Problem: Need a rigorous way of getting a closed form
The tree method: precise analysis

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We want to answer a few core questions:

1. How many nodes are there on level \( i \) (\( i = 0 \) is "root" level)?
2. At some level \( i \), how much work does a single node do? (Ignoring subtrees)
3. How many recursive levels are there?
4. How much work does the leaf level (base cases) do?
   1. How much work does a single leaf node do?
   2. How many leaf nodes are there?
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- $1$ node, $n$ work per $2$ nodes, $n^2$ work per $4$ nodes, $n^4$ work per $2^i$ nodes, $n^i$ work per $2^i$ nodes,
- $1$ work per $1$ node,
- $\text{numNodes}(i) = 2^i$,
- $\text{workPerNode}(n, i) = n^i$,
- $\text{numLevels}(n) = ?$,
- $\text{workPerLeafNode}(n) = 1$,
- $\text{numLeafNodes}(n) = ?$.
The tree method: precise analysis

1. numNodes(i) = ?
2. workPerNode(n, i) = ?
3. numLevels(n) = ?
4. workPerLeafNode(n) = ?
5. numLeafNodes(n) = ?
The tree method: precise analysis

1 node, $n$ work per

2 nodes, $\frac{n}{2}$ work per

4 nodes, $\frac{n}{4}$ work per

$2^i$ nodes, $\frac{n}{i}$ work per

$2^h$ nodes, 1 work per

1. $\text{numNodes}(i) = ?$
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5. $\text{numLeafNodes}(n) = ?$

1 node, $n$ work per node
2 nodes, $\frac{n}{2}$ work per node
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How many levels are there, exactly? Is it $\log_2(n)$?

Let's try an example. Suppose we have $T(4)$. What happens?

$T(8)$

Height is $\log_2(4) = 2$.

For this recursive function, the number of recursive levels is the same as the height.

Important: total levels, counting the base case, is the height + 1.

Important: for other recursive functions, where the base case doesn't happen at $n \leq 1$, the number of recursive levels might be different than the height.
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![Recursive Tree Diagram]

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```
   4
  /\  /\  \\
 /   \ /   \ /
2     2
   \   \   \
  1    1    1
```

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  / \  \
 2   2
 / \ / \  \
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We discovered:

1. \( \text{numNodes}(i) = 2^i \)
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Our formulas:

\[
\text{recursiveWork} = \sum_{i=0}^{\text{numLevels}(n)} \text{numNodes}(i) \cdot \text{workPerNode}(n, i)
\]

\[
\text{baseCaseWork} = \text{numLeafNodes}(n) \cdot \text{workPerLeafNode}(n)
\]

\[
\text{totalWork} = \text{recursiveWork} + \text{baseCaseWork}
\]
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$$\text{totalWork} = \text{recursiveWork} + \text{baseCaseWork}$$
The tree method: precise analysis

Solve for recursive case:

\[
\text{recursiveWork} = \sum_{i=0}^{\log_2(n)} 2^i \cdot \frac{n}{2^i}
\]

Solve for base case:

\[
\text{baseCaseWork} = \text{numLeafNodes}(n) \cdot \text{workDonePerLeafNode}(n) = n \cdot 1 = n
\]

So exact closed form is \( n \log_2(n) + n \).
The tree method: precise analysis

Solve for recursive case:

\[\text{recursiveWork} = \sum_{i=0}^{\log_2(n)} 2^i \cdot \frac{n}{2^i}\]

\[= \sum_{i=0}^{\log_2(n)} n\]

Solve for base case:

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Solve for recursive case:

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\text{recursiveWork} = \log_2(n) \sum_{i=0}^{\log_2(n)} 2^i \cdot \frac{n}{2^i}
\]

\[
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\]

\[
= n \log_2(n)
\]

Solve for base case:

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\text{baseCaseWork} = \text{numLeafNodes}(n) \cdot \text{workDonePerLeafNode}(n)
\]

\[
= n \cdot 1 = n
\]

So exact closed form is \( n \log_2(n) + n \).
Practice: Let’s go back to our old recurrence...

\[
S(n) = \begin{cases} 
2 & \text{if } n \leq 1 \\
2S\left(\frac{n}{3}\right) + n^2 & \text{otherwise}
\end{cases}
\]
The tree method: practice

\[ n^2 \]

\[ \frac{n^2}{9} \]

\[ \frac{n^4}{81} \]

\[ \frac{n^2}{81} \]

\[ \vdots \]

\[ 2 \]

\[ \frac{n^2}{81} \]

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\[ \vdots \]
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2 nodes, $\frac{n^2}{9}$ work per

4 nodes, $\frac{n^2}{3^2}$ work per

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3. \( \text{numLevels}(n) = \log_3(n) \)
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Combine into a single expression representing the total runtime.
The tree method: practice

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Combine into a single expression representing the total runtime.

$$\text{totalWork} = \left( \sum_{i=0}^{\log_3(n)} 2^i \cdot \frac{n^2}{9^i} \right) + 2n^{\log_3(2)}$$
The tree method: practice

1. numNodes\( (i) \) \quad = \quad 2^i
2. workPerNode\( (n, i) \) \quad = \quad \frac{n^2}{9^i}
3. numLevels\( (n) \) \quad = \quad \log_3(n)
4. workPerLeafNode\( (n) \) \quad = \quad 2
5. numLeafNodes\( (n) \) \quad = \quad 2^{\text{numLevels}(n)} = 2^{\log_3(n)} = n^{\log_3(2)}

Combine into a single expression representing the total runtime.

\[
\text{totalWork} = \left( \sum_{i=0}^{\log_3(n)} 2^i \cdot \frac{n^2}{9^i} \right) + 2n^{\log_3(2)}
\]

\[
= n^2 \sum_{i=0}^{\log_3(n)} \frac{2^i}{9^i} + 2n^{\log_3(2)}
\]
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\]

\[
= n^2 \sum_{i=0}^{\log_3(n)} \left( \frac{2}{9} \right)^i + 2n^{\log_3(2)}
\]
The finite geometric series

We have:

\[ n^2 \sum_{i=0}^{\log_3(n)} \left( \frac{2}{9} \right)^i + 2n^{\log_3(2)} \]
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We have: \[ n^2 \sum_{i=0}^{\log_3(n)} \left( \frac{2}{9} \right)^i + 2n^{\log_3(2)} \]

The finite geometric series identity: \[ \sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r} \]
The finite geometric series

We have: \[ n^2 \sum_{i=0}^{\log_3(n)} \left( \frac{2}{9} \right)^i + 2n^{\log_3(2)} \]

The finite geometric series identity: \[ \sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r} \]

Plug and chug:

\[
\text{totalWork} = n^2 \sum_{i=0}^{\log_3(n)} \left( \frac{2}{9} \right)^i + 2n^{\log_3(2)} \\
= n^2 \sum_{i=0}^{\log_3(n)+1-1} \left( \frac{2}{9} \right)^i + 2n^{\log_3(2)} \\
= n^2 \frac{1 - \left( \frac{2}{9} \right)^{\log_3(n)+1}}{1 - \frac{2}{9}} + 2n^{\log_3(2)}
\]
Applying the finite geometric series

With a bunch of effort...

\[
\text{totalWork} = n^2 \frac{1 - \left( \frac{2}{9} \right)^{\log_3(n)+1}}{1 - \frac{2}{9}} + 2n^{\log_3(2)}
\]

\[
= \frac{9}{7} n^2 \left( 1 - \frac{2}{9} \left( \frac{2}{9} \right)^{\log_3(n)} \right) + 2n^{\log_3(2)}
\]

\[
= \frac{9}{7} n^2 - \frac{2}{7} n^2 \left( \frac{2}{9} \right)^{\log_3(n)} + 2n^{\log_3(2)}
\]

\[
= \frac{9}{7} n^2 - \frac{2}{7} n^2 n^{\log_3(2/9)} + 2n^{\log_3(2)}
\]

\[
= \frac{9}{7} n^2 - \frac{2}{7} n^2 n^{\log_3(2)-2} + 2n^{\log_3(2)}
\]

\[
= \frac{9}{7} n^2 - \frac{2}{7} n^{\log_3(2)} + 2n^{\log_3(2)}
\]

\[
= \frac{9}{7} n^2 + \frac{12}{7} n^{\log_3(2)}
\]
Is there an easier way?
The master theorem

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If we want to find an exact closed form, no. Must use either the unfolding technique or the tree technique.
The master theorem

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If we want to find an exact closed form, no. Must use either the unfolding technique or the tree technique.

If we want to find a big-Θ bound, yes.
The master theorem

Suppose we have a recurrence of the following form:

\[
T(n) = \begin{cases} 
    d & \text{if } n = 1 \\
    aT\left(\frac{n}{b}\right) + n^c & \text{otherwise}
\end{cases}
\]
The master theorem

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T(n) = \begin{cases} 
  d & \text{if } n = 1 \\
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\end{cases}
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Then...

- If \( \log_b(a) < c \), then \( T(n) \in \Theta(n^c) \)
- If \( \log_b(a) = c \), then \( T(n) \in \Theta(n^c \log(n)) \)
- If \( \log_b(a) > c \), then \( T(n) \in \Theta(n^{\log_b(a)}) \)
The master theorem

Given:

\[ T(n) = \begin{cases} 
  d & \text{if } \log_b(a) < c, \text{ then } T(n) \in \Theta(n^c) \\
  aT\left(\frac{n}{b}\right) + n^c & \text{if } \log_b(a) = c, \text{ then } T(n) \in \Theta(n^c \log(n)) \\
  & \text{if } \log_b(a) > c, \text{ then } T(n) \in \Theta(n^{\log_b(a)}) 
\end{cases} \]

Then...

Sanity check: try checking merge sort.

We have \(a = 2\), \(b = 2\), and \(c = 1\). We know \(\log_b(a) = \log_2(2) = 1 = c\), therefore merge sort is \(\Theta(n \log(n))\).

Sanity check: try checking \(S(n) = 2S\left(\frac{n}{3}\right) + n^2\).

We have \(a = 2\), \(b = 3\), and \(c = 2\). We know \(\log_3(2) \leq 1 < 2 = c\), therefore \(S(n) \in \Theta(n^2)\).
The master theorem

Given: \[ T(n) = \begin{cases} \frac{d}{a} & \text{if } \log_b(a) < c, \text{ then } T(n) \in \Theta(n^c) \\ aT\left(\frac{n}{b}\right) + nc & \text{if } \log_b(a) = c, \text{ then } T(n) \in \Theta(n^c \log(n)) \\ & \text{if } \log_b(a) > c, \text{ then } T(n) \in \Theta(n^{\log_b(a)}) \end{cases} \]

Then...

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Given: \( T(n) = \begin{cases} 
  d & \text{if } \log_b(a) < c, \text{ then } T(n) \in \Theta(n^c) \\
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\end{cases} \)

Then...

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The master theorem

Given:  

\[ T(n) = \begin{cases} 
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  aT\left(\frac{n}{b}\right) + nc & \text{if } \log_b(a) = c, \\
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\end{cases} \]

Then...

\[ \text{If } \log_b(a) < c, \text{ then } T(n) \in \Theta(n^c) \]
\[ \text{If } \log_b(a) = c, \text{ then } T(n) \in \Theta(n^c \log(n)) \]
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We have \( a = 2, \ b = 2, \) and \( c = 1. \) We know
\( \log_b(a) = \log_2(2) = 1 = c, \) therefore merge sort is \( \Theta(n \log(n)). \)

Sanity check: try checking \( S(n) = 2S(n/3) + n^2. \)
The master theorem

Given:

\[ T(n) = \begin{cases} 
  d & \text{Then...} \\
  aT\left(\frac{n}{b}\right) + n^c & \text{If } \log_b(a) < c, \text{ then } T(n) \in \Theta(n^c) \\
  & \text{If } \log_b(a) = c, \text{ then } T(n) \in \Theta(n^c \log(n)) \\
  & \text{If } \log_b(a) > c, \text{ then } T(n) \in \Theta(n^{\log_b(a)}) 
\end{cases} \]

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We have \( a = 2, b = 2, \) and \( c = 1. \) We know \( \log_b(a) = \log_2(2) = 1 = c, \) therefore merge sort is \( \Theta(n \log(n)) \).

Sanity check: try checking \( S(n) = 2S(n/3) + n^2. \)

We have \( a = 2, b = 3, \) and \( c = 2. \) We know \( \log_3(2) \leq 1 < 2 = c, \) therefore \( S(n) \in \Theta(n^2) \).
The master theorem: intuition

**Intuition, the \( \log_b(a) < c \) case:**

1. We do work more rapidly than we divide.
2. So, more of the work happens near the “top”, which means that the \( n^c \) term dominates.
The master theorem: intuition

**Intuition, the** $\log_b(a) > c$ **case:**

1. We divide more rapidly than we do work.
2. So, most of the work happens near the “bottom”, which means the work done in the leaves dominates.
3. Note: Work in leaves is about
   
   $d \cdot a^{\text{height}} = d \cdot a^{\log_b(n)} = d \cdot n^{\log_b(a)}$. 
Intuition, the $\log_b(a) = c$ case:

1. Work is done roughly equally throughout tree.
2. Each level does about the same amount of work, so we approximate by just multiplying work done on first level by the height: $n^c \log_b(n)$. 
A few final thoughts about sorting...
**Problem:** Quick sort, in the best case, is pretty fast – often a constant factor faster than merge sort. But in the worst case, it’s $O(n^2)$!
Hybrid sorts

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**Hybrid sort**

A sorting algorithm which combines two or more other sorting algorithms.
Example: Introsort

Core idea: Combine quick sort with heap sort

(Why heap sort? It’s also $O(n \log(n))$, and can be implemented in-place.)
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Punchline: worst-case runtime is now $O(n \log(n))$, not $O(n^2)$. Used by various C++ implementations, used by Microsoft’s .NET framework.
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**Adaptive sorts**

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**Idea:** Modify our strategy based on what the input data actually looks like!

**Adaptive sort**

A sorting algorithm that *adapts* to patterns and pre-existing order in the input.

Most adaptive sorts take advantage of how real-world data is often partially sorted.
Example: Timsort

Core idea: Combine merge sort with insertion sort

(Who’s Tim? A Python core developer)
Example: Timsort

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1. Works by looking for “sorted runs”.
2. Will use insertion sort to merge two small runs and merge sort to merge two large runs
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Used by Python and Java.
Linear sorts (aka ‘Niche’ sorts)

Basic idea: Can we do better than $O(n \log(n))$ if we assume more things about the input list?
Counting sort

- **Assumption:** Input is a list of ints where every item is between 0 and \( k \)
- **Worst-case runtime:**

How would you implement this? Hint: start by creating a temp array of length \( k \). Take inspiration from how hash tables work.
Counting sort

**Assumption:** Input is a list of ints where every item is between 0 and $k$

**Worst-case runtime:**

How would you implement this? Hint: start by creating a temp array of length $k$. Take inspiration from how hash tables work.

**The algorithm:**

1. Create a temp array named `arr` of length $k$.
2. Loop through the input array. For each number `num`, run
   ```python
   arr[num] += 1
   ```
3. The temp array will now contain how many times we say each number in the input list.
4. Iterate through it to create the output list.
Counting sort

**Assumption:** Input is a list of ints where every item is between 0 and $k$

**Worst-case runtime:** $O(n + k)$

How would you implement this? Hint: start by creating a temp array of length $k$. Take inspiration from how hash tables work.

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Other interesting linear sorts:

▶ Radix sort
  ▶ Assumes items in list are some kind of sequence-like thing such as strings or ints (which is a sequence of digits)
  ▶ Assumes each “digit” is also sortable
  ▶ Sorts all the digits in the 1s place, then the 2s place...
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  - Assumes each “digit” is also sortable
  - Sorts all the digits in the 1s place, then the 2s place...

- **Bucket sort**
  - A generalization of counting sort
  - Assumes items are randomly and uniformly distributed across a range of possibilities
Introduction to graphs
What is a graph?

In this class, by "graph", we mean the graph data structure.
What is a graph?

This is a graph:

![Graph Diagram]

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What is a graph?

This is a graph:

This is also a graph:
What is a graph?

This is a graph:

```
A -- B
 |   |
|   |
D -- C
```

This is also a graph:

```
2    y
|    |
|    |
8    |
|    |
|    |
8    |
```

In this class, by “graph”, we mean the graph data structure.
Graph: formal definition

<table>
<thead>
<tr>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>A <strong>graph</strong> is a pair ( G = (V, E) ), where...</td>
</tr>
<tr>
<td>- ( V ) is a set of <strong>vertices</strong></td>
</tr>
<tr>
<td>- ( E ) is a set of <strong>edges</strong> (pairs of vertices)</td>
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Notes:

- Vertices are the circle things, edges are the lines
- The words “node” and “vertex” are synonyms
Graph: formal definition examples

Examples:

- \( V = \{ a \} \), \( E = \{ \} \)
- \( V = \{ b, c, d \} \), \( E = \{ (b, c), (c, d) \} \)
- \( V = \{ e, f, g, h \} \), \( E = \{ (e, f), (f, g), (g, h), (h, e), (e, g), (f, h) \} \)
Applications of graphs

**In a nutshell:**

Graphs let us model the “relationship” between items.

If that seems like a very general definition, that’s because graphs are a very general concept!
Applications of graphs

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Graphs let us model the “relationship” between items.

If that seems like a very general definition, that’s because graphs are a very general concept!

Core insight:

- Graphs are an abstract concept that appear in many different ways
- Many problems can be modeled as a graph problem

Some examples...
Application: Airline flight graph

Questions:
- What is the cheapest/shortest/etc flight from A to B?
- Is the route the airline offering me actually the cheapest route?
- What happens if a city is snowed in – how can we reroute flights?
Application: Airline flight graph

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http://allthingsgraphed.com/public/images/airline-google-earth.png
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Questions:
- Why does this graph look clustered?
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Application: Social media polarization
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Questions: how to ideas flow between bloggers? Right now? Over time? Who’s the most influential within a given party? In general?

Application: Analyzing code

Questions:
- Which files import which ones?
- Which files are most used and should be optimized?
- What if two files import each other?
- If a file has a security vulnerability, how might it propagate?

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Application of graphs

Other interesting questions:

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Application of graphs

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Other interesting questions:

- How does Google work?
- How do I solve Sudoku efficiently?
- How do genes in my DNA influence and regulate each other?
- How can I allocate registers to variables in a program?
- How similar/dissimilar are words? Based on spelling? Based on meaning?
How would you model the following using graphs? Decide what you think the vertices are and what the edges are:

- **Maps (e.g. Google Maps)**
- **Web pages**
- **A running program**
- **Courses at UW**
- **A family tree**
How would you model the following using graphs? Decide what you think the vertices are and what the edges are:

- **Maps (e.g. Google Maps)**
  Idea: vertices are intersections, edges are roads. How do we model traffic? Paths for cyclists vs cars? One-way roads?

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- **A family tree**
  Idea: model people as vertices, and relations as edges. Or the other way around: model events like birth or divorce as vertices, and make people the edges connecting events?
Is there a “graph” ADT?

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Well, what operations belong to the ADT? Hmm, lots of ideas (getEdge(v), reachableFrom(...), centrality(...), etc..)
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Well, what operations belong to the ADT? Hmm, lots of ideas (getEdge(v), reachableFrom(...), centrality(...), etc..)

**Observation:** It’s very unclear what the “standard operations” are. Instead, what we do is think about graphs abstractly, and think about which algorithms are relevant to the problems at hand.
**Undirected graphs**

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This means that \((x, y) \in E\) implies that \((y, x) \in E\). (Often, we treat these two pairs as equivalent and only include one).

**Degree of a vertex**

The **degree** of some vertex \(v\) is the number of edges containing that vertex. So, the degree is the number of adjacent vertices.
Undirected graphs

Undirected graph

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![Diagram of an undirected graph]

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**Directed graph**

![Directed graph diagram](image-url)

- **In-degree of a vertex**:
  The in-degree of a vertex $v$ is the number of edges that point to $v$.

- **Out-degree of a vertex**:
  The out-degree of a vertex $v$ is the number of edges that start at $v$. 
In a **directed graph**, edges *do* have a direction: are one-way.

Now, \((x, y)\) and \((y, x)\) mean different things.
Directed graphs

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![Directed graph example]

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The out-degree of \(v\) is the number of edges that start at \(v\).
Self-loops and parallel edges

**Self-loop**

A **self-loop** is an edge that starts and ends at the same vertex.

![Self-loop diagram](image)
### Self-loops and parallel edges

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![Self-loop example](image)

#### Parallel edges
Two edges are **parallel** if they both start and end at the same vertices.

![Parallel edges example](image)
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Whether we allow or disallow self-loops and parallel edges depends on what we’re trying to model.
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**Simple graph**

A graph with no self-loops and no parallel edges.
Some questions

Questions:

- In an undirected graph, is it possible to have a vertex with a degree of zero?

- In a directed graph, a vertex with an in-degree and out-degree of both zero?
Questions:

- In an undirected graph, is it possible to have a vertex with a degree of zero?
  Yes.

- In a directed graph, a vertex with an in-degree and out-degree of both zero?
  Yes.