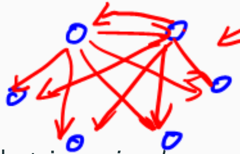
CSE 373: More on graphs; DFS and BFS

Michael Lee

Wednesday, Feb 14, 2018

Warmup

Warmup: $x = 6 \rightarrow$



6×5

Discuss with your neighbor:

- Remind your neighbor: what is a *simple graph*?

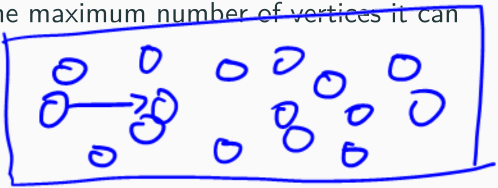
A simple graph has no...
(1) self-loops and (2) parallel edges

- Suppose we have a *simple, directed graph* with x nodes. What is the maximum number of edges it can have, in terms of x?

$x(x-1)$

- Now, suppose we have a different simple, undirected graph with y edges. What is the maximum number of vertices it can have, in terms of y?

∞



Warmup:

Discuss with your neighbor:

- Remind your neighbor: what is a *simple graph*?
  A simple graph is a graph that has no self-loops and no parallel edges.
- Suppose we have a *simple, directed graph* with x nodes. What is the maximum number of edges it can have, in terms of x?
  Each vertex can connect to $x - 1$ other vertices, so $x(x - 1)$.
- Now, suppose we have a different simple, undirected graph with y edges. What is the maximum number of vertices it can have, in terms of y?
  Infinite: just keep adding nodes with no edges attached.

Warmup:

Some follow-up questions:

$6+5+4 \dots$

- Suppose we have a *simple, undirected graph* with x nodes. What is the maximum number of edges it can have? What if the graph is not simple?

simple undirected

$$\sum_{i=1} i = \frac{x(x-1)}{2}$$

$$\frac{x(x-1)}{2}$$

not simple (directed or undirected)

∞

- Now, suppose we have a different simple, undirected graph with y edges. What is the maximum number of vertices it can have? What if the graph is not simple?

∞ | ∞

Warmup:

Some follow-up questions:

- Suppose we have a *simple, undirected graph* with x nodes. What is the maximum number of edges it can have? What if the graph is not simple?
  If the graph is simple, the max number of edges is exactly half of what it would be if the graph were directed. So, $\dfrac{x(x-1)}{2}$.
  If the graph is not simple, it's infinite: assuming $x > 0$, we can just keep adding more and more self-loops. Note that if $x = 0$, there can't be any edges at all.
- Now, suppose we have a different simple, undirected graph with y edges. What is the maximum number of vertices it can have? What if the graph is not simple?
  Either way, it's still infinite, for the same reasons given previously.

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What did we learn?

- In graphs with no restrictions, number of edges and number of vertices are independent.

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Dense graph

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To put it another way, dense graphs have "lots of edges"

Sparse graph

If $|E| \in \mathcal{O}\left(|V|\right)$, we sau the graph is **sparse**.

To put it another way, sparse graphs have "few" edges.

How do we represent graphs in code?

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Two common approaches, with different tradeoffs:

- Adjacency matrix
- Adjacency list

Adjacency matrix

Core idea:

- Assign each node a number from 0 to $|V| - 1$
- Create a $|V| \times |V|$ nested array of booleans or ints
- If $(x, y) \in E$, then `nestedArray[x][y] == true`

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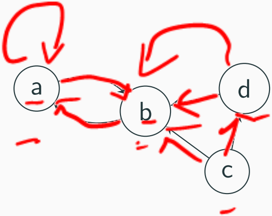

	a	b	c	d
a	T	T	F	F
b	T T		T	
c				T
d		T		

Adjacency ~~list~~ matrix

What is the worst-case runtime to:

- Get out-edges:
- Get in-edges:

$O(|V|)$

- Decide if an edge exists:
- Insert an edge:
- Delete an edge:

$O(1)$

How much space do we use? $O(|V|^2)$

Is this better for sparse or dense graphs?

Can we handle self-loops ✓ and ~~parallel edges~~?

Adjacency list

What is the worst-case runtime to:

- Get out-edges: $\mathcal{O}(|V|)$
- Get in-edges: $\mathcal{O}(|V|)$
- Decide if an edge exists: $\mathcal{O}(1)$
- Insert an edge: $\mathcal{O}(1)$
- Delete an edge: $\mathcal{O}(1)$

How much space do we use? $\mathcal{O}\left(|V|^2\right)$

Is this better for sparse or dense graphs? Dense ones

Can we handle self-loops and parallel edges? Self-loops yes, parallel edges, not easily

Adjacency list

Core idea:

- Assign each node a number from 0 to $|V| - 1$
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Core idea:

set

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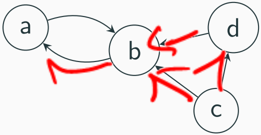

How do we tell if $c \to b$ exists?

Adjacency list

We can store edges using either sets or lists. Answer these questions for both.

What is the worst-case runtime to:

- Get out-edges:
- Get in-edges:
- Decide if an edge exists:
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list	set
$O(1)$	$O(1)$
$O(\lvert V\rvert+\lvert E\rvert)$	$O(\lvert V\rvert+\lvert E\rvert)$
$O(\lvert E\rvert)$	$O(1)$
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How much space do we use?

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Which do we pick?

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Observations:

- Most graphs are sparse
- If we implement adjacency lists using sets, we can get comparable worst-case performance

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So which do we pick?

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So by default, pick adjacency lists.

Walks and paths

Walk

A **walk** is a list of vertices $v_0, v_1, v_2, \ldots, v_n$ where if i is some int where $0 \leq i < v_n$, every pair $(v_i, v_{i+1}) \in E$ is true.

More intuitively, a walk is one continous line following the edges.

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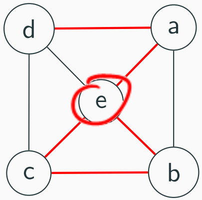

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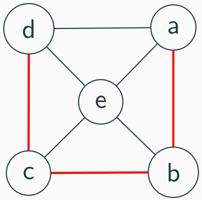

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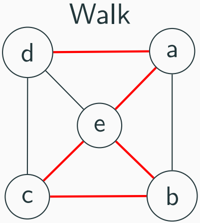

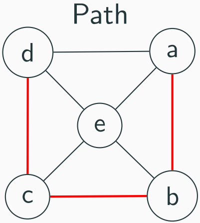

Connected components

Connected graph

A graph is **connected** if every vertex is connected to every other vertex via some path.

E.g.: if we pick up the graph and shake it, nothing flies off.

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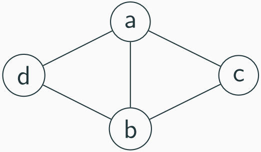

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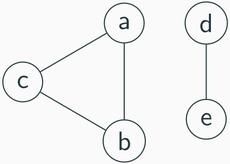

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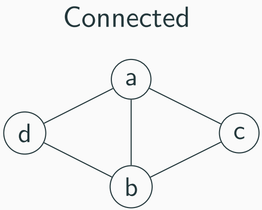
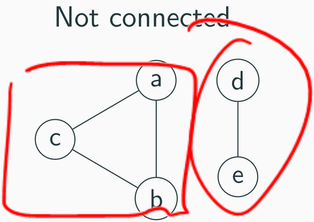




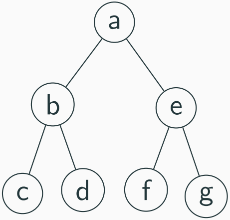
Connected component

A connected component of a graph is any *subgraph* (part of a graph) where all vertices are connected to each other.

Note: A connected graph has only one connected component.

Trees vs graphs

Is this a graph or tree?



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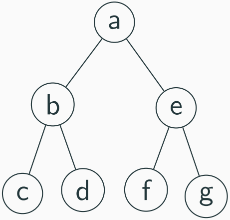

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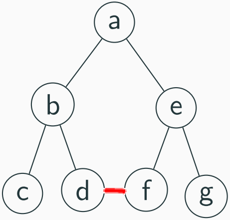

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A tree is a *connected* and *acyclic* graph.

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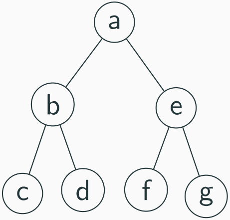

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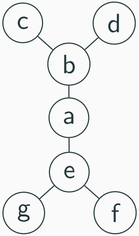

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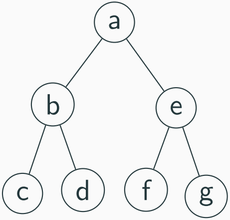

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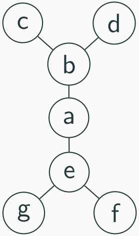

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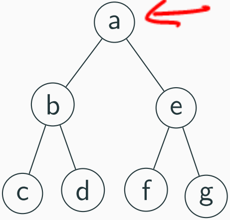

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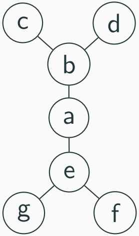

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Tree

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Rooted tree

A rooted tree is a tree where we call one special node the "root".

Detecting if a graph is connected

Question: How can we tell if a graph is connected or not?

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Idea: Let's just find out! Pick a node and see if there's a path to every other node!

Breadth-first search (BFS)

Breadth-first traversal, core idea:

1. Pick some node and “mark” it (or save it in a set, etc...)

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Pseudocode, version 1:

```
search(v):
    visited = empty set
    queue.enqueue(v)

    while (queue is not empty):
        curr = queue.dequeue()
        for (w : v.neighbors()):
            queue.enqueue(w)
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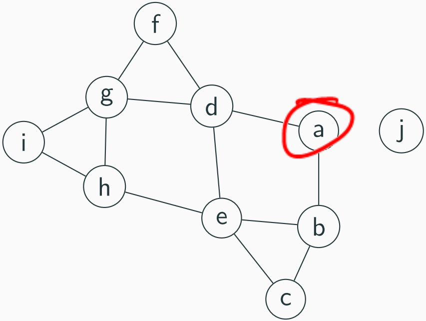

Current node:

Queue: a,

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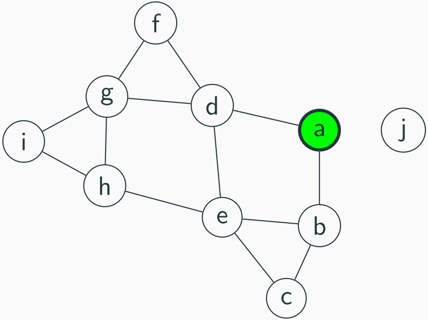

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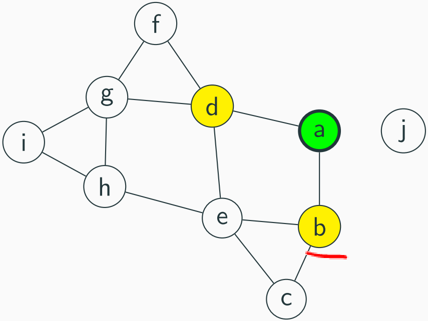

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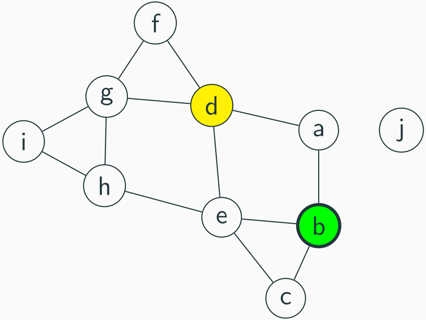

Current node: b

Queue: d,

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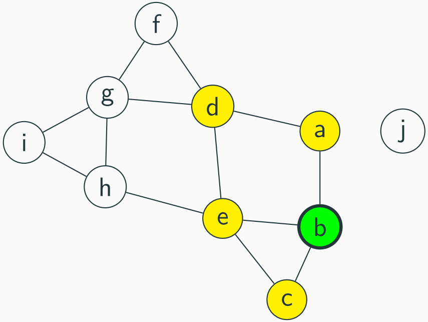

Current node: b

Queue: d, a, b, e,

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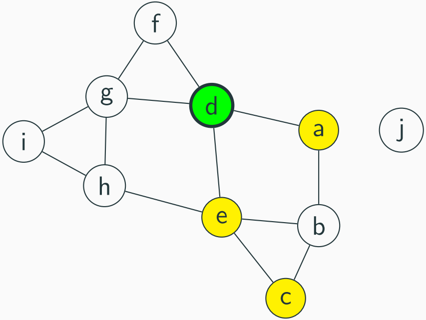

Current node: d

Queue: a, b, e,

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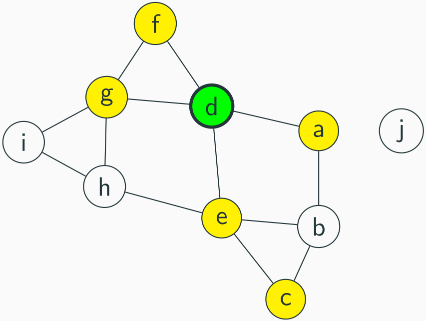

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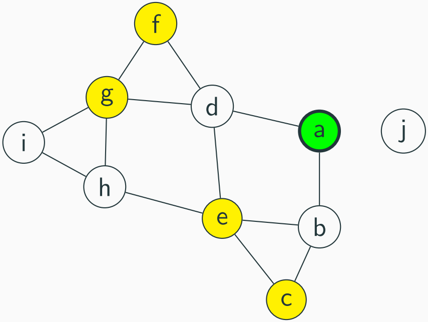

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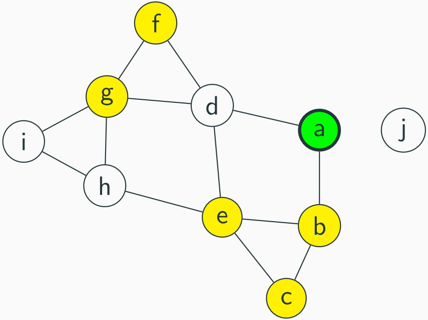

Current node: a

Queue: e, f, g,

What went wrong?

A broken traversal

Problem: We're re-visiting nodes we already visited!

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A fix: Keep track of nodes we've already visited in a set!

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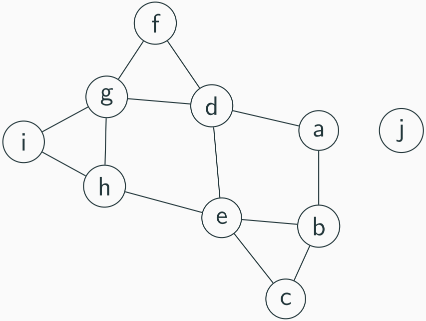

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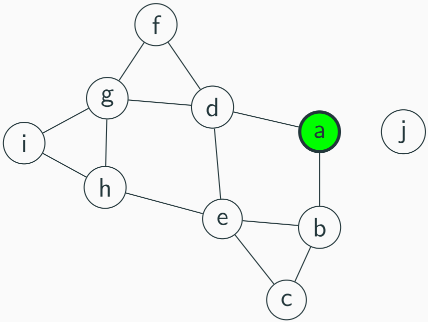


Current node: a

Queue:

Visited: a,

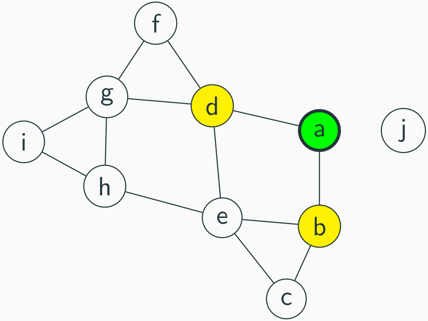
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: a

Queue: b, d,

Visited: a, b, d,

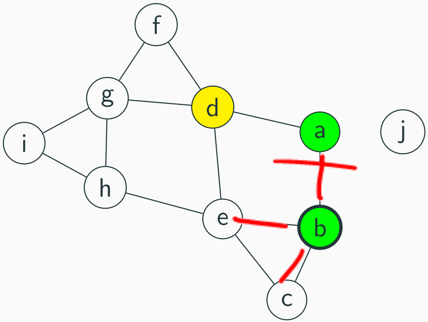
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: b

Queue: d,

Visited: a, b, d,

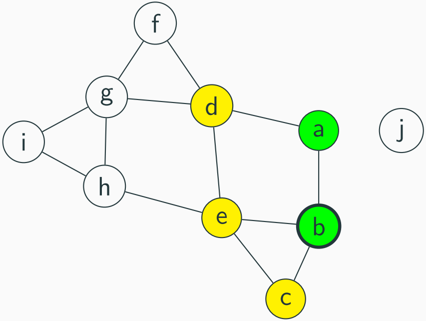
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: b

Queue: d, c, e,

Visited: a, b, d, c, e,

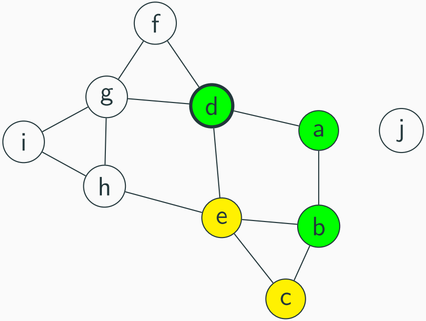
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: d

Queue: c, e,

Visited: a, b, d, c, e,

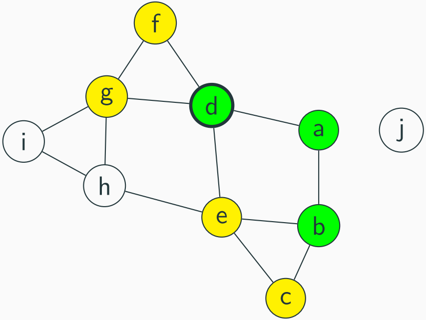
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: d

Queue: c, e, f, g,

Visited: a, b, d, c, e, f, g,

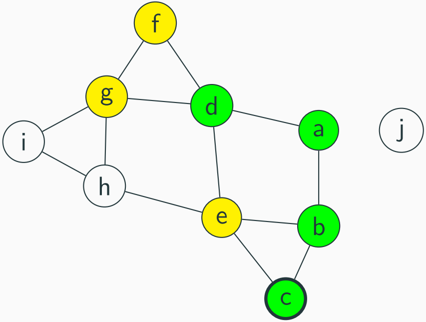
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: c

Queue: e, f, g,

Visited: a, b, d, c, e, f, g,

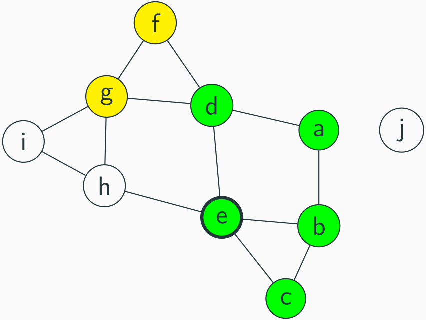
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: e

Queue: f, g,

Visited: a, b, d, c, e, f, g,

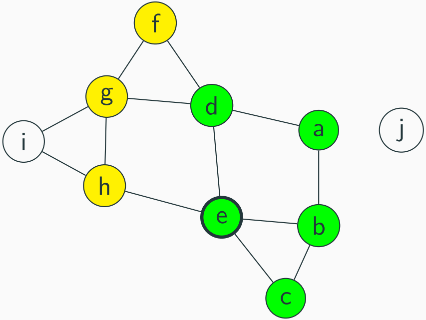
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: e

Queue: f, g, h,

Visited: a, b, d, c, e, f, g, h,

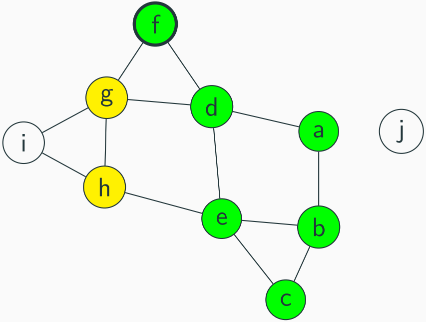
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: f

Queue: g, h,

Visited: a, b, d, c, e, f, g, h,

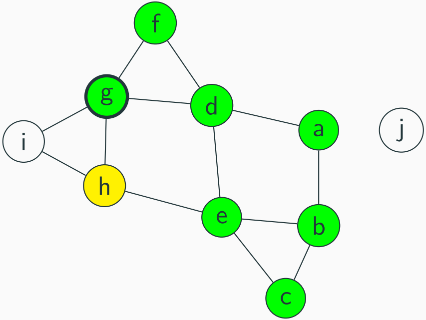
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: g

Queue: h,

Visited: a, b, d, c, e, f, g, h,

Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
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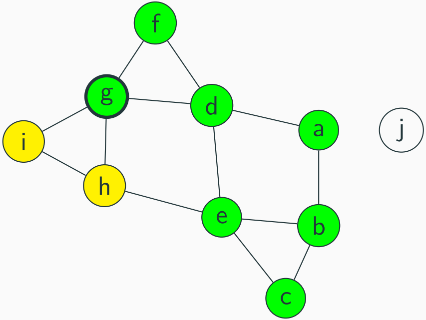
        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: g

Queue: h, i,

Visited: a, b, d, c, e, f, g, h, i,

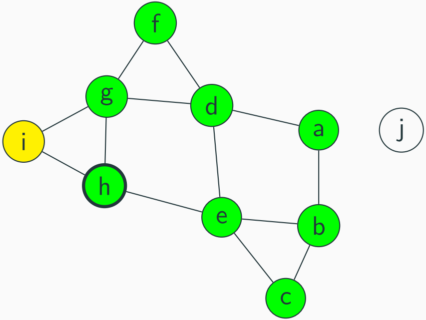
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: h

Queue: i,

Visited: a, b, d, c, e, f, g, h, i,

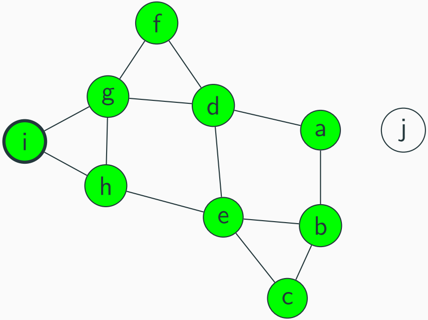
Breadth-first search (BFS) example

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Current node: i

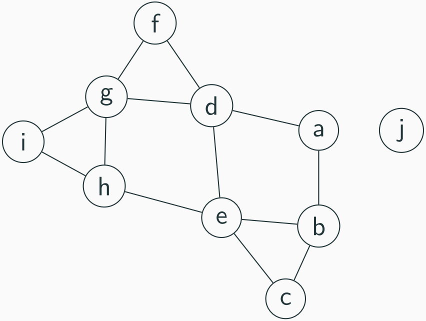
$O(|V| + |E|)$

Queue:

Visited: a, b, d, c, e, f, g, h, i,

An interesting property...

Note: We visited the nodes in “rings” – maintained a gradually growing “frontier” of nodes.



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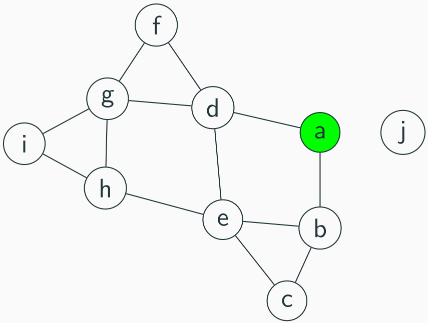

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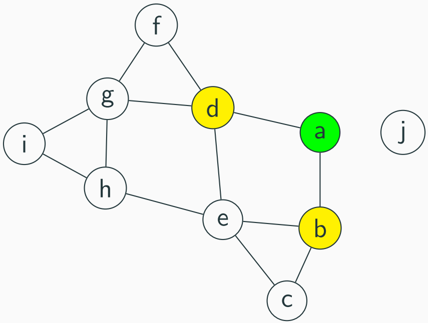

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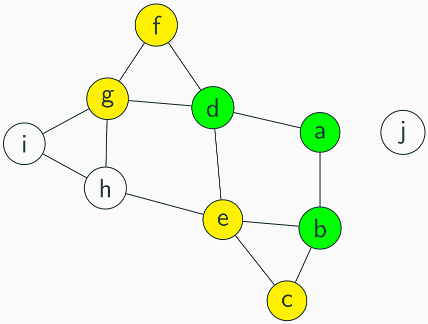

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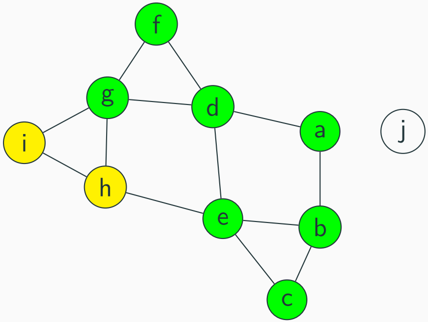

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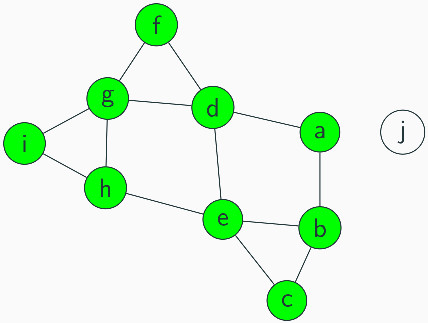


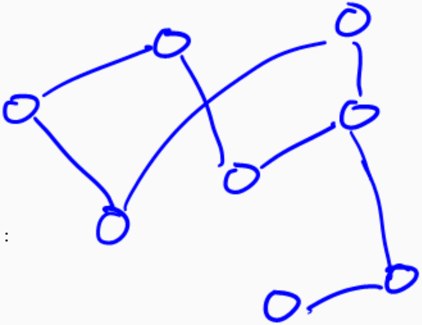
BFS analysis

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(curr)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```



Questions:

- What is the worst-case runtime? (Let $|V|$ be the number of vertices, let $|E|$ be the number of edges)
- What is the worst-case amount of memory used?

BFS analysis

```
search(v):
    visited = empty set

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        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```

Questions:

- What is the worst-case runtime? (Let $|V|$ be the number of vertices, let $|E|$ be the number of edges)
  We visit each vertex once, and each edge once, so $\mathcal{O}(|V| + |E|)$.
- What is the worst-case amount of memory used?

Other applications of BFS

Describe how you would use or modify BFS to solve the following:

- Determine if some graph is also a tree.
- Print out all the elements in a tree level by level.
- Find the shortest path from one node to another.

Depth-first search (DFS)

Question: Why a queue? Can we use other data structures?

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Answer: Yes! Any kind of list-like thing that supports appends and removes works! For example, what if we try using a stack?

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The BFS algorithm:

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(v)

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        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```

Depth-first search (DFS)

Question: Why a queue? Can we use other data structures?

Answer: Yes! Any kind of list-like thing that supports appends and removes works! For example, what if we try using a stack?

The BFS algorithm:

```
search(v):
    visited = empty set

    queue.enqueue(v)
    visited.add(v)

    while (queue is not empty):
        curr = queue.dequeue()

        for (w : v.neighbors()):
            if (w not in visited):
                queue.enqueue(w)
                visited.add(curr)
```

The DFS algorithm:

```
search(v):
    visited = empty set
    stack.push(v)
    visited.add(v)

    while (stack is not empty):
        curr = stack.pop()
        visited.add(curr)

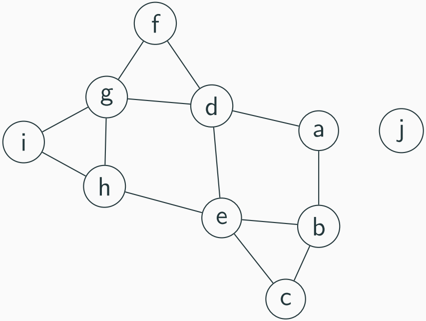
        for (w : v.neighbors()):
            if (w not in visited):
                stack.push(w)
                visited.add(v)
```

Depth-first search (DFS) example

```
search(v):
    visited = empty set
    stack.push(v)

    while (stack is not empty):
        curr = stack.pop()
        visited.add(curr)

        for (w : v.neighbors()):
            if (w not in visited):
                stack.push(w)
```



Current node:

Stack:

Visited: