CSE 373: More on graphs; DFS and BFS

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Wednesday, Feb 14, 2018

Warmup:

Discuss with your neighbor:

- Remind your neighbor: what is a *simple graph*?

- Suppose we have a *simple, directed graph* with x nodes. What is the maximum number of edges it can have, in terms of x?

- Now, suppose we have a different simple, undirected graph with y edges. What is the maximum number of vertices it can have, in terms of y?

Warmup:

Discuss with your neighbor:

- Remind your neighbor: what is a *simple graph*?
  A simple graph is a graph that has no self-loops and no parallel edges.
- Suppose we have a *simple, directed graph* with x nodes. What is the maximum number of edges it can have, in terms of x?
  Each vertex can connect to $x - 1$ other vertices, so $x(x - 1)$.
- Now, suppose we have a different simple, undirected graph with y edges. What is the maximum number of vertices it can have, in terms of y?
  Infinite: just keep adding nodes with no edges attached.

Warmup:

Some follow-up questions:

- Suppose we have a *simple, undirected graph* with x nodes. What is the maximum number of edges it can have? What if the graph is not simple?

- Now, suppose we have a different simple, undirected graph with y edges. What is the maximum number of vertices it can have? What if the graph is not simple?

Warmup:

Some follow-up questions:

- Suppose we have a *simple, undirected graph* with x nodes. What is the maximum number of edges it can have? What if the graph is not simple?
  If the graph is simple, the max number of edges is exactly half of what it would be if the graph were directed. So, $\frac{x(x-1)}{2}$.
  If the graph is not simple, it's infinite: assuming $x > 0$, we can just keep adding more and more self-loops. Note that if $x = 0$, there can't be any edges at all.
- Now, suppose we have a different simple, undirected graph with y edges. What is the maximum number of vertices it can have? What if the graph is not simple?
  Either way, it's still infinite, for the same reasons given previously.

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What did we learn?

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Dense graph

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Sparse graph

If $|E| \in \mathcal{O}\left(|V|\right)$, we sau the graph is **sparse**.

To put it another way, sparse graphs have "few" edges.

How do we represent graphs in code?

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Two common approaches, with different tradeoffs:

- Adjacency matrix
- Adjacency list

Adjacency matrix

Core idea:

- Assign each node a number from 0 to $|V| - 1$
- Create a $|V| \times |V|$ nested array of booleans or ints
- If $(x, y) \in E$, then `nestedArray[x][y] == true`

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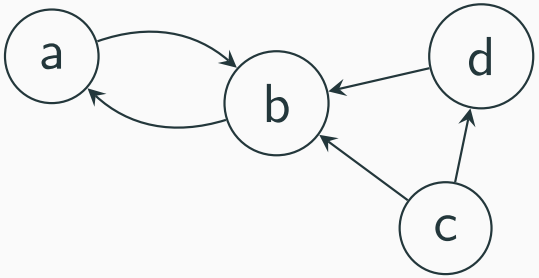

	a	b	c	d
a				
b				
c				
d				

Adjacency list

What is the worst-case runtime to:

- Get out-edges:
- Get in-edges:
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How much space do we use?

Is this better for sparse or dense graphs?

Can we handle self-loops and parallel edges?

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What is the worst-case runtime to:

- Get out-edges: $\mathcal{O}(|V|)$
- Get in-edges: $\mathcal{O}(|V|)$
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- Insert an edge: $\mathcal{O}(1)$
- Delete an edge: $\mathcal{O}(1)$

How much space do we use? $\mathcal{O}\left(|V|^2\right)$

Is this better for sparse or dense graphs? Dense ones

Can we handle self-loops and parallel edges? Self-loops yes, parallel edges, not easily

Adjacency list

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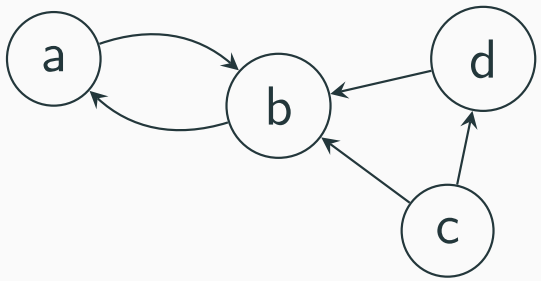

Adjacency list

We can store edges using either sets or lists. Answer these questions for both.

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So by default, pick adjacency lists.

Walks and paths

Walk

A **walk** is a list of vertices $v_0, v_1, v_2, \ldots, v_n$ where if i is some int where $0 \leq i < v_n$, every pair $(v_i, v_{i+1}) \in E$ is true.

More intuitively, a walk is one continous line following the edges.

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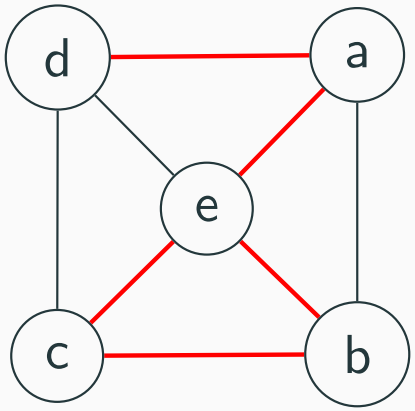

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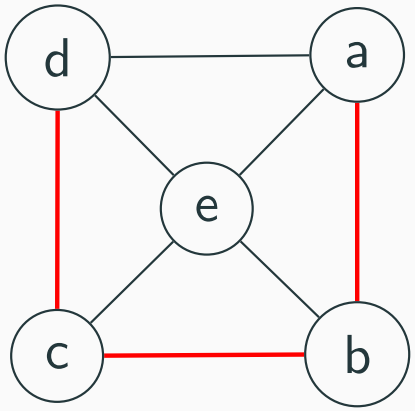

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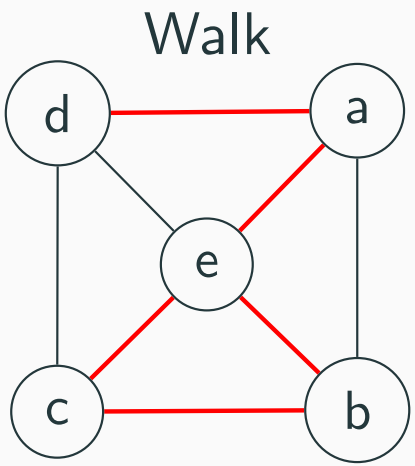

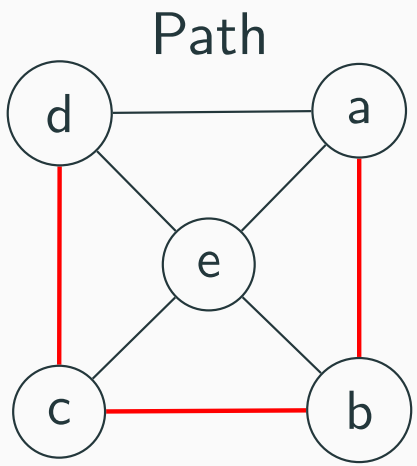

Connected components

Connected graph

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E.g.: if we pick up the graph and shake it, nothing flies off.

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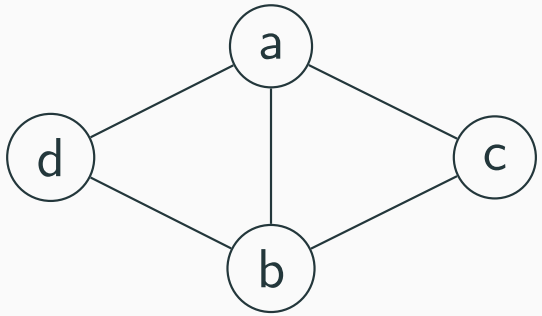

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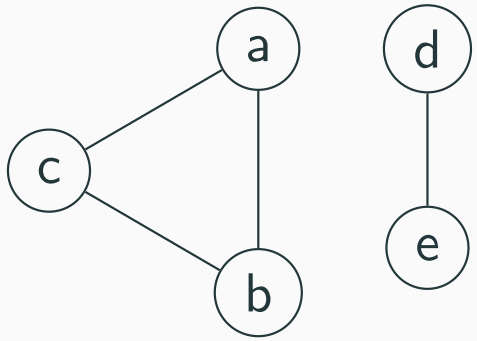

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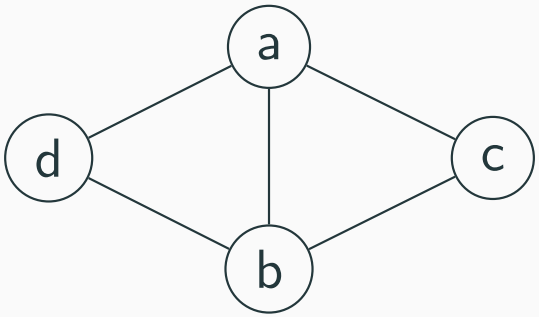
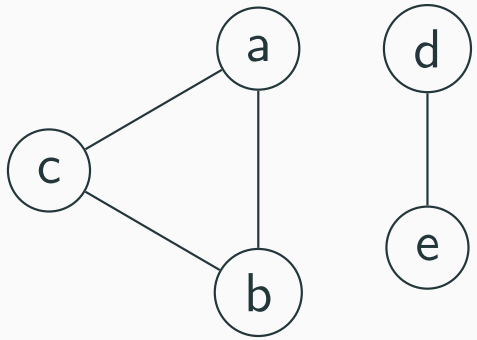


Not connected



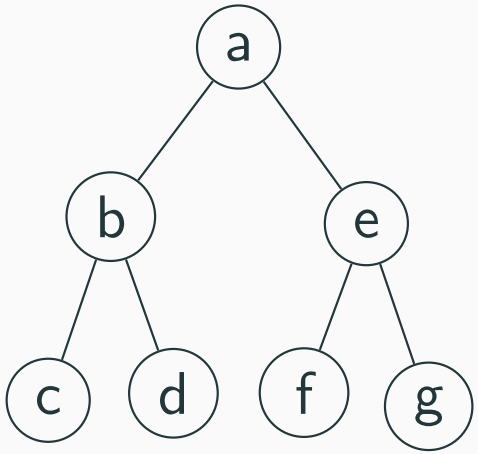
Connected component

A connected component of a graph is any *subgraph* (part of a graph) where all vertices are connected to each other.

Note: A connected graph has only one connected component.

Trees vs graphs

Is this a graph or tree?



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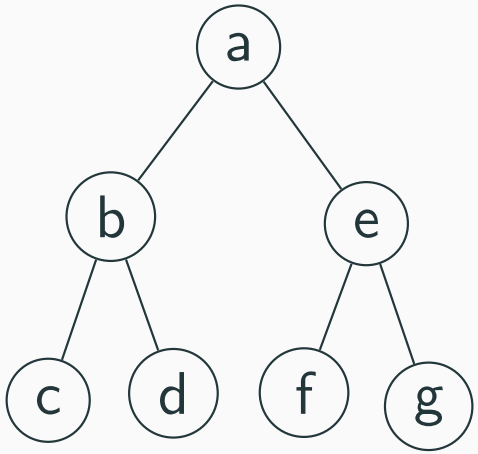

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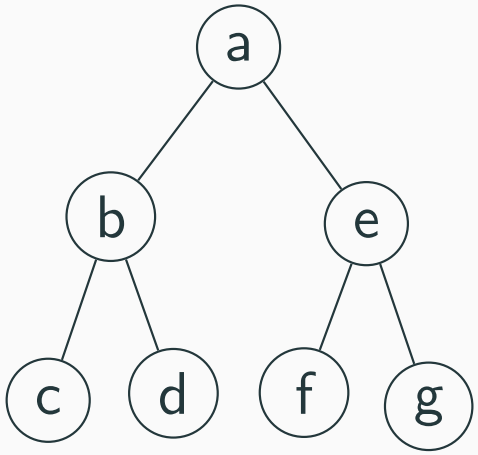

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Tree

A tree is a *connected* and *acyclic* graph.

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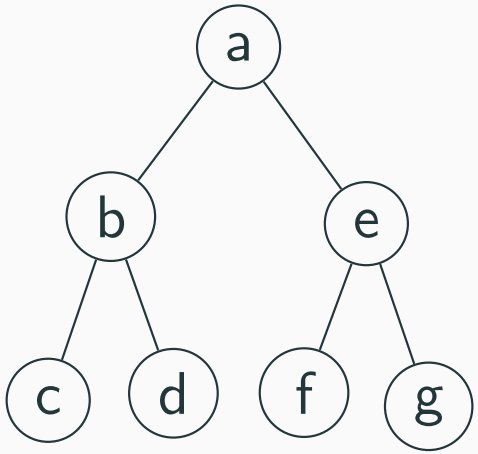

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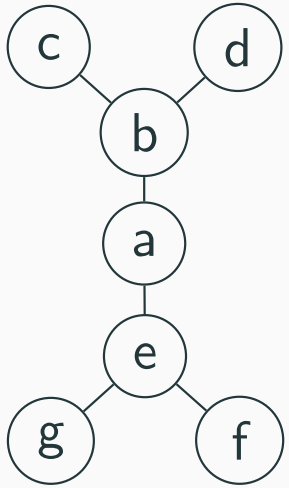

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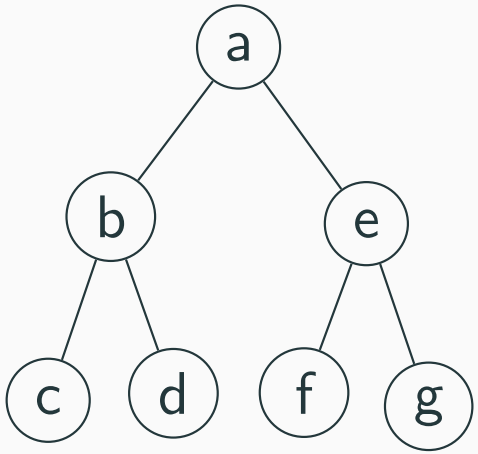

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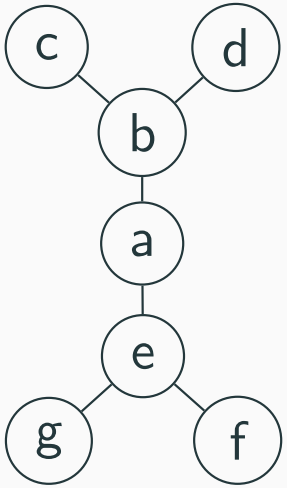

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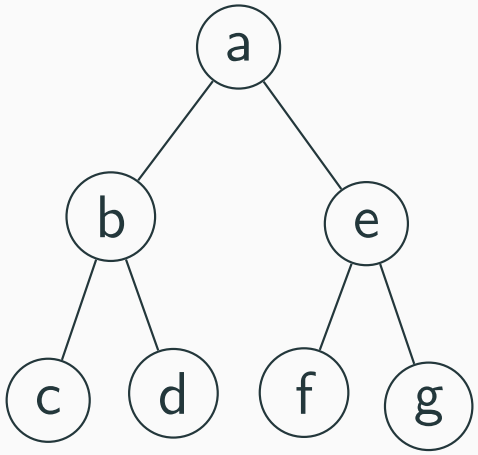

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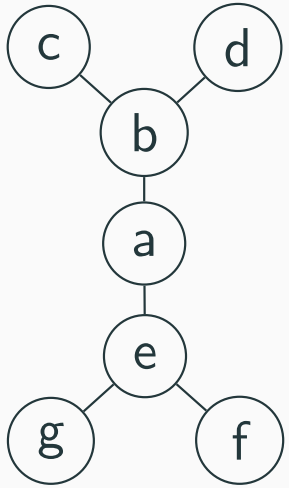

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Tree

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Rooted tree

A rooted tree is a tree where we call one special node the "root".

Detecting if a graph is connected

Question: How can we tell if a graph is connected or not?

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Idea: Let's just find out! Pick a node and see if there's a path to every other node!

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Breadth-first traversal, core idea:

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Pseudocode, version 1:

```
search(v):
    visited = empty set
    queue.enqueue(v)

    while (queue is not empty):
        curr = queue.dequeue()
        for (w : v.neighbors()):
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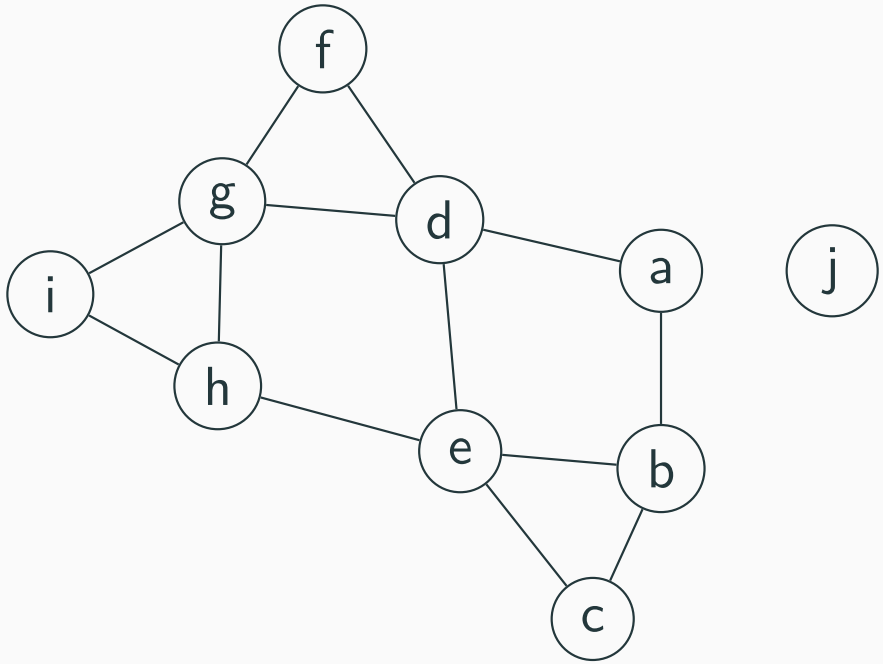

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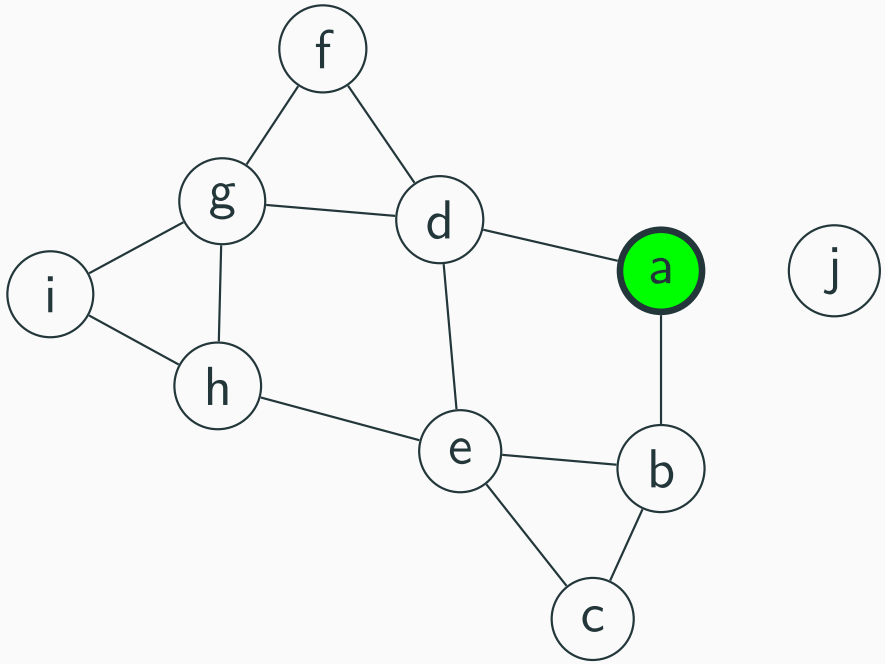

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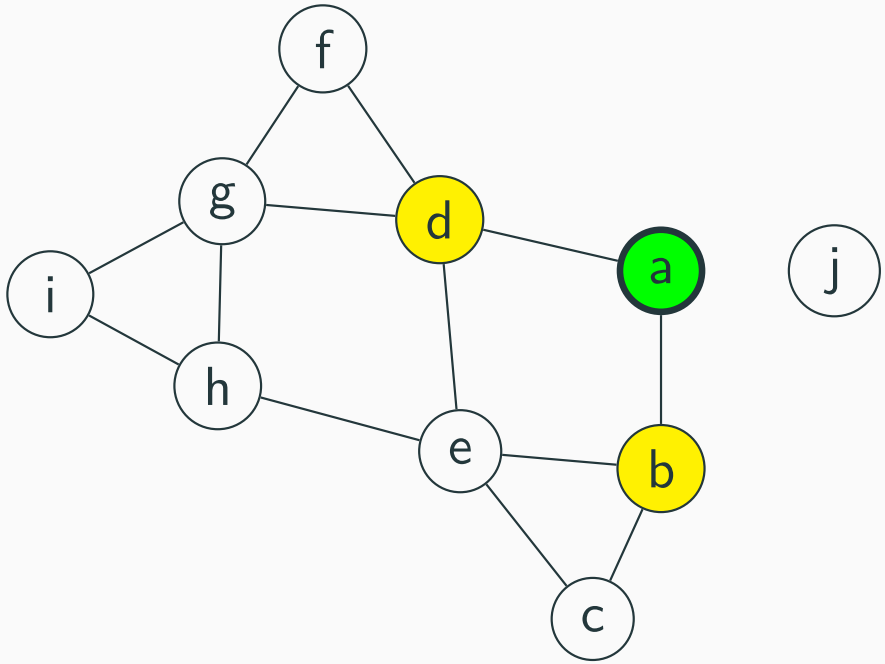

Current node: a

Queue: b, d,

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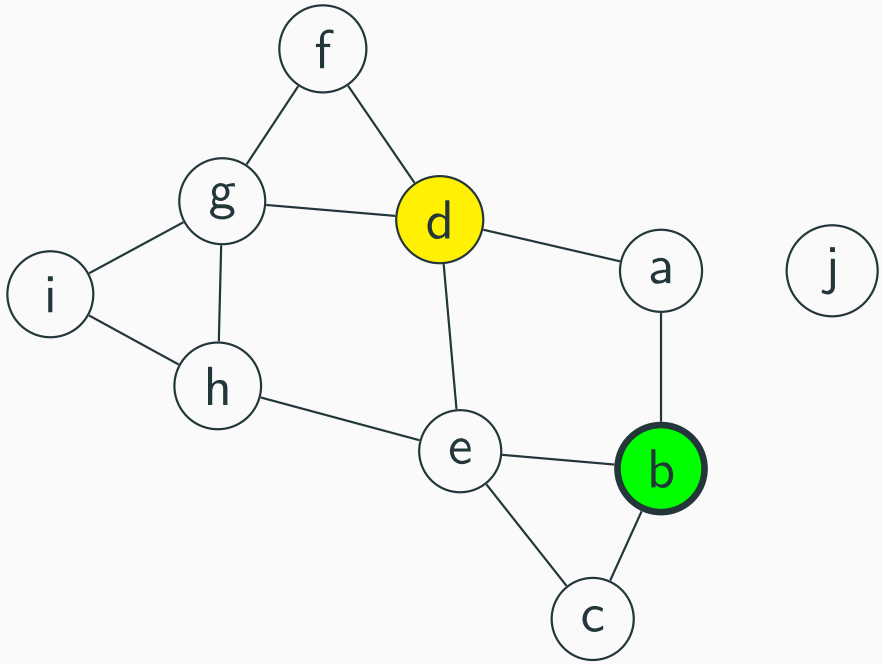

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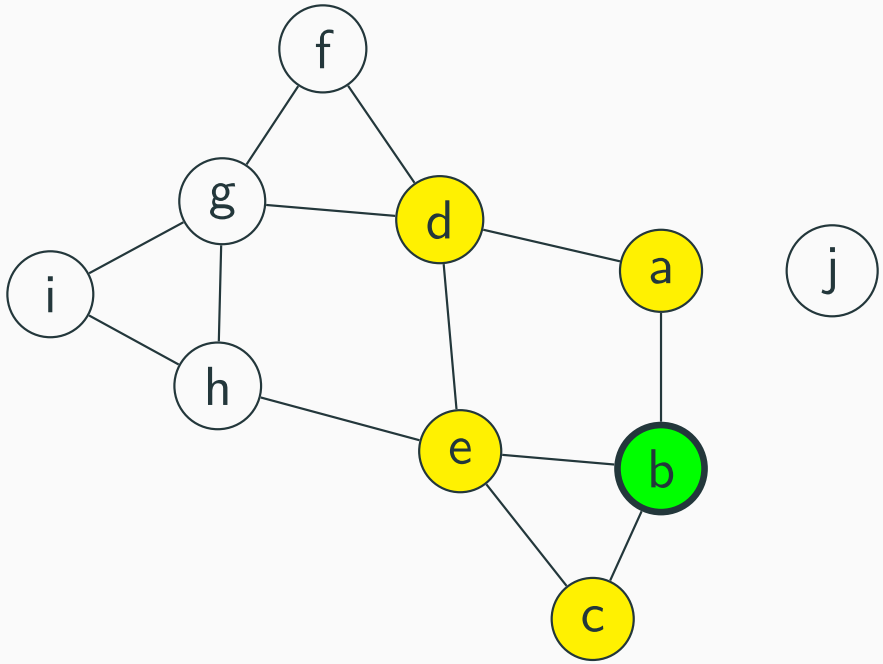

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Queue: d, a, b, e,

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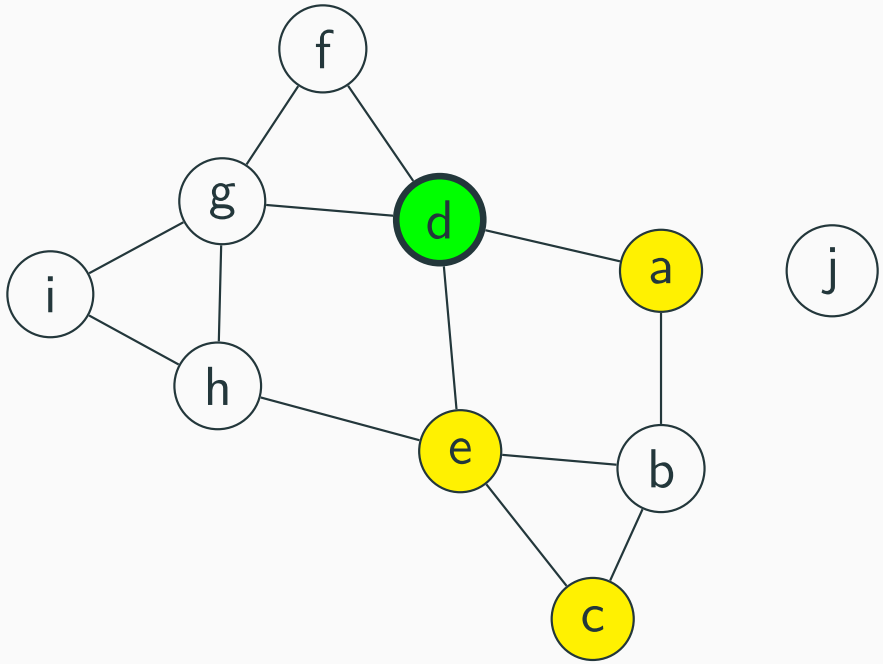

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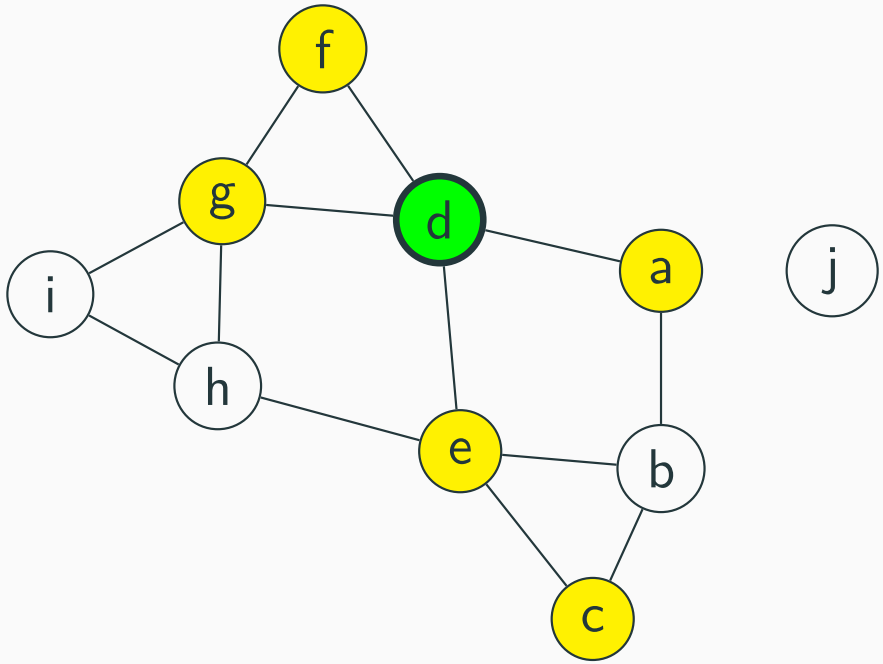

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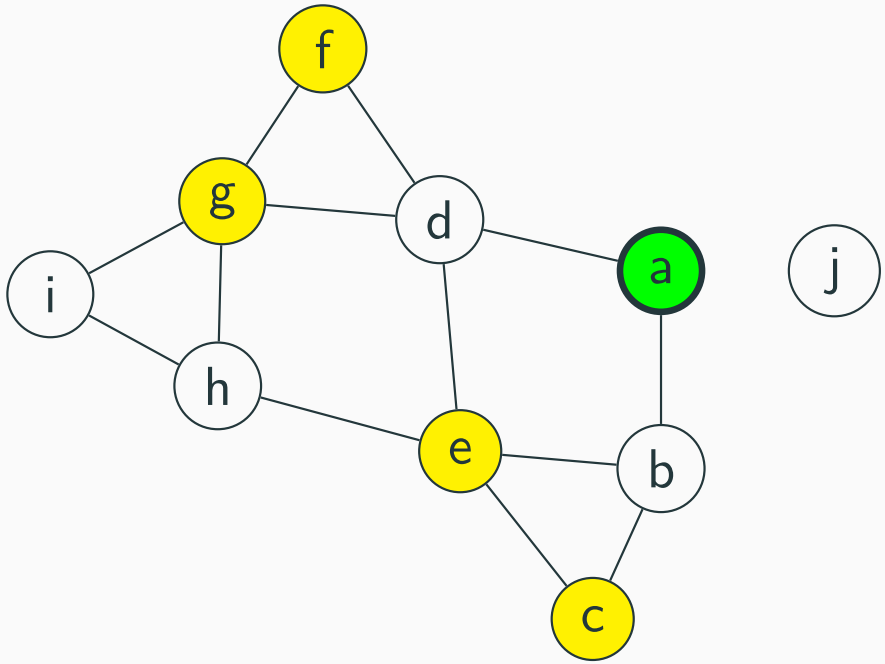

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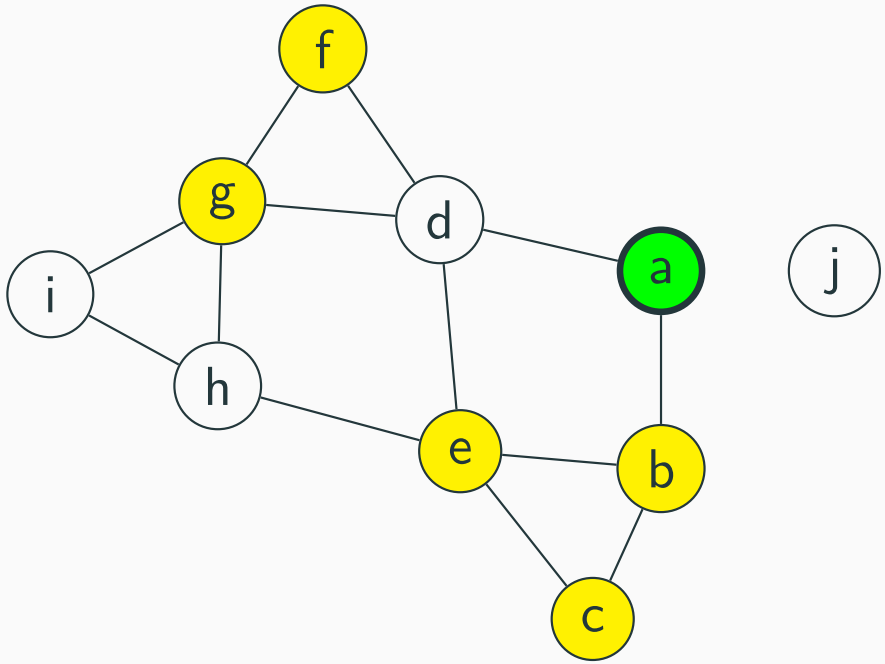

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Queue: e, f, g,

What went wrong?

A broken traversal

Problem: We're re-visiting nodes we already visited!

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A fix: Keep track of nodes we've already visited in a set!

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                visited.add(curr)
```
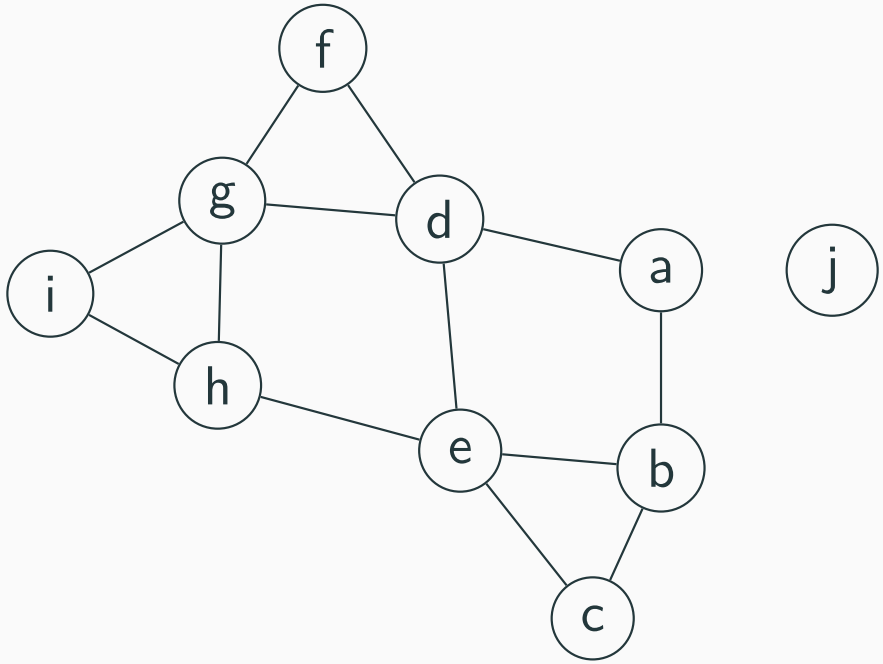



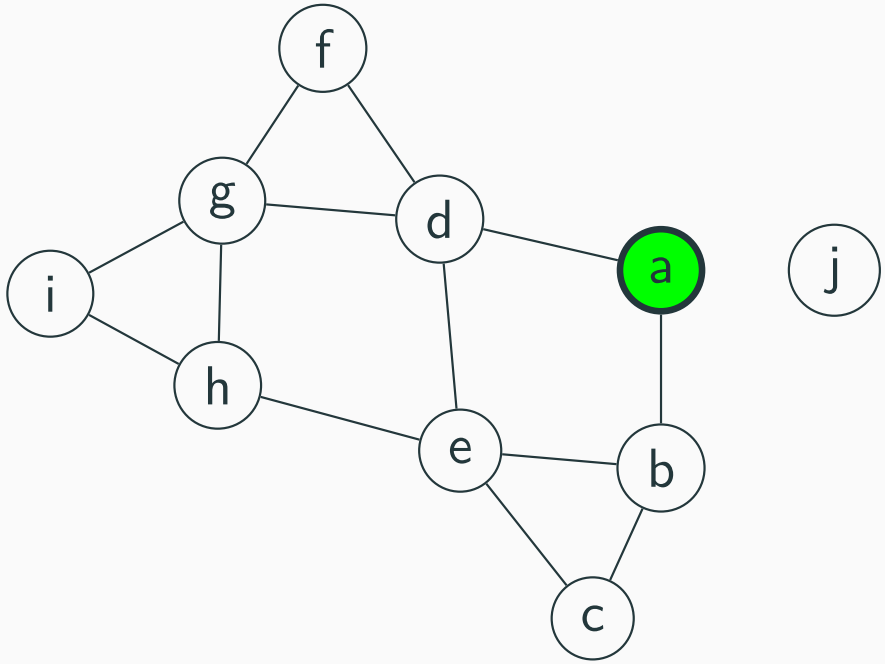
Current node:

Queue: a,

Visited: a,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



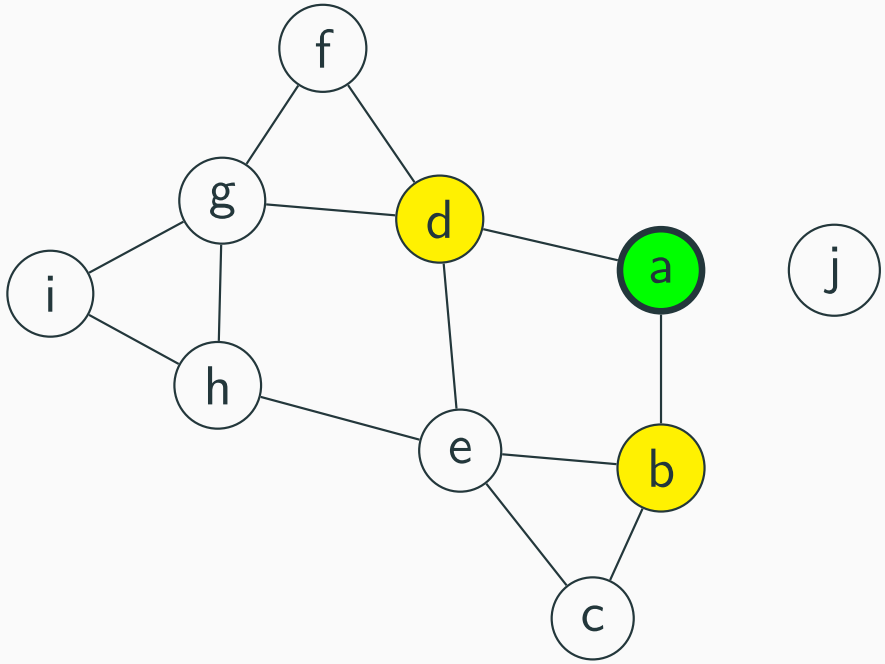
Current node: a

Queue:

Visited: a,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



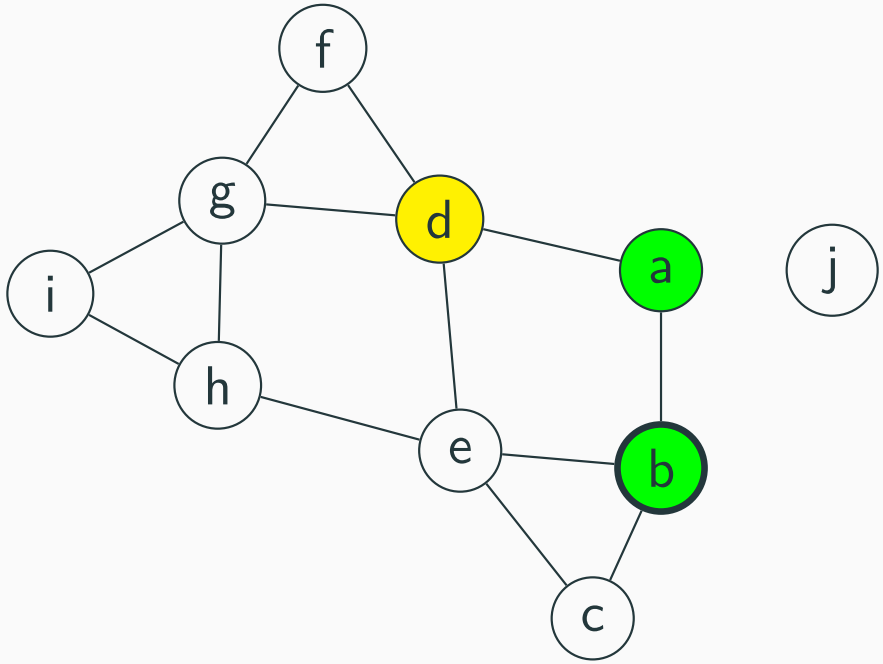
Current node: a

Queue: b, d,

Visited: a, b, d,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



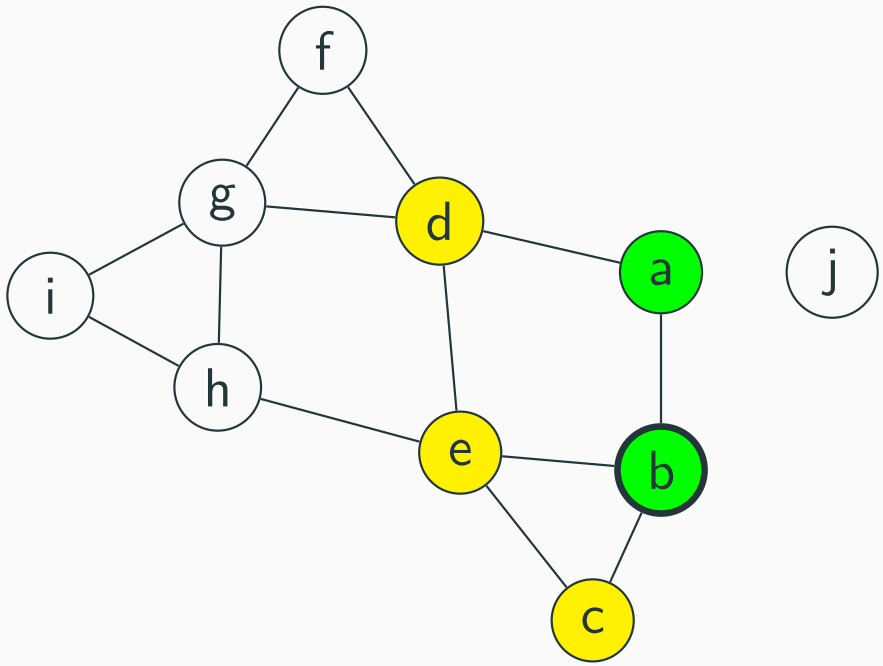
Current node: b

Queue: d,

Visited: a, b, d,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



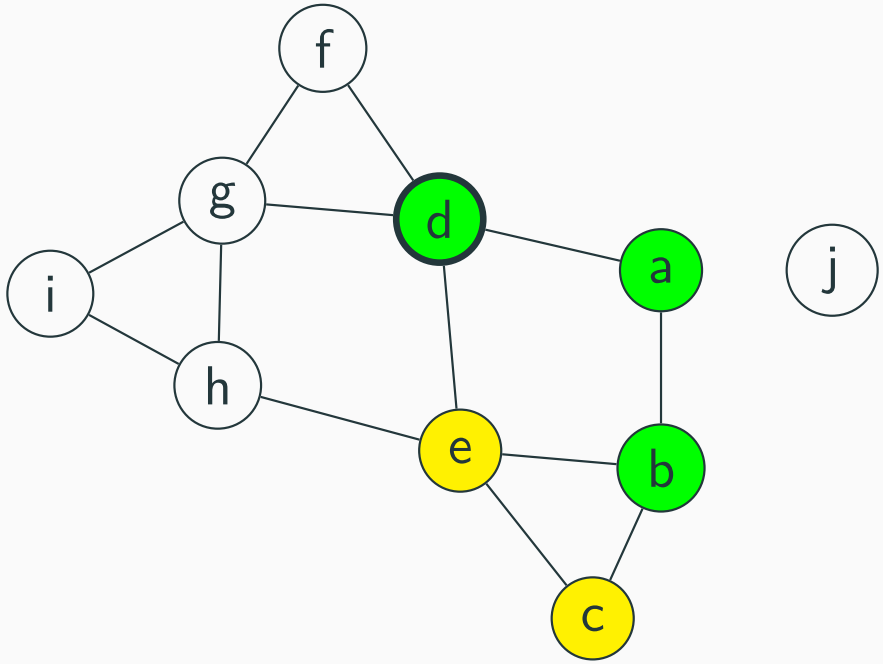
Current node: b

Queue: d, c, e,

Visited: a, b, d, c, e,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



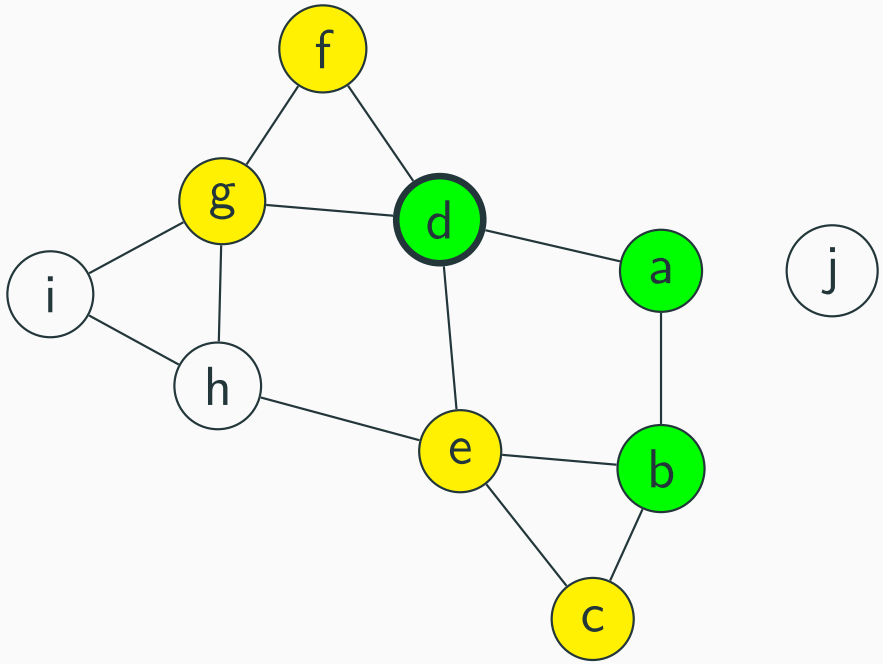
Current node: d

Queue: c, e,

Visited: a, b, d, c, e,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



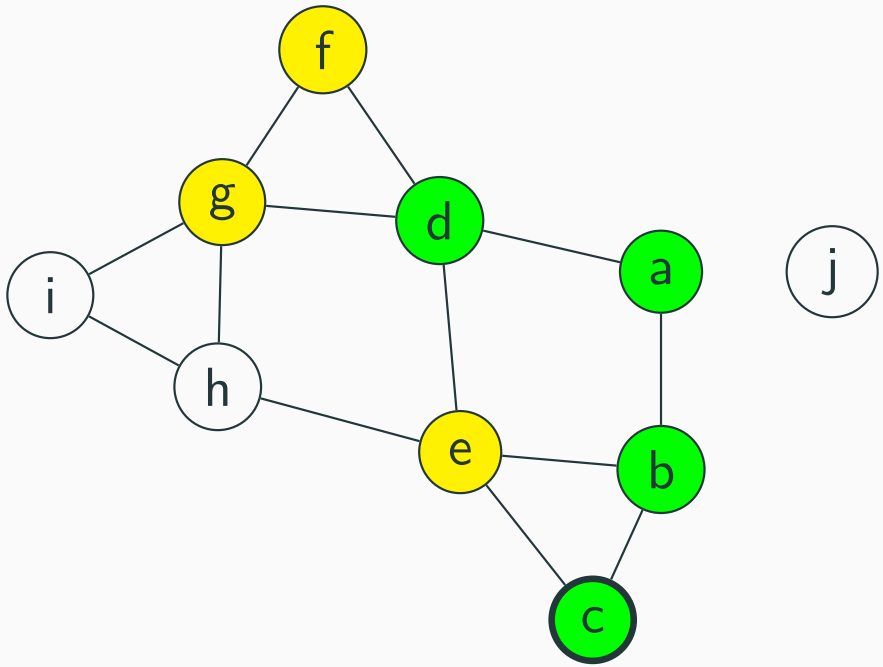
Current node: d

Queue: c, e, f, g,

Visited: a, b, d, c, e, f, g,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



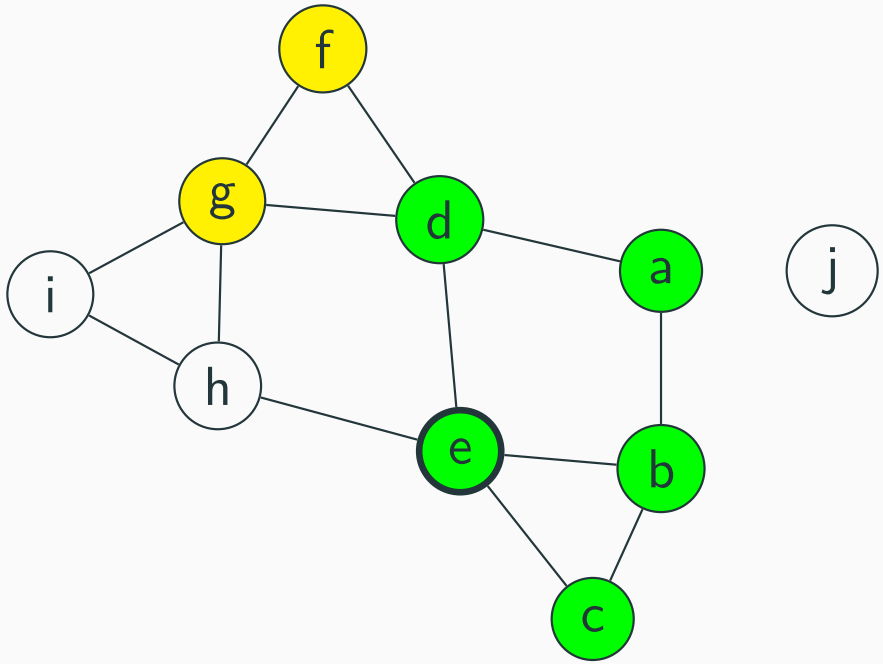
Current node: c

Queue: e, f, g,

Visited: a, b, d, c, e, f, g,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



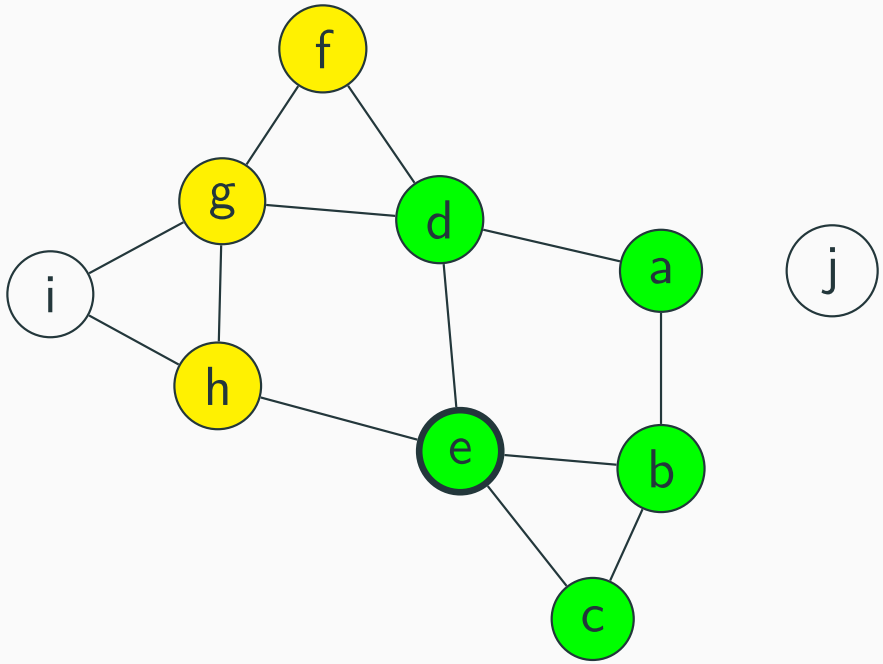
Current node: e

Queue: f, g,

Visited: a, b, d, c, e, f, g,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



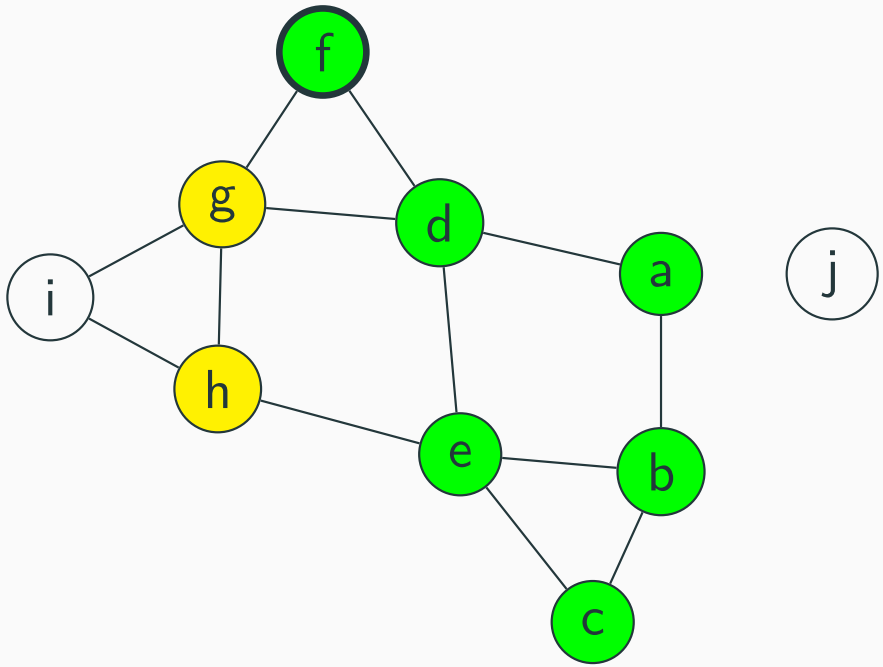
Current node: e

Queue: f, g, h,

Visited: a, b, d, c, e, f, g, h,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



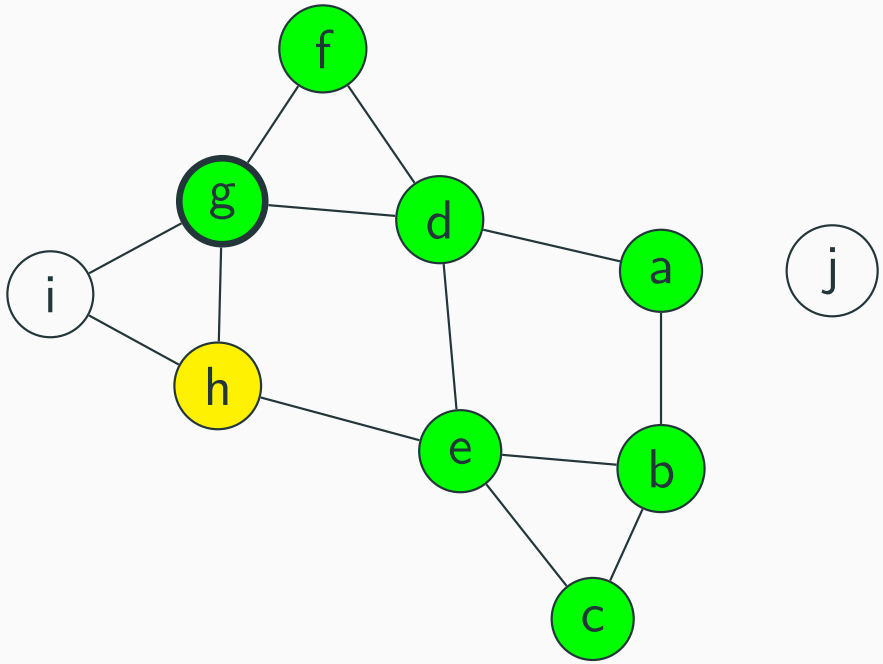
Current node: f

Queue: g, h,

Visited: a, b, d, c, e, f, g, h,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



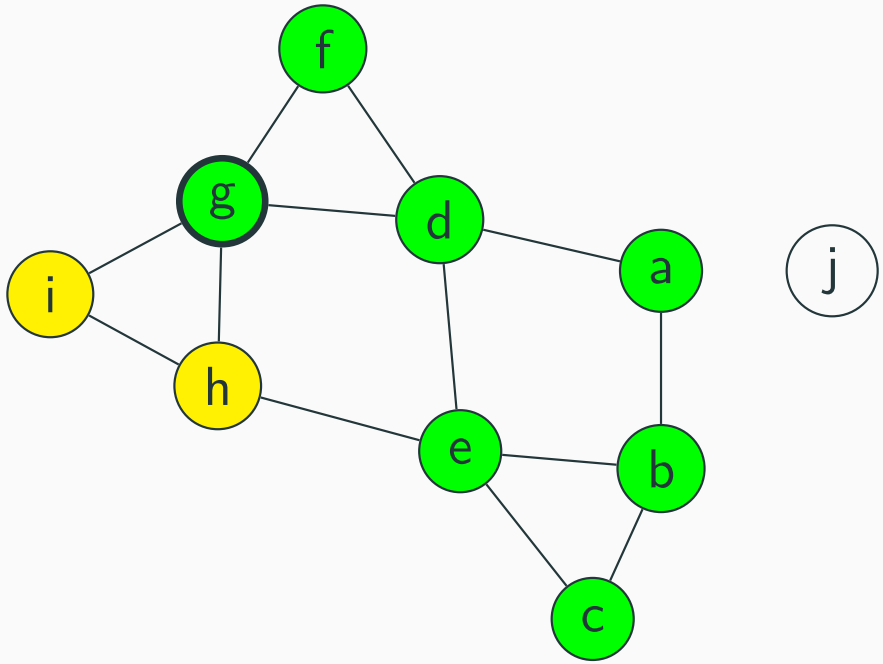
Current node: g

Queue: h,

Visited: a, b, d, c, e, f, g, h,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



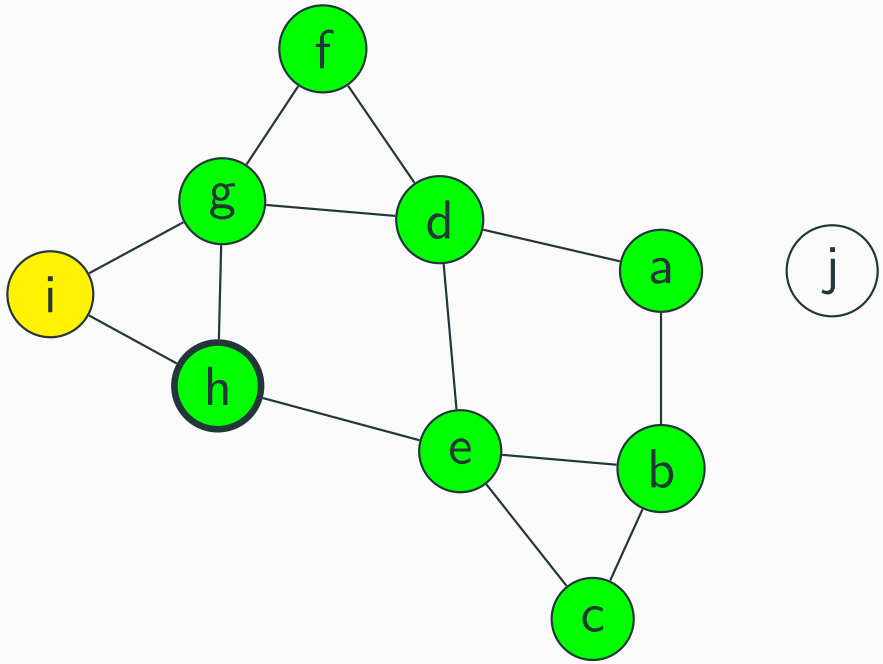
Current node: g

Queue: h, i,

Visited: a, b, d, c, e, f, g, h, i,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



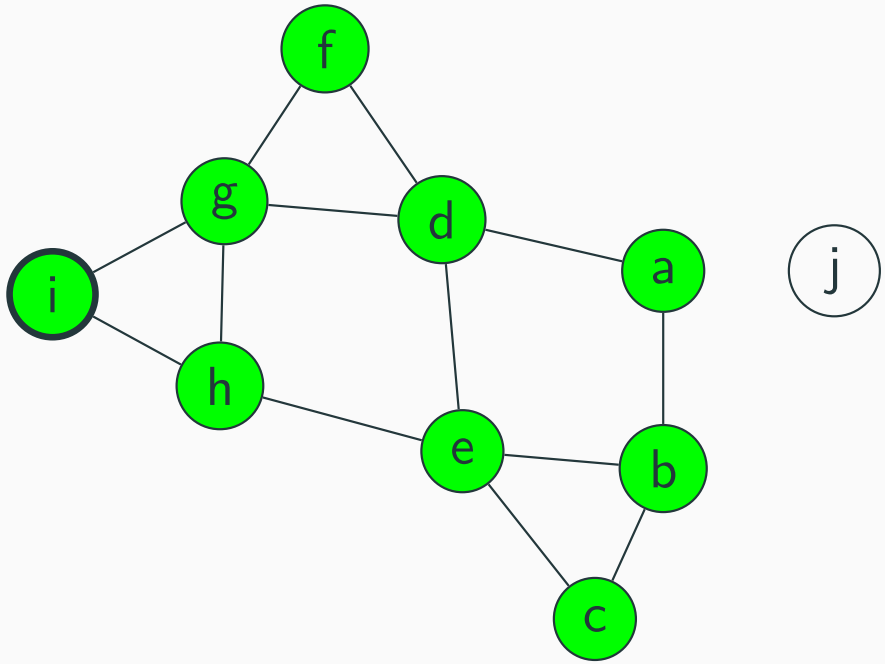
Current node: h

Queue: i,

Visited: a, b, d, c, e, f, g, h, i,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



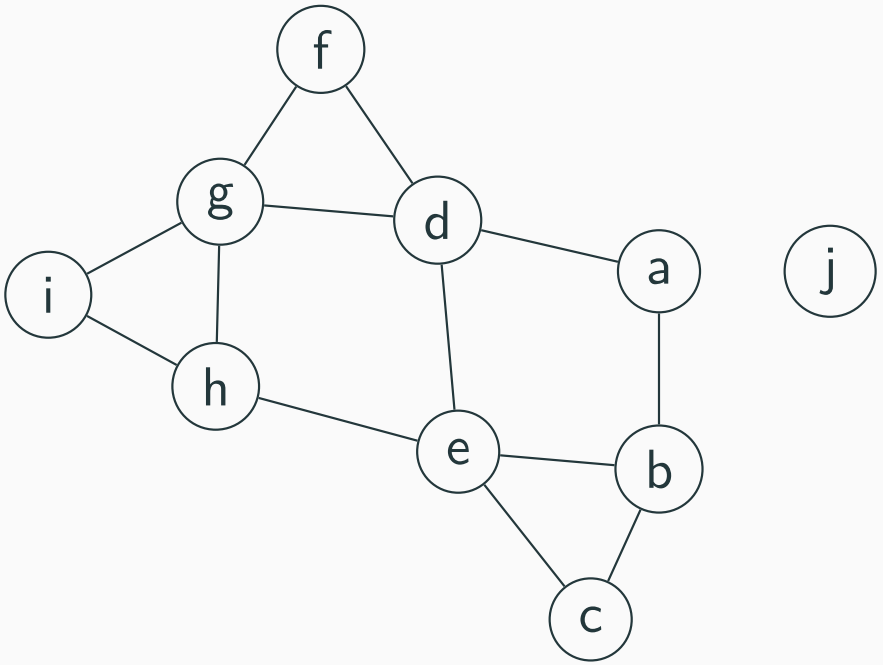
Current node: i

Queue:

Visited: a, b, d, c, e, f, g, h, i,

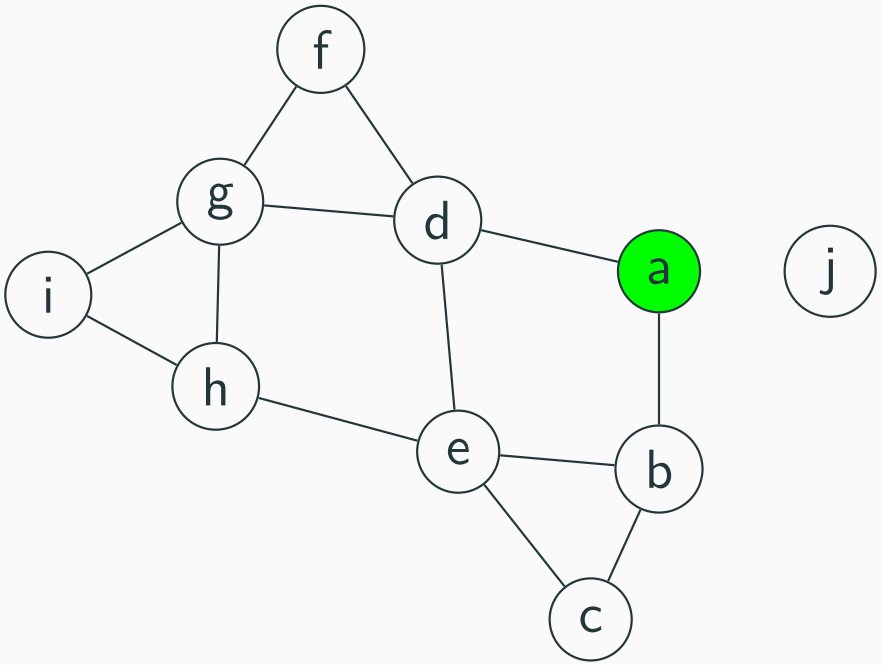
An interesting property...

Note: We visited the nodes in "rings" – maintained a gradually growing "frontier" of nodes.



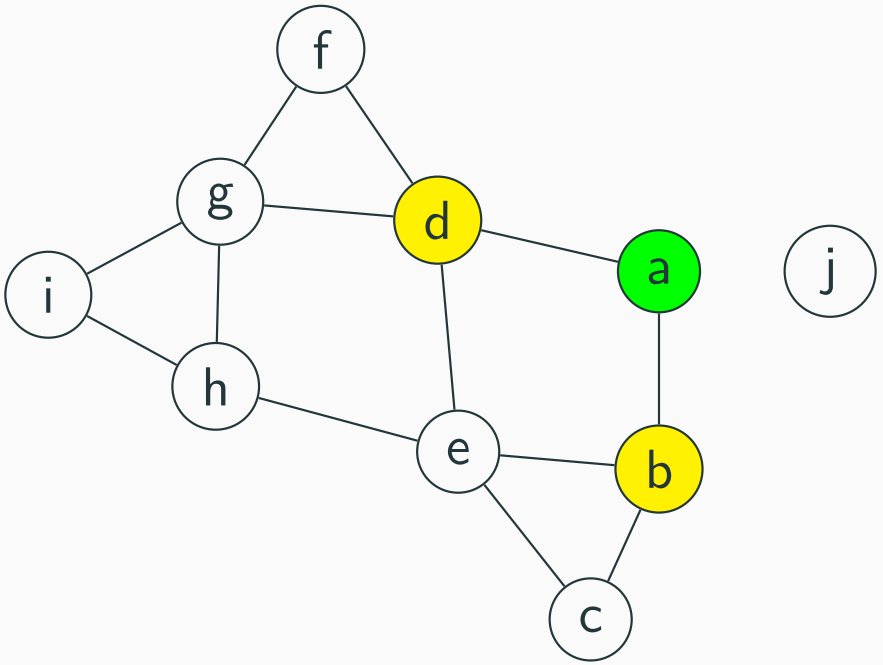
An interesting property...

Note: We visited the nodes in "rings" – maintained a gradually growing "frontier" of nodes.



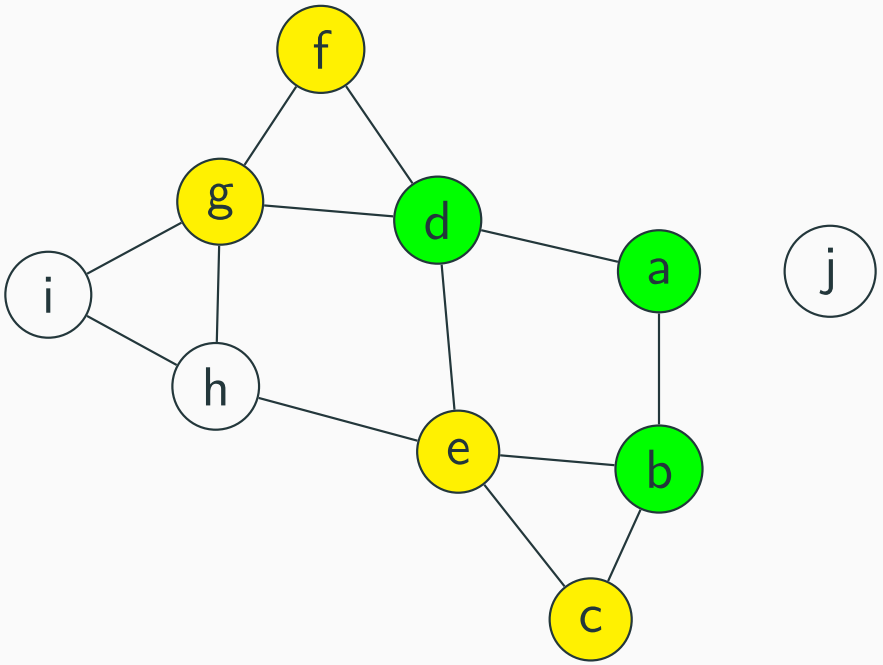
An interesting property...

Note: We visited the nodes in "rings" – maintained a gradually growing "frontier" of nodes.



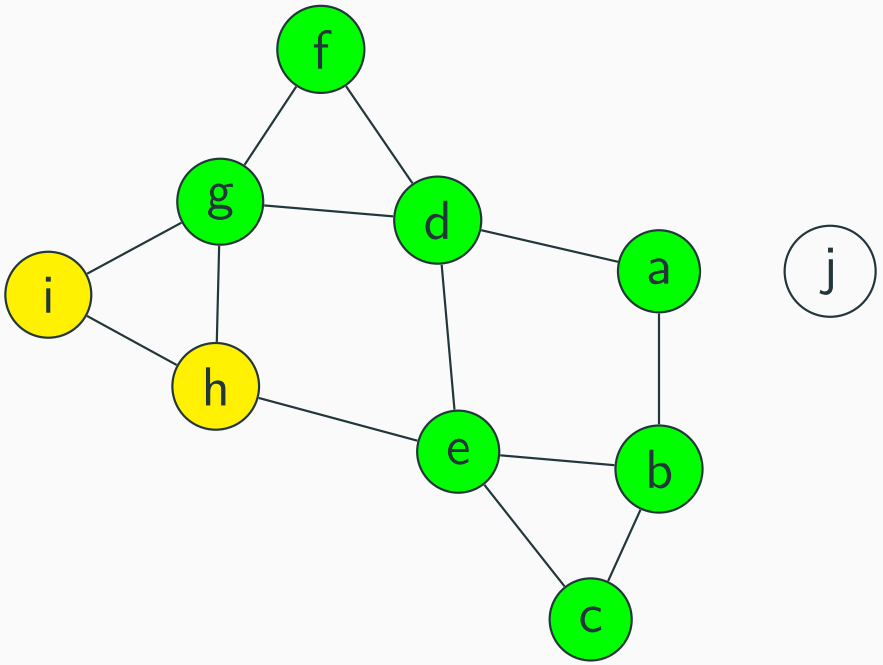
An interesting property...

Note: We visited the nodes in "rings" – maintained a gradually growing "frontier" of nodes.



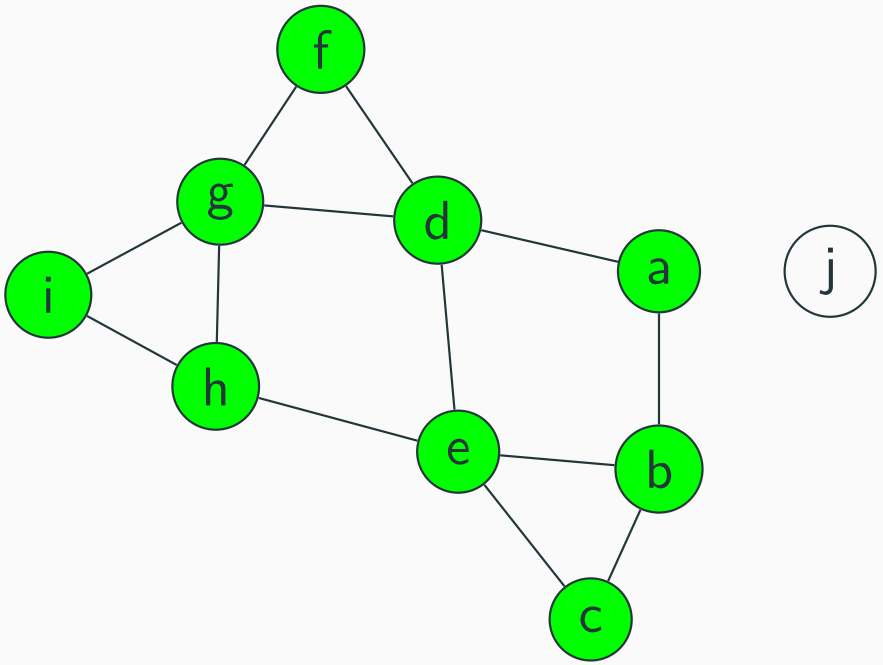
An interesting property...

Note: We visited the nodes in "rings" – maintained a gradually growing "frontier" of nodes.



An interesting property...

Note: We visited the nodes in "rings" – maintained a gradually growing "frontier" of nodes.



Pseudocode

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```

BFS analysis

Questions:

- What is the worst-case runtime? (Let $|V|$ be the number of vertices, let $|E|$ be the number of edges)

- What is the worst-case amount of memory used?

BFS analysis

Questions:

- What is the worst-case runtime? (Let $|V|$ be the number of vertices, let $|E|$ be the number of edges)
  We visit each vertex once, and each edge once, so $\mathcal{O}(|V| + |E|)$.
- What is the worst-case amount of memory used?
  Whatever the largest "horizon size" is. In the worst case, the horizon will contain $|V| - 1$ nodes, so $\mathcal{O}(|V|)$.

Note: $\mathcal{O}(|V| + |E|)$ is also called "graph linear".

Other applications of BFS

Describe how you would use or modify BFS to solve the following:

- Determine if some graph is also a tree.
- Print out all the elements in a tree level by level.
- Find the shortest path from one node to another.