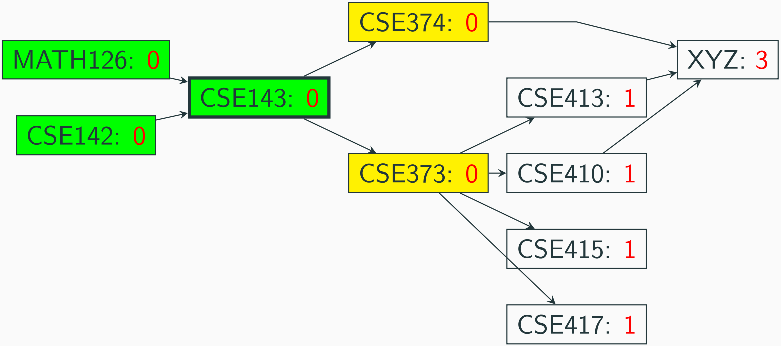

;

Output: CSE142, MATH126, CSE143,

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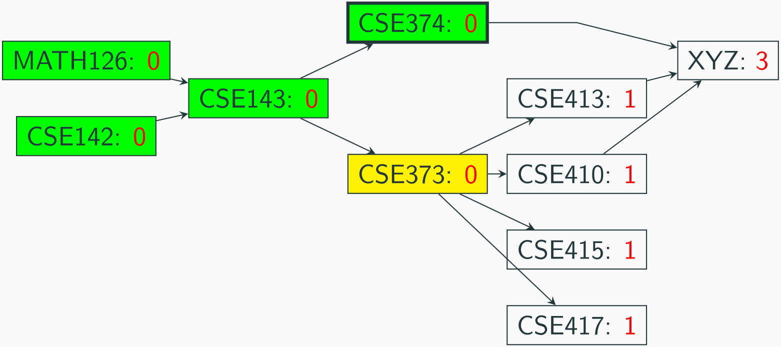

;

Output: CSE142, MATH126, CSE143, CSE374,

Topological sort: Example 1

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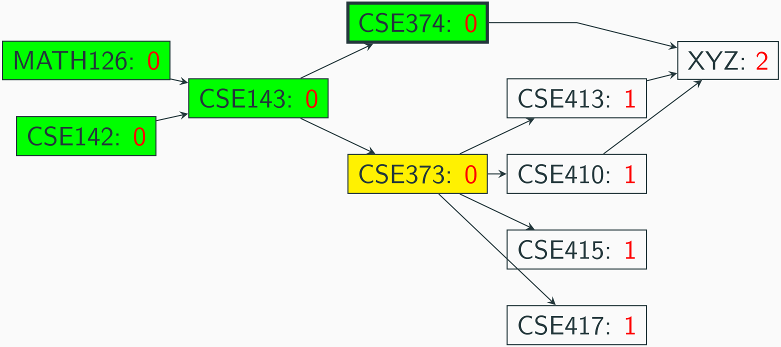

;

Output: CSE142, MATH126, CSE143, CSE374,

Topological sort: Example 1

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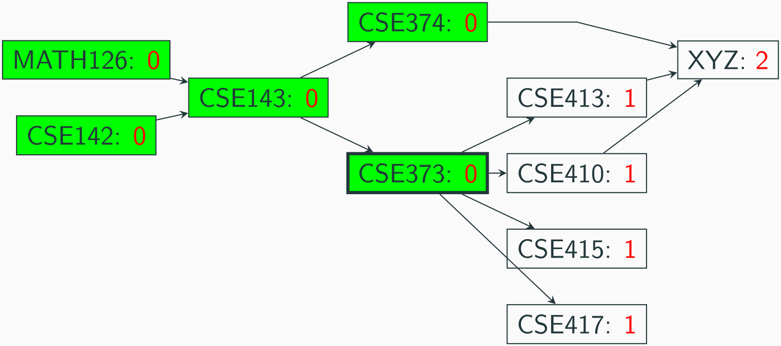

;

Output: CSE142, MATH126, CSE143, CSE374, CSE373,

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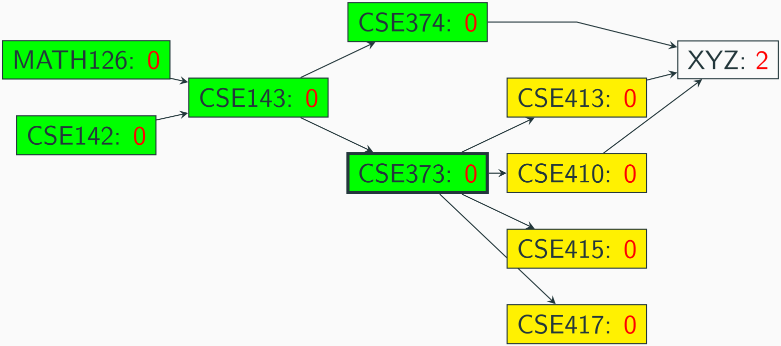

;

Output: CSE142, MATH126, CSE143, CSE374, CSE373,

Topological sort: Example 1

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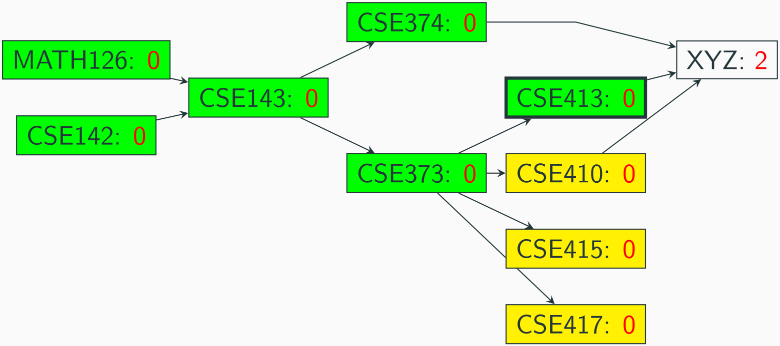

;

Output: CSE142, MATH126, CSE143, CSE374, CSE373, CSE413,

Topological sort: Example 1

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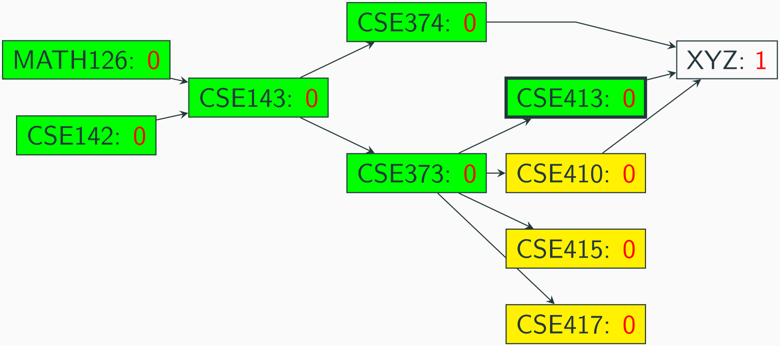

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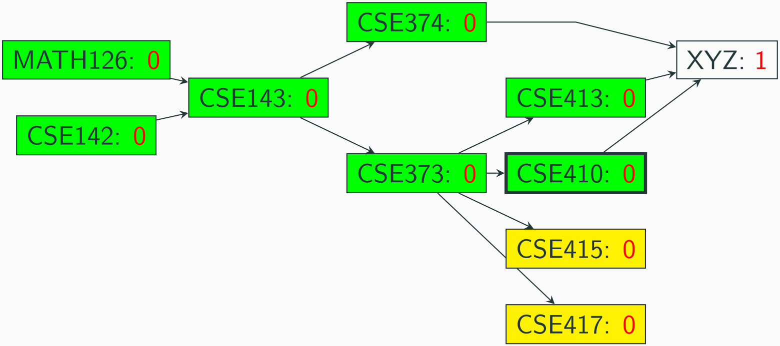

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Output: CSE142, MATH126, CSE143, CSE374, CSE373, CSE413, CSE410,

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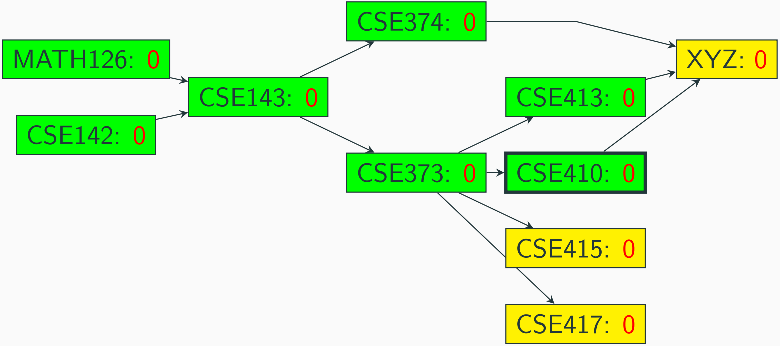

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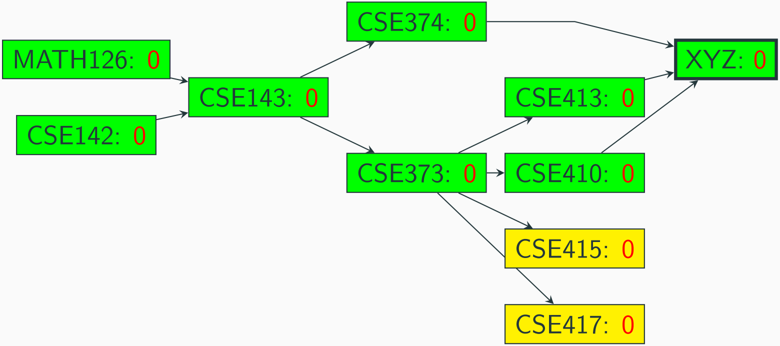

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Output: CSE142, MATH126, CSE143, CSE374, CSE373, CSE413, CSE410, XYZ,

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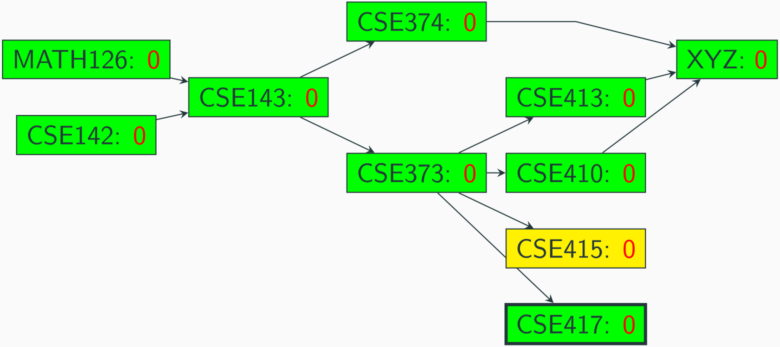

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Output: CSE142, MATH126, CSE143, CSE374, CSE373, CSE413, CSE410, XYZ, CSE417,

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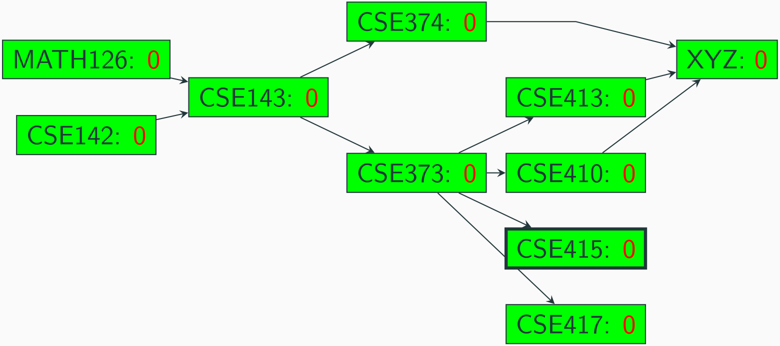

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Output: CSE142, MATH126, CSE143, CSE374, CSE373, CSE413, CSE410, XYZ, CSE417, CSE415

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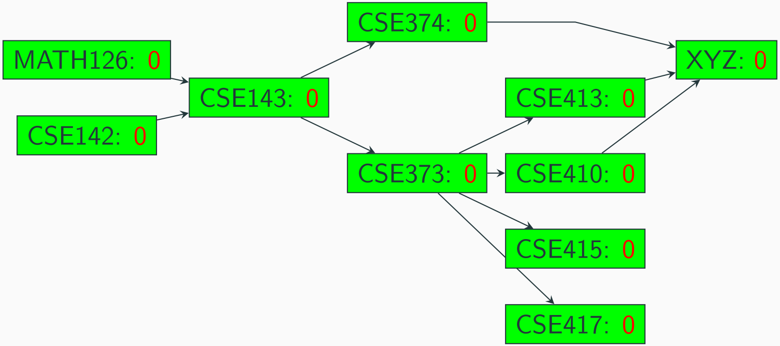

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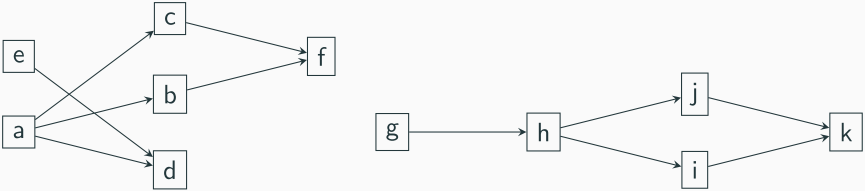
Topological sort: Example 2

Now you try. List one possible output:



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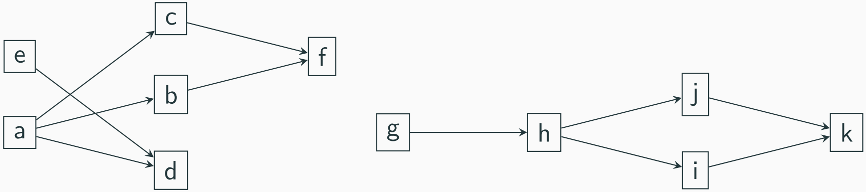

One possible answer: a, b, g, c, e, h, d, i, f, j, k

Topological sort: Algorithm

Our algorithm so far:

Setup

- Look at each vertex and record its in-degree somewhere

Core loop

- Choose an arbitrary vertex a with in-degree 0
- Output a and conceptually remove it from the graph
- For each vertex b adjacent to a, decrement the in-degree of b
- Repeat

Topological sort: Algorithm

One possible implementation:

```
def toposort(graph):
    indegrees = new HashMap<Vertex, Integer>()
    visited = new HashSet<Vertex>()
    output = new AnyList<Vertex>()

    compute all indegrees and add to dictionary

    while (we still need to visit vertices):
        current = getNextVertex(indegrees, visited)
        add current to both visited and output

        for (v : current.allNeighbors()):
            indegrees[v] -= 1

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def getNextVertex(indegrees, visited):
    for (node, num : indegrees):
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Questions:

Worst-case runtime?

$\mathcal{O}\left(|V|^2 + |E|\right)$

Is this optimal?

Maybe not. Do we really need to look at each node multiple times? Can we somehow get $\mathcal{O}\left(|V| + |E|\right)$?

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How can we improve this?

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How can we improve this?

- Can we get rid of the inner loop somehow?
- Would using different/more data structures help?
- Can we collect additional information somewhere else?

Topological sort: Algorithm 2

Insight: When we're updating the indegrees, we already know which nodes now have an indegree of zero!

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def toposort(graph):
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    stack = new Stack<Vertex>();

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    while (we still need to visit vertices):
        current = stack.pop()
        add current to both visited and output

        for (v : current.allNeighbors()):
            indegrees[v] -= 1
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                stack.push(v)
    return output
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Question: Does this actually work?

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Question: Does this actually work?

Answer: No, there's a bug! The stack is initially empty, so first pop fails.

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    compute all indegrees and add to dictionary
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    while (we still need to visit vertices):
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Question: Can we improve this algorithm even more?

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Question: Can we improve this algorithm even more?

Answer: Why do we need the visited set?

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Question: What's the worst-case runtime now?

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Question: What's the worst-case runtime now?

Answer: $\mathcal{O}(|V| + |E|)$

And now, for something completely different...

Minimum spanning trees

Punchline: a MST of a graph connects all the vertices together while minimizing the number of edges used (and their weights).

Minimum spanning trees

Given a connected, undirected graph $G = (V, E)$, a **minimum spanning tree** is a *subgraph* $G' = (V', E')$ such that...

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Given a connected, undirected graph $G = (V, E)$, a **minimum spanning tree** is a *subgraph* $G' = (V', E')$ such that...

- $V = V'$ (G' is *spanning*)

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Given a connected, undirected graph $G = (V, E)$, a **minimum spanning tree** is a *subgraph* $G' = (V', E')$ such that...

- $V = V'$ (G' is *spanning*)
- There exists a path from any vertex to any other one
- The sum of the edge weights in E' is *minimized*.

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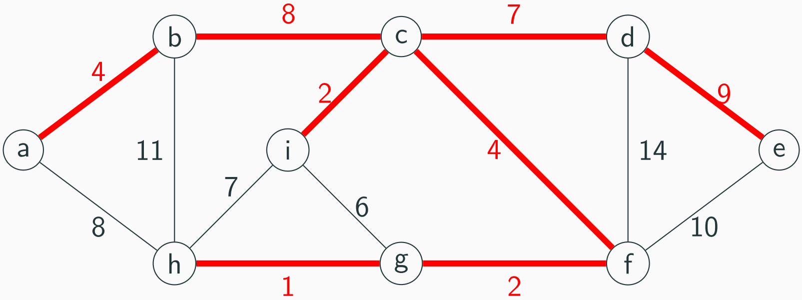
- $V = V'$ (G' is *spanning*)
- There exists a path from any vertex to any other one
- The sum of the edge weights in E' is *minimized*.

In order for a graph to have a MST, the graph must...

- ...be connected – there is a path from a vertex to any other vertex. (Note: this means $|V| \leq |E|$).
- ...be undirected.

Minimum spanning trees: example

An example of an minimum spanning tree (MST):



Minimum spanning trees: Applications

Example questions:

- We want to connect phone lines to houses, but laying down cable is expensive. How can we minimize the amount of wire we must install?

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Minimum spanning trees: Applications

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Other applications:

- Implement efficient multiple constant multiplication
- Minimizing number of packets transmitted across a network
- Machine learning (e.g. real-time face verification)
- Graphics (e.g. image segmentation)

Minimum spanning trees: properties

Some questions...

- Can a valid MST contain a cycle?

- If we take a valid MST and remove an edge, is it still an MST?

- If we take a valid MST and add an edge, is it still an MST?

- If there are V vertices, how many edges are contained in the minimum spanning tree?

Minimum spanning trees: properties

Some questions...

- Can a valid MST contain a cycle?
  Answer: no. If there's a cycle, we can always remove one edge to break the cycle while still leaving all nodes connected.
- If we take a valid MST and remove an edge, is it still an MST?
  Answer: No. If we're already using the fewest edges possible, removing an edge would make the nodes no longer connected.

- If we take a valid MST and add an edge, is it still an MST?
  Answer: No. Since all the edges are already connected, this would introduce a cycle.
- If there are V vertices, how many edges are contained in the minimum spanning tree?
  Answer: $|V| - 1$

Minimum spanning trees: algorithm

Design question: how would you implement an algorithm to find the MST of some graph, assuming the edges all have the same weight?

Minimum spanning trees: algorithm

Design question: how would you implement an algorithm to find the MST of some graph, assuming the edges all have the same weight?

One idea: run DFS, and keep all the edges that don't connect back to an already-visited vertex.

Another idea: iterate through the edges, and add an edge as long as it doesn't introduce a cycle.

Minimum spanning tree: coming up next

Next time:

How do we account for edge weights?

- **Prim's algorithm:** Traverse through graph, and add nodes

Minimum spanning tree: coming up next

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- **Kruskal's algorithm:** Iterate through edges, and add edges

Minimum spanning tree: coming up next

Next time:

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In both cases, we avoid adding nodes/edges that introduce a cycle, and need to figure out how to pick the "best" node or edge.