CSE 373: Master method, finishing sorts, intro to graphs

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Monday, Feb 12, 2018

The tree method: precise analysis

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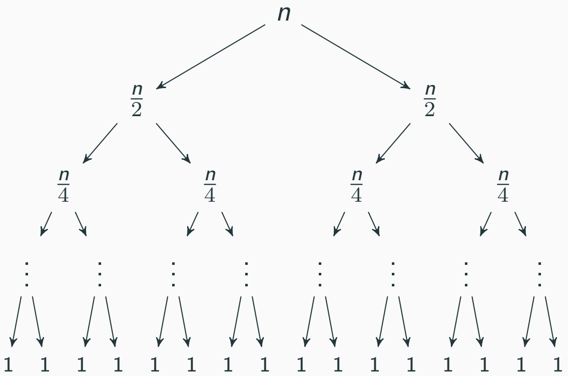

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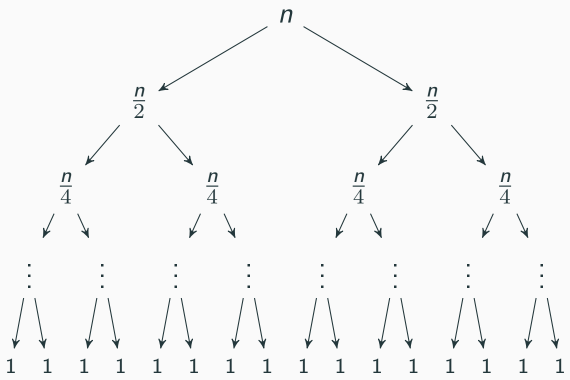

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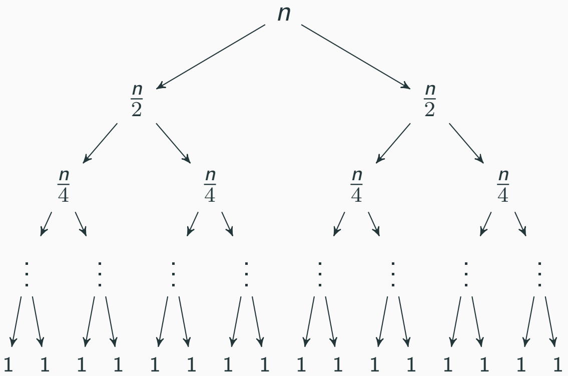

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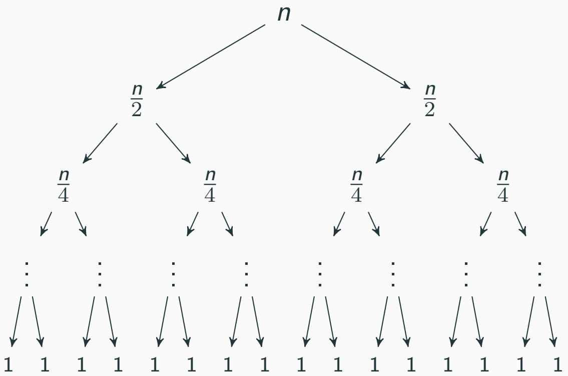

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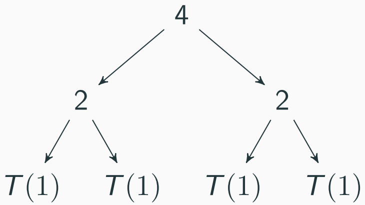

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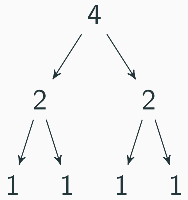

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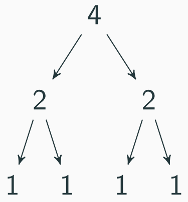

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For this recursive function, num recursive levels is same as height.

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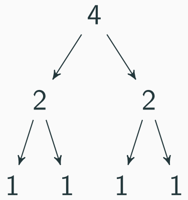

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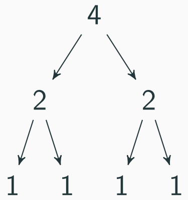

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Important: for other recursive functions, where base case doesn't happen at $n \leq 1$, num recursive levels might be different then

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$$\text{totalWork} = \text{recursiveWork} + \text{baseCaseWork}$$

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So exact closed form is $n \log_2(n) + n$.

The tree method: practice

Practice: Let's go back to our old recurrence...

$$S(n) = \begin{cases} 2 & \text{if } n \leq 1 \\ 2S\left(n/3\right) + n^2 & \text{otherwise} \end{cases}$$

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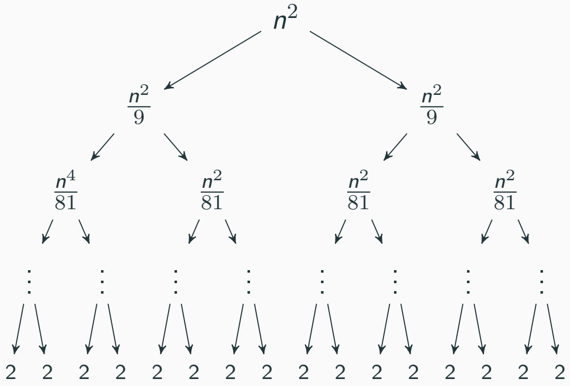

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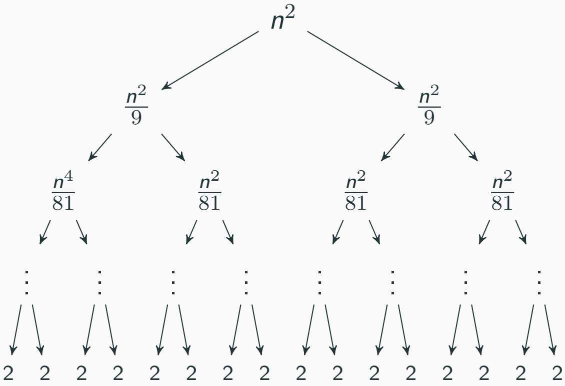

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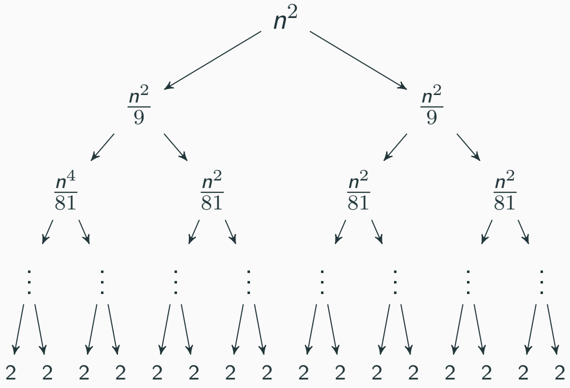

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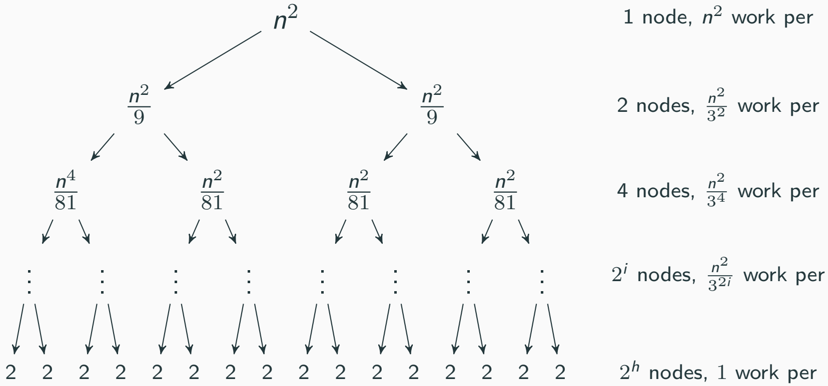

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&= n^2 \sum_{i=0}^{\log_3(n)} \frac{2^i}{9^i} + 2n^{\log_3(2)} \\
&= n^2 \sum_{i=0}^{\log_3(n)} \left(\frac{2}{9} \right)^i + 2n^{\log_3(2)}
\end{aligned}
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The finite geometric series

We have: $n^2 \sum_{i=0}^{\log_3(n)} \left(\frac{2}{9}\right)^i + 2n^{\log_3(2)}$

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Plug and chug:

$$\begin{aligned}
\text{totalWork} &= n^2 \sum_{i=0}^{\log_3(n)} \left(\frac{2}{9}\right)^i + 2n^{\log_3(2)} \\
&= n^2 \sum_{i=0}^{\log_3(n)+1-1} \left(\frac{2}{9}\right)^i + 2n^{\log_3(2)} \\
&= n^2 \frac{1-\left(\frac{2}{9}\right)^{\log_3(n)+1}}{1-\frac{2}{9}} + 2n^{\log_3(2)}
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Applying the finite geometric series

With a bunch of effort...

$$
\begin{aligned}
\text{totalWork} &= n^2 \frac{1 - \left(\frac{2}{9}\right)^{\log_3(n)+1}}{1 - \frac{2}{9}} + 2n^{\log_3(2)} \\
&= \frac{9}{7} n^2 \left(1 - \frac{2}{9}\left(\frac{2}{9}\right)^{\log_3(n)}\right) + 2n^{\log_3(2)} \\
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&= \frac{9}{7} n^2 - \frac{2}{7} n^2 n^{\log_3(2/9)} + 2n^{\log_3(2)} \\
&= \frac{9}{7} n^2 - \frac{2}{7} n^2 n^{\log_3(2)-2} + 2n^{\log_3(2)} \\
&= \frac{9}{7} n^2 - \frac{2}{7} n^{\log_3(2)} + 2n^{\log_3(2)} \\
&= \frac{9}{7} n^2 + \frac{12}{7} n^{\log_3(2)}
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If we want to find a big-Θ bound, yes.

The master theorem

Suppose we have a recurrence of the following form:

$$T(n) = \begin{cases} d & \text{if } n = 1 \\ aT\left(\frac{n}{b}\right) + n^c & \text{otherwise} \end{cases}$$

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Then...

- If $\log_b(a) < c$, then $T(n) \in \Theta\left(n^c\right)$
- If $\log_b(a) = c$, then $T(n) \in \Theta\left(n^c \log(n)\right)$
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Sanity check: try checking $S(n) = 2S(n/3) + n^2$.

We have $a = 2$, $b = 3$, and $c = 2$. We know $\log_3(2) \leq 1 < 2 = c$, therefore $S(n) \in \Theta\left(n^2\right)$.

The master theorem: intuition

Intuition, the $\log_b(a) < c$ case:

1. We do work more rapidly then we divide.
2. So, more of the work happens near the "top", which means that the n^c term dominates.

The master theorem: intuition

Intuition, the $\log_b(a) > c$ case:

1. We divide more rapidly then we do work.
2. So, most of the work happens near the "bottom", which means the work done in the leaves dominates.
3. Note: Work in leaves is about $d \cdot a^{\text{height}} = d \cdot a^{\log_b(n)} = d \cdot n^{\log_b(a)}$.

Intuition, the $\log_b(a) = c$ case:

1. Work is done roughly equally throughout tree.
2. Each level does about the same amount of work, so we approximate by just multiplying work done on first level by the height: $n^c \log_b(n)$.

Parting thoughts on sorting

A few final thoughts about sorting...

Hybrid sorts

Problem: Quick sort, in the best case, is pretty fast – often a constant factor faster then merge sort. But in the worst case, it's $\mathcal{O}\left(n^2\right)$!

Hybrid sorts

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Hybrid sort

A sorting algorithm which combines two or more other sorting algorithms.

Example: Introsort

Core idea: Combine quick sort with heap sort

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Used by various C++ implementations, used by Microsoft's .NET framework.

Adaptive sorts

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Adaptive sort

A sorting algorithm that adapts to patterns and pre-existing order in the input.

Most adaptive sorts take advantage of how real-world data is often partially sorted.

Example: Timsort

Core idea: Combine merge sort with insertion sort

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Used by Python and Java.

Linear sorts (aka 'Niche' sorts)

Basic idea: Can we do better then $\mathcal{O}\left(n \log(n)\right)$ if we assume more things about the input list?

Counting sort

- **Assumption:** Input is a list of ints where every item is between 0 and k
- **Worst-case runtime:**

How would you implement this? Hint: start by creating a temp array of length k. Take inspiration from how hash tables work.

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The algorithm:

1. Create a temp array named `arr` of length k.
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Other linear sorts

Other interesting linear sorts:

- **Radix sort**
  - Assumes items in list are some kind of sequence-like thing such as strings or ints (which is a sequence of digits)
  - Assumes each "digit" is also sortable
  - Sorts all the digits in the 1s place, then the 2s place...

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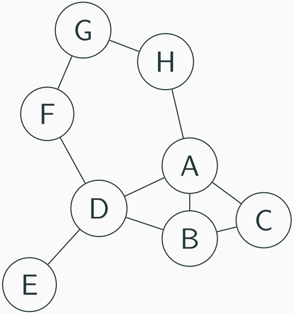
- **Bucket sort**
  - A generalization of counting sort
  - Assumes items are randomly and uniformly distributed across a range of possibilities

Introduction to graphs

What is a graph?

What is a graph?

This is a graph:



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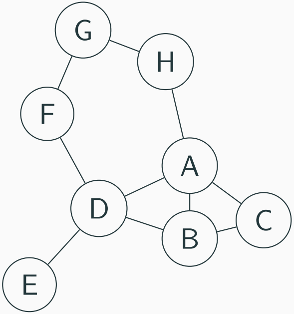

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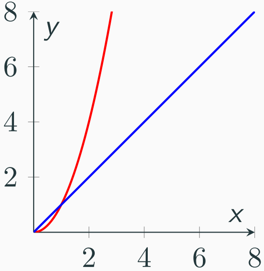

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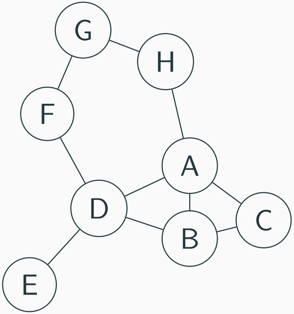

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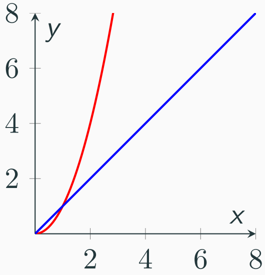


In this class, by "graph", we mean the graph *data structure*.

Graph: formal definition

Graph

A **graph** is a pair $G = (V, E)$, where...

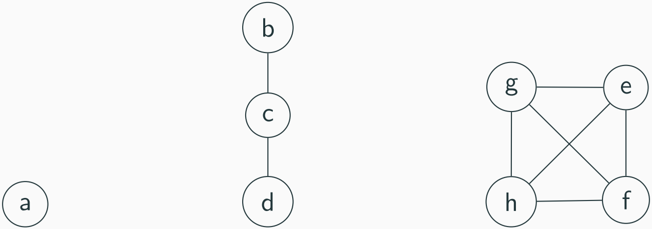
- V is a set of **vertices**
- E is a set of **edges** (pairs of vertices)

Notes:

- Vertices are the circle things, edges are the lines
- The words "node" and "vertex" are synonyms

Graph: formal definition examples

Examples:



$$V = \{a\} \qquad E = \{\}$$

$$V = \{b, c, d\} \qquad E = \{(b, c), (c, d)\}$$

$$V = \{e, f, g, h\}$$
$$E = \{(e, f), (f, g), (g, h), (h, e), (e, g), (f, h)\}$$

Applications of graphs

In a nutshell:

Graphs let us model the "relationship" between items.

If that seems like a very general definition, that's because graphs are a very general concept!

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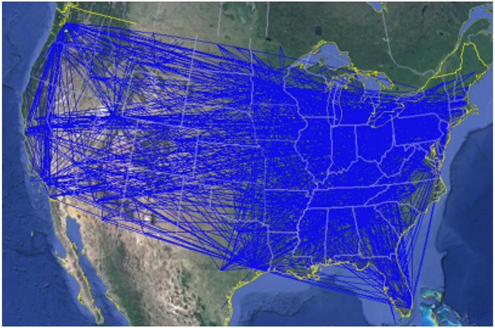
Core insight:

- Graphs are an abstract concept that appear in many different ways
- Many problems can be modeled as a graph problem

Some examples...

Application: Airline flight graph



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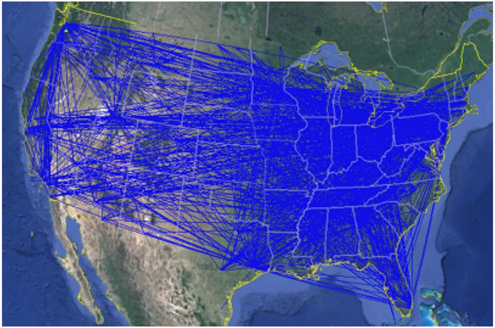


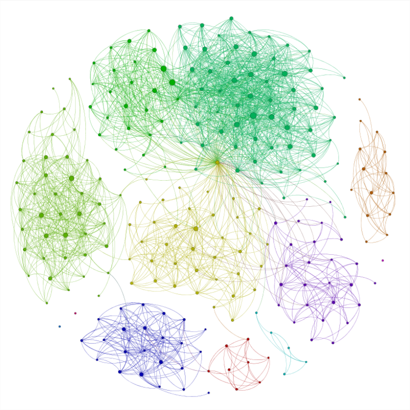
Questions: What is the cheapest/shortest/etc flight from A to B?

Is the route the airline offering me actually the cheapest route?

What happens if a city is snowed in – how can we reroute flights?

http://allthingsgraphed.com/public/images/airline-google-earth.png

Application: Social media graph



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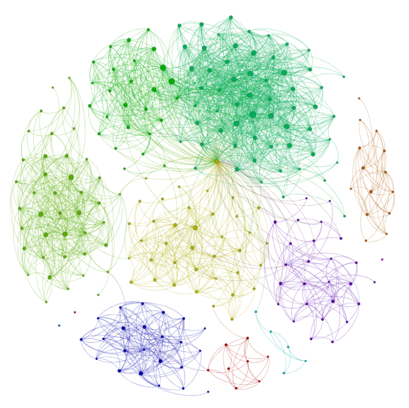
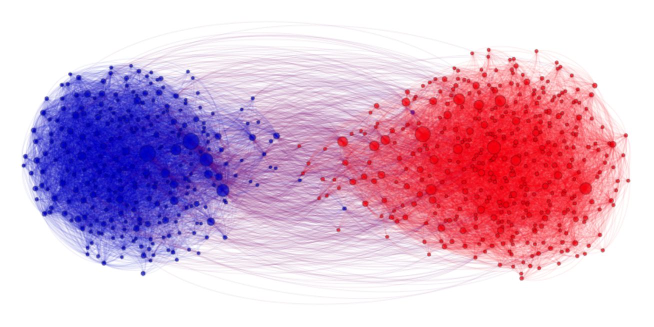


Questions: Why does this graph look clustered? Why are all my friends more popular then me? Who do my friends know? If I want to hire somebody to promote my product, who do I pick?

http://allthingsgraphed.com/public/images/facebook/facebook-friend-graph.png

Application: Social media polarization



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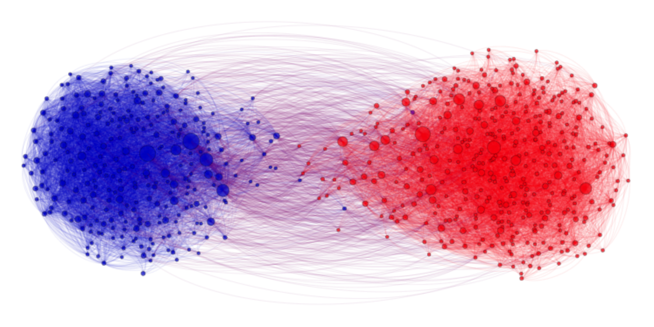
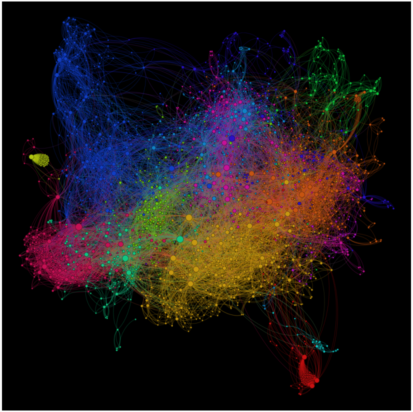
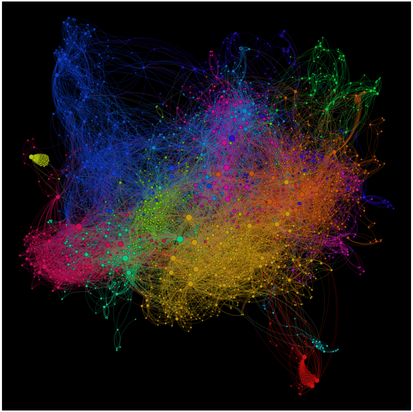


Questions: how to ideas flow between bloggers? Right now? Over time? Who's the most influential within a given party? In general?

http://allthingsgraphed.com/public/images/political-blogs-2004/left-right.png

Application: Analyzing code





Questions: which files import which ones? Which files are most used and should be optimized? What if two files import each other? If a file has a security vulnerability, how might it propagate?

http://allthingsgraphed.com/public/images/java-call-graph/cassandra-import-graph.png

Application of graphs

Other interesting questions:

- How does Google work?

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- How similar/dissimilar are words? Based on spelling? Based on meaning?

Modeling problems with graph

How would you model the following using graphs? Decide what you think the vertices are and what the edges are:

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- Web pages
- A running program
- Courses at UW
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  Idea: model people as vertices, and relations as edges. Or the other way around: model events like birth or divorce as vertices, and make people the edges connecting events?

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Well, what operations belong to the ADT? Hmm, lots of ideas (`getEdge(v)`, `reachableFrom(...)`, `centrality(...)`, etc..)

Observation: It’s very unclear what the “standard operations” are

Instead, what we do is think about graphs abstractly, and think about which algorithms are relevant to the problems at hand.

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In a **undirected graph**, edges have no direction: are two-way

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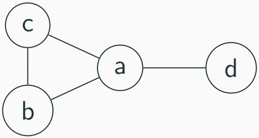

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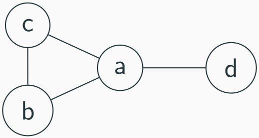

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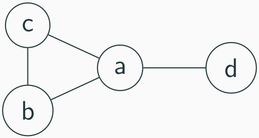

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Degree of a vertex

The **degree** of some vertex v is the number of edges containing that vertex.

So, the degree is the number of adjacent vertices.

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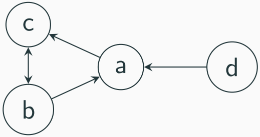

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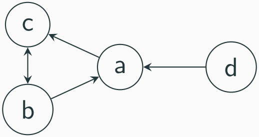

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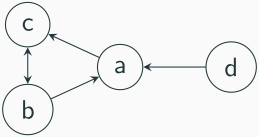

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In-degree of a vertex

The **in-degree** of v is the number of edges that point to v.

Out-degree of a vertex

The **out-degree** of v is the number of edges that start at v.

Self-loops and parallel edges

Self-loop

A **self-loop** is an edge that starts and ends at the same vertex.

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Parallel edges

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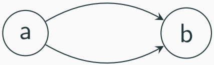

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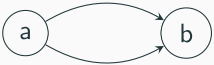

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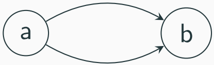

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Simple graph

A graph with no self-loops and no parallel edges.

Some questions

Questions:

- In an undirected graph, is it possible to have a vertex with a degree of zero?
- In a directed graph, a vertex with an in-degree and out-degree of both zero?

Some questions

Questions:

- In an undirected graph, is it possible to have a vertex with a degree of zero?
  Yes.
- In a directed graph, a vertex with an in-degree and out-degree of both zero?
  Yes.