CSE 373: More sorts, tree method, the master method

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Technique: Divide-and-Conquer

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Example template

```
algorithm(input) {
    if (small enough) {
        CONQUER, solve, and return input
    } else {
        DIVIDE input into multiple pieces
        RECURSE on each piece
        COMBINE and return results
    }
}
```

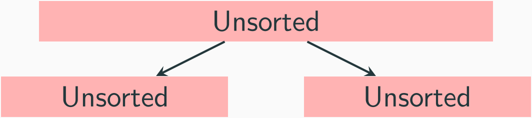
Merge sort: Core pieces

Divide:

Unsorted

Merge sort: Core pieces

Divide: Split array roughly into half



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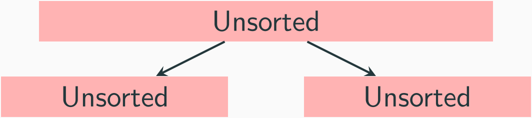

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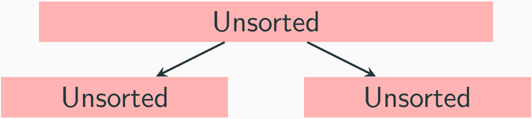

Conquer: Return array when length ≤ 1

Merge sort: Core pieces

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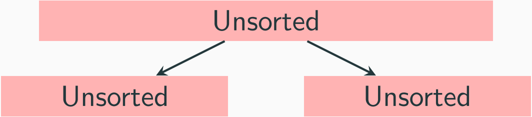

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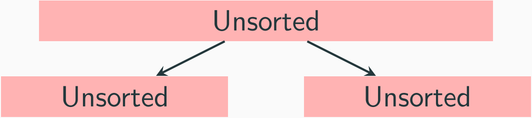
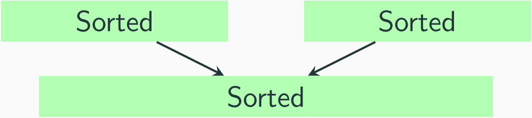


Conquer: Return array when length ≤ 1

Combine: Combine two sorted arrays using `merge`



Merge sort: Summary

Core idea: split array in half, sort each half, merge back together.

If the array has size 0 or 1, just return it unchanged.

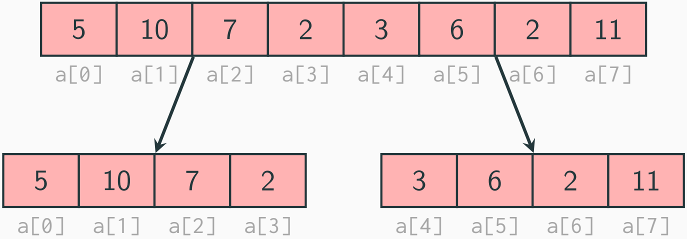
Pseudocode

```
sort(input) {
    if (input.length < 2) {
        return input;
    } else {
        smallerHalf = sort(input[0, ..., mid]);
        largerHalf = sort(input[mid + 1, ...]);
        return merge(smallerHalf, largerHalf);
    }
}
```

Merge sort: Example

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
5	10	7	2	3	6	2	11

Merge sort: Example



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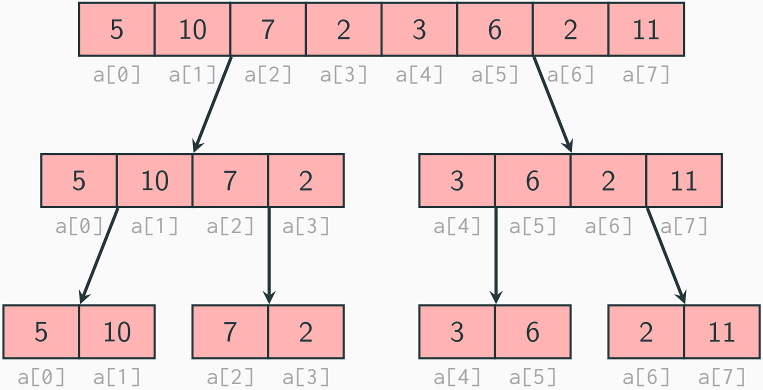

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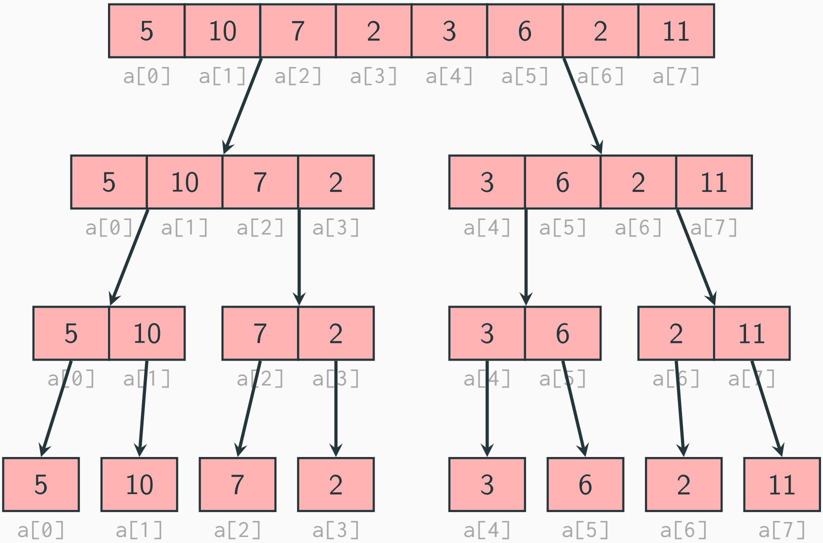

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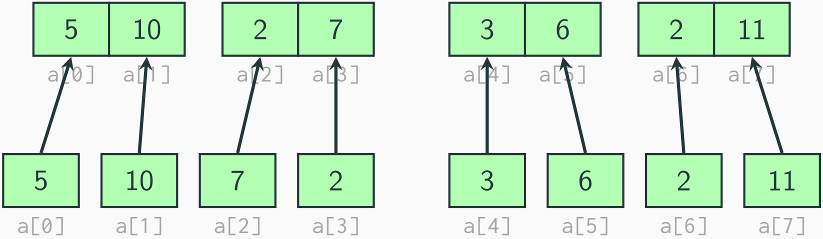

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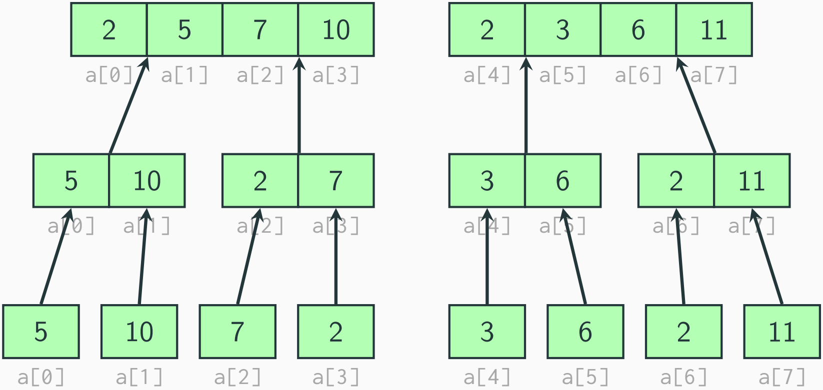

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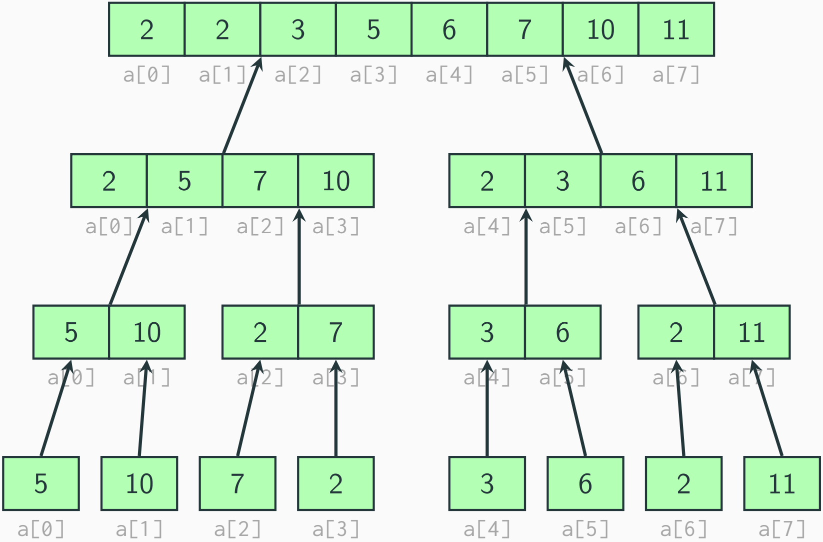

Merge sort: Analysis

Pseudocode

```
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    if (input.length < 2) {
        return input;
    } else {
        smallerHalf = sort(input[0, ..., mid]);
        largerHalf = sort(input[mid + 1, ...]);
        return merge(smallerHalf, largerHalf);
    }
}
```

Best case runtime?

Worst case runtime?

Merge sort: Analysis

Best and worst case

We always subdivide the array in half on each recursive call, and merge takes $\mathcal{O}(n)$ time to run. So, the best and worst case runtime is the same:

$$T(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ 2T(n/2) + n & \text{otherwise} \end{cases}$$

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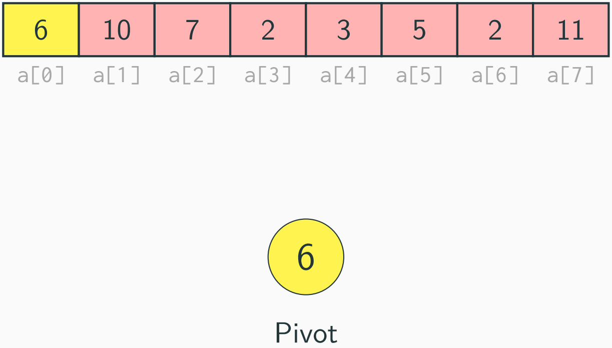
Spoiler alert: this is $\Theta(n \log(n))$

Quick sort: Divide step

6	10	7	2	3	5	2	11
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]

Quick sort: Divide step



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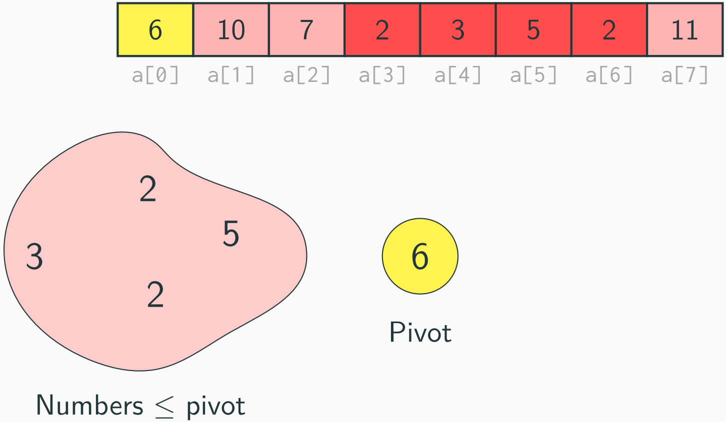

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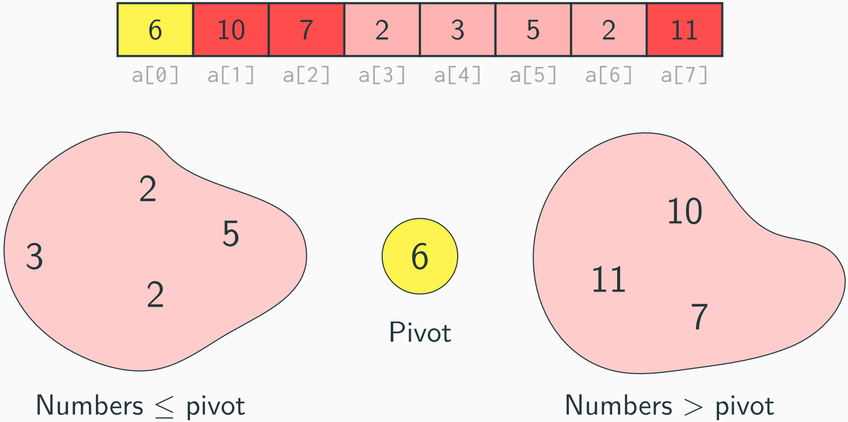
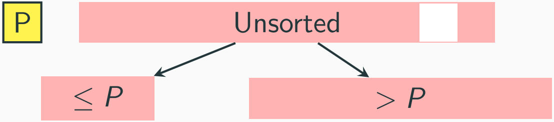


Quick sort: Core pieces

Divide: Pick a pivot, partition into groups



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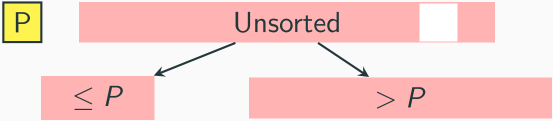
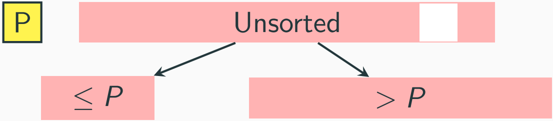


Conquer:

Quick sort: Core pieces

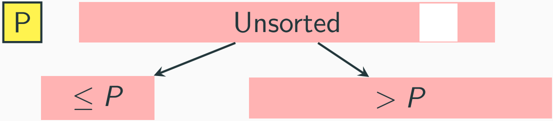
Divide: Pick a pivot, partition into groups



Conquer: Return array when length ≤ 1

Quick sort: Core pieces

Divide: Pick a pivot, partition into groups

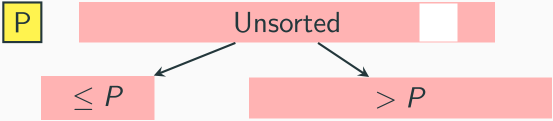


Conquer: Return array when length ≤ 1

Combine:

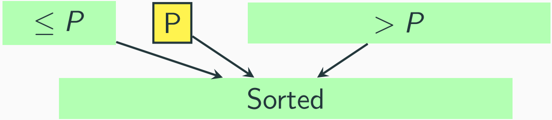
Quick sort: Core pieces

Divide: Pick a pivot, partition into groups



Conquer: Return array when length ≤ 1

Combine: Combine sorted portions and the pivot



Quick sort: Summary

Core idea: Pick some item from the array and call it the **pivot**. Put all items **smaller** in the pivot into one group and all items **larger** in the other and recursively sort. If the array has size 0 or 1, just return it unchanged.

Pseudocode

```
sort(input) {
    if (input.length < 2) {
        return input;
    } else {
        pivot = getPivot(input);
        smallerHalf = sort(getSmaller(pivot, input));
        largerHalf = sort(getBigger(pivot, input));
        return smallerHalf + pivot + largerHalf;
    }
}
```

Quick sort: Example

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]
20	50	70	10	60	40	30

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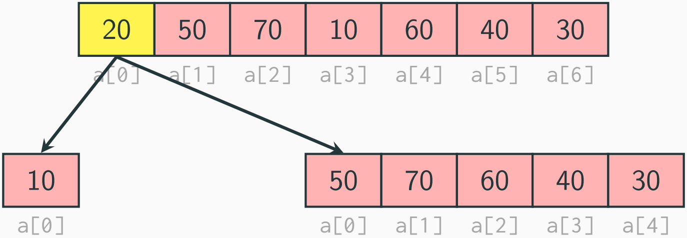

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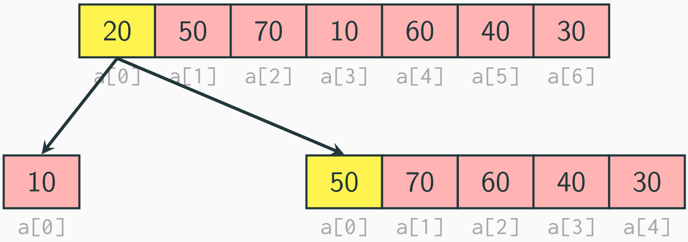

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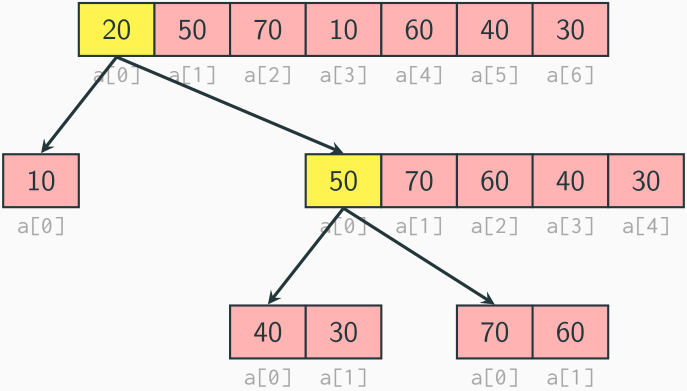

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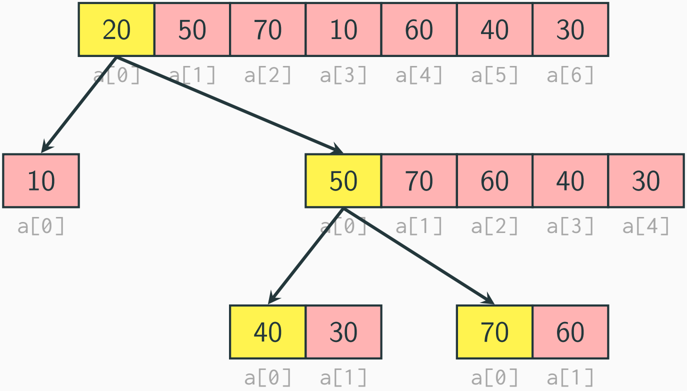

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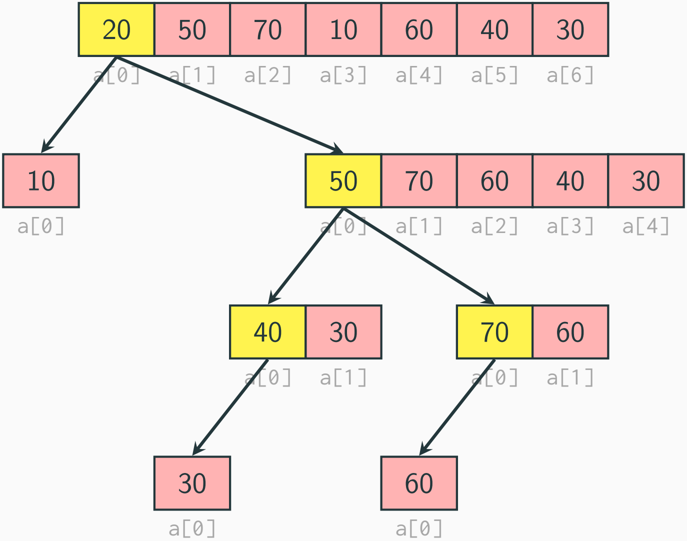

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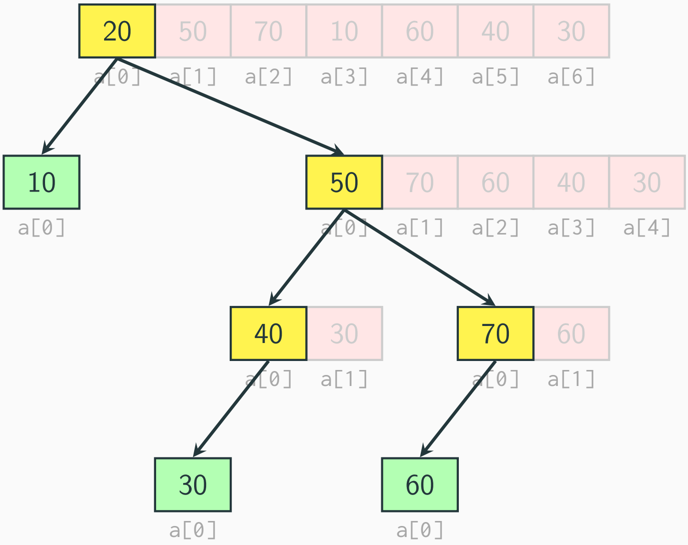

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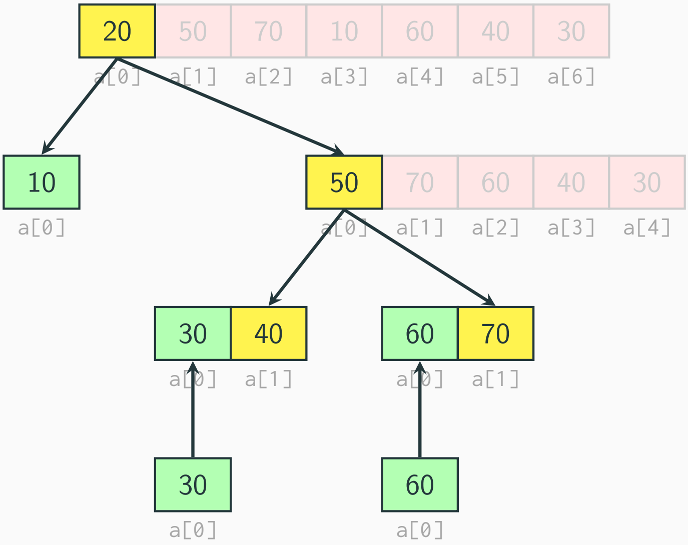

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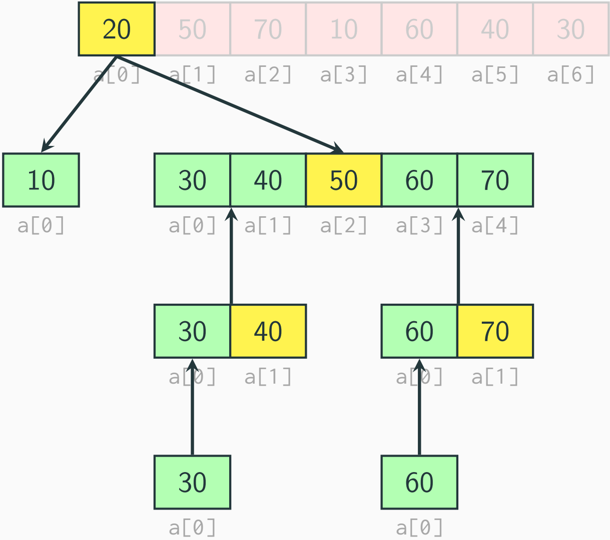

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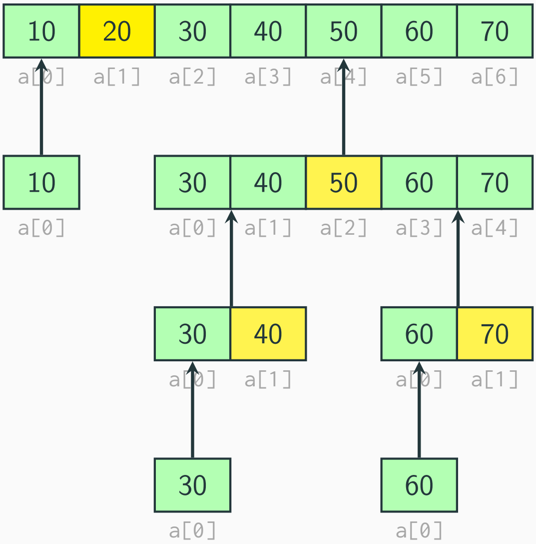

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Best case runtime?

Worst case runtime?

Quick sort: Analysis

Best case analysis

In the **best** case, we always pick the **median** element.

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(Spoiler alert: this is $\Theta(n \log(n))$)

Quick sort: Analysis

Worst case analysis

In the **worst** case, we always end up picking the **minimum** or **maximum** element.

$$T(n) = \begin{cases} T(n-1) + n & \text{if } n > 1 \\ 1 & \text{otherwise} \end{cases}$$

So, the worst-case runtime is $\Theta\left(n^2\right)$.

Quick sort: Analysis

Best case analysis

In the **best** case, we always pick the **median** element, so the best-case runtime is $\Theta(n \log(n))$.

Worst case analysis

In the **worst** case, we always end up picking the **minimum** or **maximum** element, so, the worst-case runtime is $\Theta\left(n^2\right)$.

Average case runtime

Usually, we'll pick a **random** element, which makes the runtime $\Theta(n \log(n))$.

Quick sort: Unresolved questions

How do we pick a pivot?

How do we partition?

Quick sort: Unresolved questions

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- Worst case? Pick the minimum or the maximum. The work will shrink by only 1 on each recursive call.

How do we partition?

Quick sort: Unresolved questions

How do we pick a pivot?

- Worst case? Pick the minimum or the maximum. The work will shrink by only 1 on each recursive call.
- Ideally? Pick the median. The work will split in half on each recursive call.

How do we partition?

Quick sort: Picking a pivot

How do we find the median?

Quick sort: Picking a pivot

How do we find the median?

- Idea: pick the first item in the array
  - Problem: what if the array is already sorted?
  - (Real world data often is partially sorted)
  - But hey, it's speedy ($\mathcal{O}(1)$)

Quick sort: Picking a pivot

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These seem like bad ideas :(

Quick sort: Picking a pivot

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  - Random number generation can sometimes be expensive/fraught with peril

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- Idea: pick the median of first, middle, and last

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These seem like good ideas :)

Quick sort: Unresolved questions

How do we pick a pivot?

How do we partition?

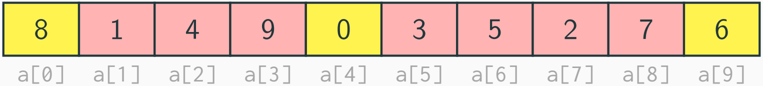
Quick sort: Partitioning (using median-of-three pivot)

Find the lo, med, and hi

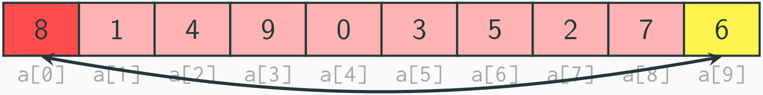
8	1	4	9	0	3	5	2	7	6
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Quick sort: Partitioning (using median-of-three pivot)

Find the lo, med, and hi



Find the median of the three and swap with front



Quick sort: Partitioning (using median-of-three pivot)

Find the `lo`, `med`, and `hi`

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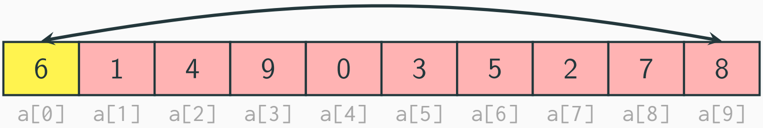
8	1	4	9	0	3	5	2	7	6

Final result: pivot is now at index 0

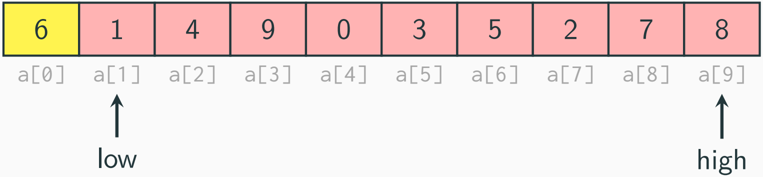
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	9	0	3	5	2	7	8

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:



Partitioning:



$1 \leq 6$

$8 > 6$

Quick sort: Partitioning (using median-of-three pivot)

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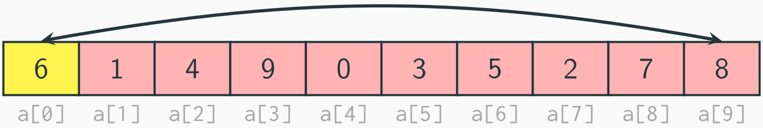

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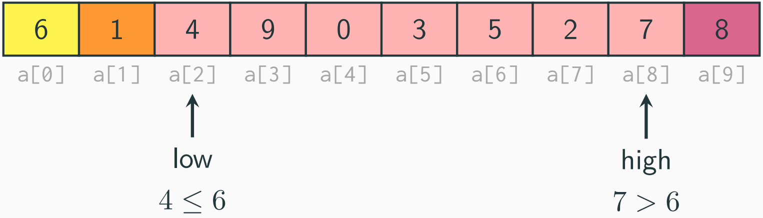

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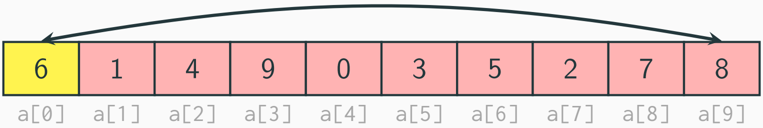
Array after moving pivot:



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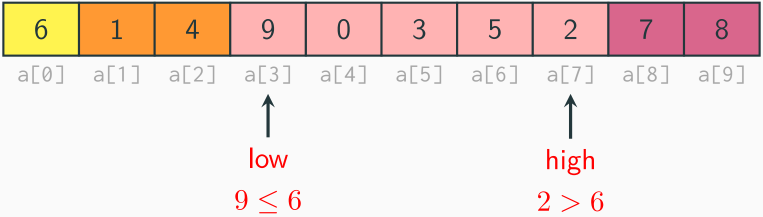

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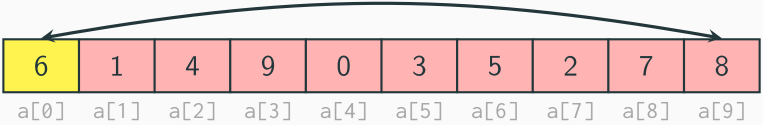

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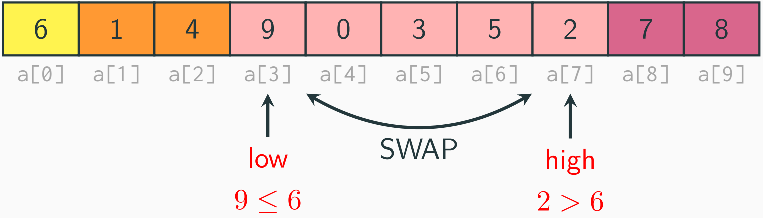

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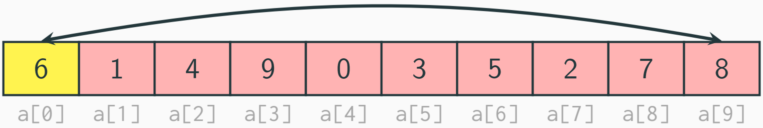

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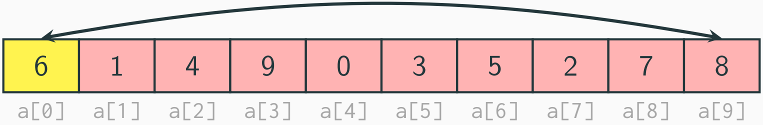
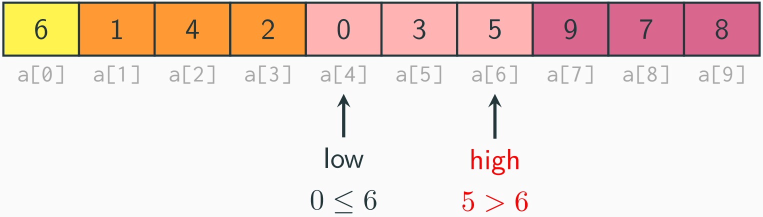
Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:



Partitioning:



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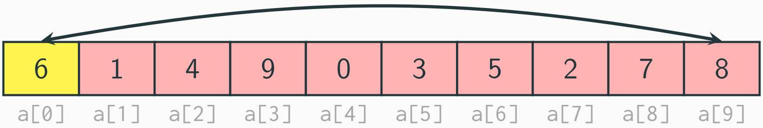

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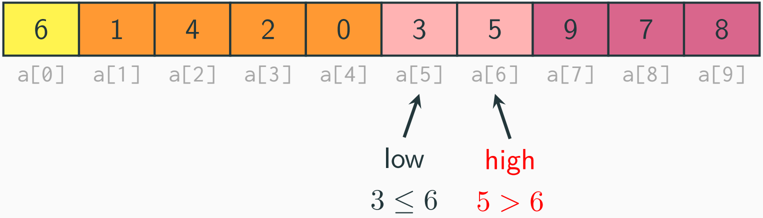

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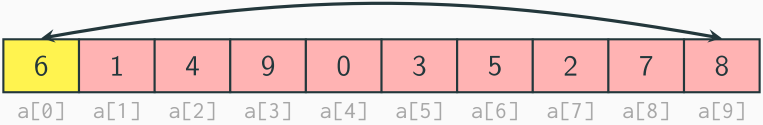

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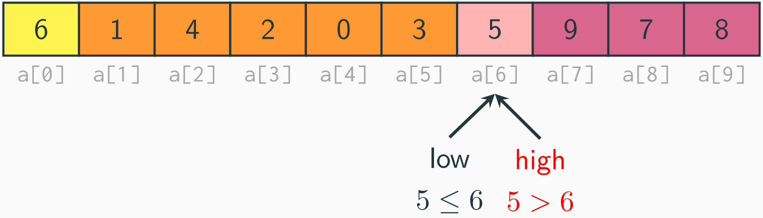

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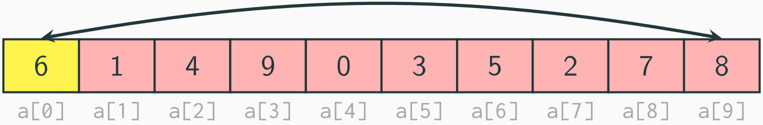

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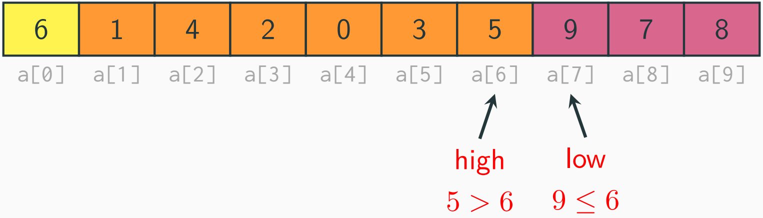

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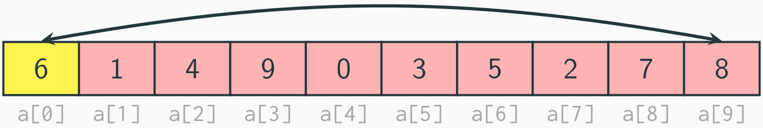

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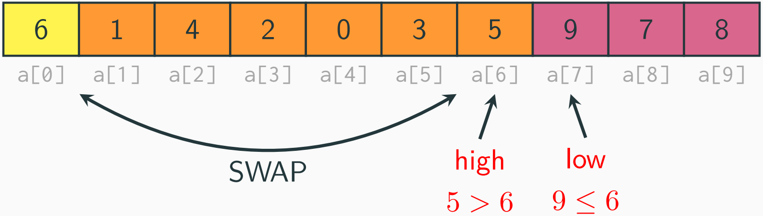

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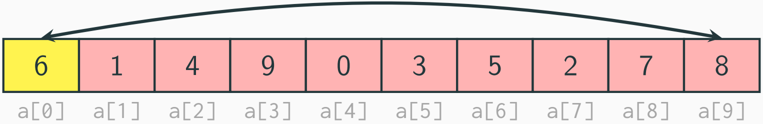

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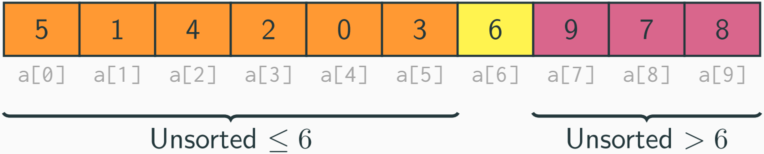
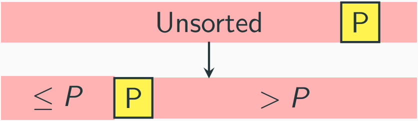


Quick sort: Core pieces revisited

Divide: Pick a pivot, partition in-place into groups



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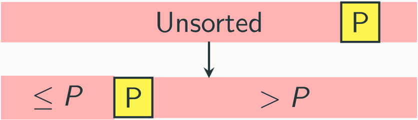

Conquer: When subarray is length ≤ 1, do nothing

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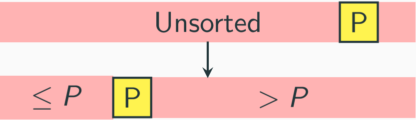
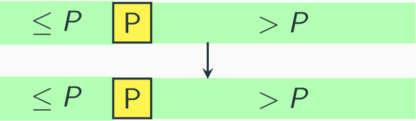


Conquer: When subarray is length ≤ 1, do nothing

Combine: Do nothing; already done!



Analyzing recurrences, part 2

So, merge sort and quick sort are both:

$$T(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ 2T(n/2) + n & \text{otherwise} \end{cases}$$

Analyzing recurrences, part 2

So, merge sort and quick sort are both:

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I claim $T(n) \in \Theta(n \log(n))$. How can we show this?

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$$
\begin{aligned}
T(n) &= n + 2T\left(\frac{n}{2}\right) \\
&= n + 2\left(\frac{n}{2} + 2T\left(\frac{n}{4}\right)\right)
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&= n + n + n + 8T\left(\frac{n}{8}\right) \\
&= \underbrace{n + n + \cdots + n}_{\text{about } \log(n) \text{ times}} + n
\end{aligned}$$

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&= n\log(n)
\end{aligned}
$$

The tree method: overview

Core idea:

1. Draw what the work looks like visually, as a *tree*

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The tree method: overview

Core idea:

1. Draw what the work looks like visually, as a *tree*
2. Use the visualization to help us analyze the overall behavior
3. Either find the closed form, or construct a summation that we can simplify to get the closed form

The tree method: example

Step 1: Start with the function, let n be the input value

$$\boxed{T(n)}$$

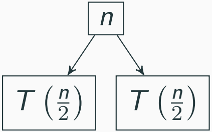
The tree method: example

Step 2: Replace with definition

$$\boxed{T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + n}$$

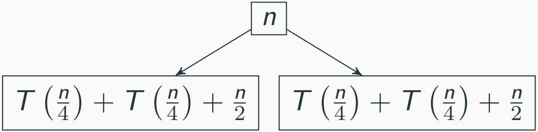
The tree method: example

Step 3: Stick each recursive call into a *subtree*



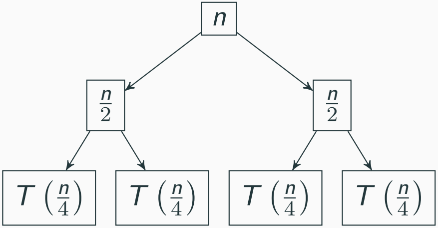
The tree method: example

Step 4: Replace with definition



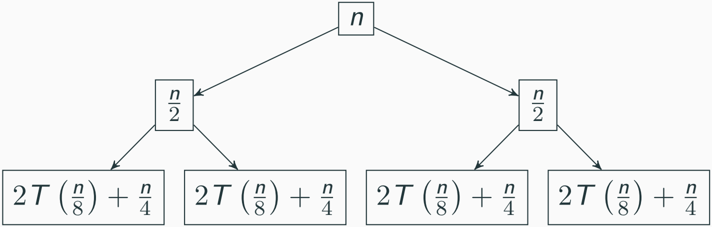
The tree method: example

Repeat step 3 (move recursive call to subtrees):



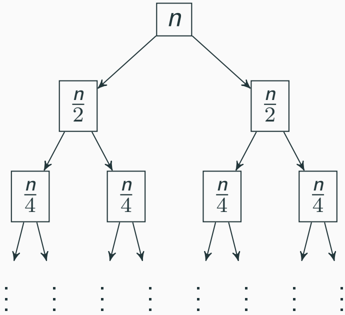
The tree method: example

Repeat step 4 (replace recursive call with definition):



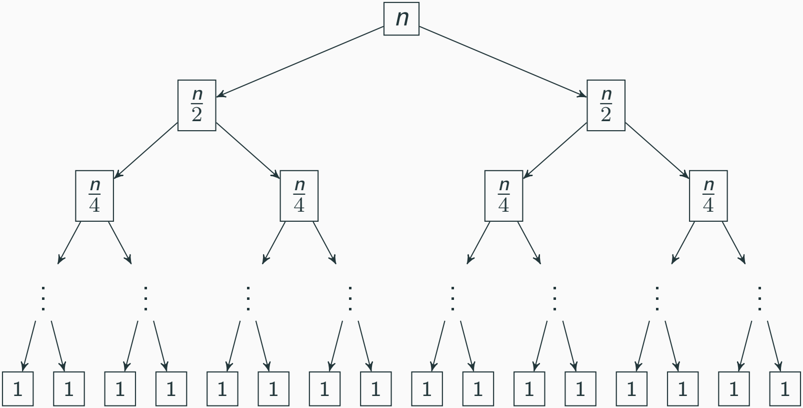
The tree method: example

Repeat...



The tree method: example

Final step: how much work does each base case do?



The tree method: analysis

Now, let's add everything up!

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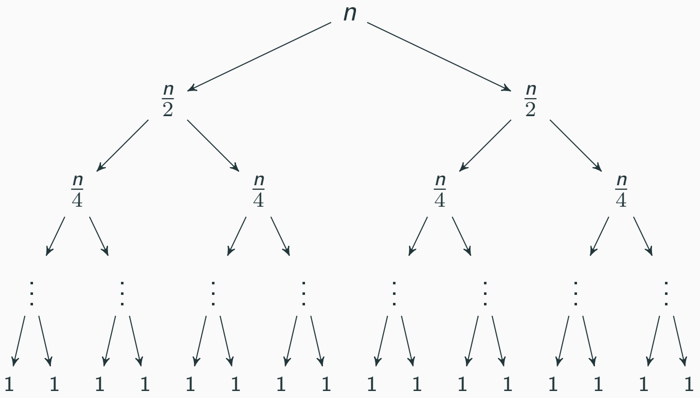
How much work is done per level?



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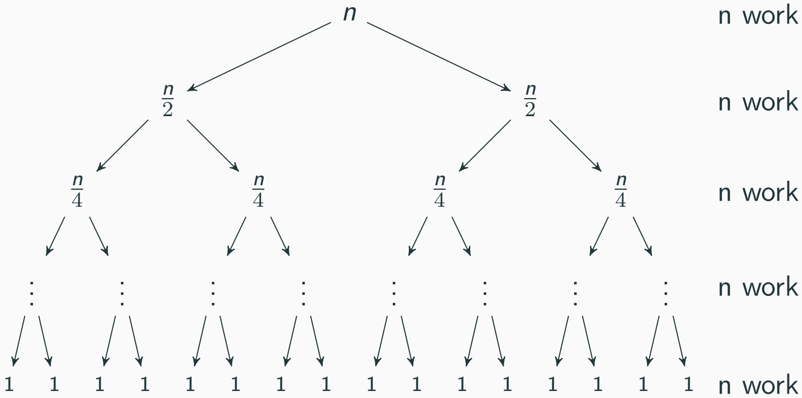

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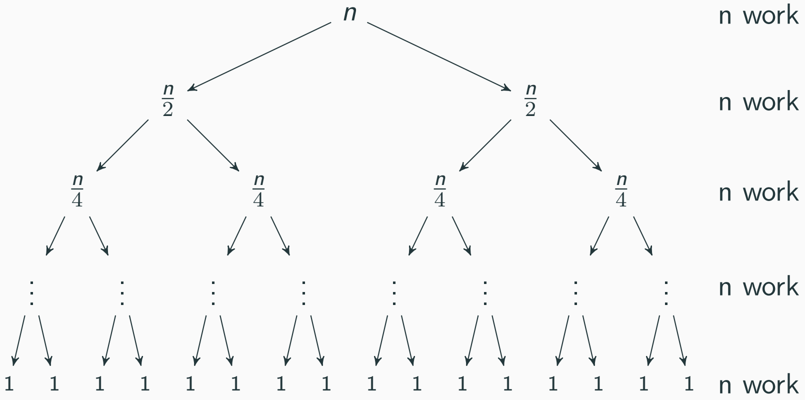

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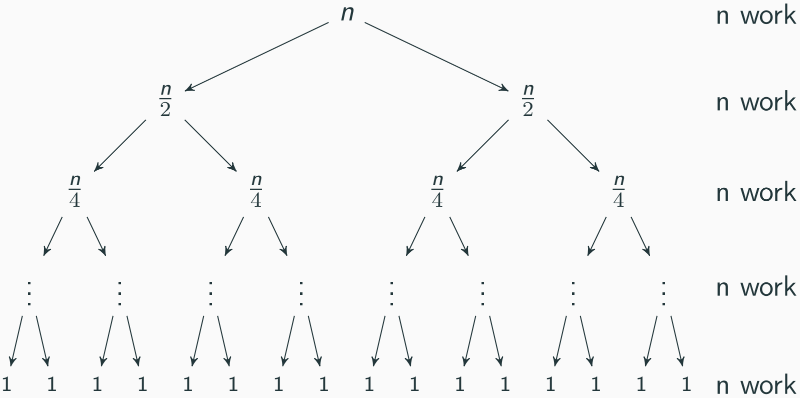

Height is roughly $\log_2(n)$, so total work is about $n \log_2(n)$.

The tree method: practice

Consider the following recurrence:

$$S(n) = \begin{cases} 2 & \text{if } n \leq 1 \\ 2S\left(n/3\right) + n^2 & \text{otherwise} \end{cases}$$

The tree method: practice

Consider the following recurrence:

$$S(n) = \begin{cases} 2 & \text{if } n \leq 1 \\ 2S\left(n/3\right) + n^2 & \text{otherwise} \end{cases}$$

Draw a tree to help you visualize the work done.

The tree method: practice

Step 1: Start with the function, let n be the input value

$$\boxed{S(n)}$$

The tree method: practice

Step 2: Replace with definition

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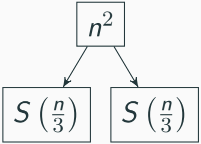


The tree method: practice

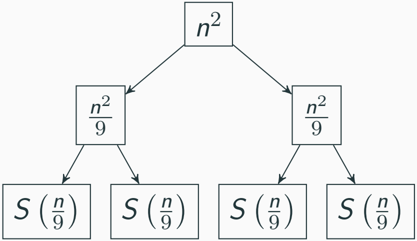
Step 4: Replace with definition

$$n^2$$

$$S\left(\frac{n}{9}\right) + S\left(\frac{n}{9}\right) + \frac{n^2}{9} \qquad S\left(\frac{n}{9}\right) + S\left(\frac{n}{9}\right) + \frac{n^2}{9}$$

The tree method: practice

Repeat step 3 (move recursive call to subtrees):



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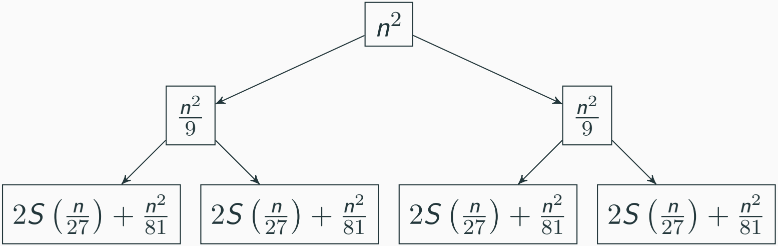
Repeat step 4 (replace recursive call with definition):



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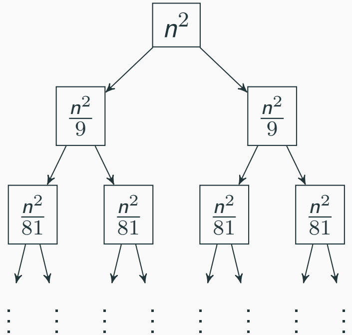

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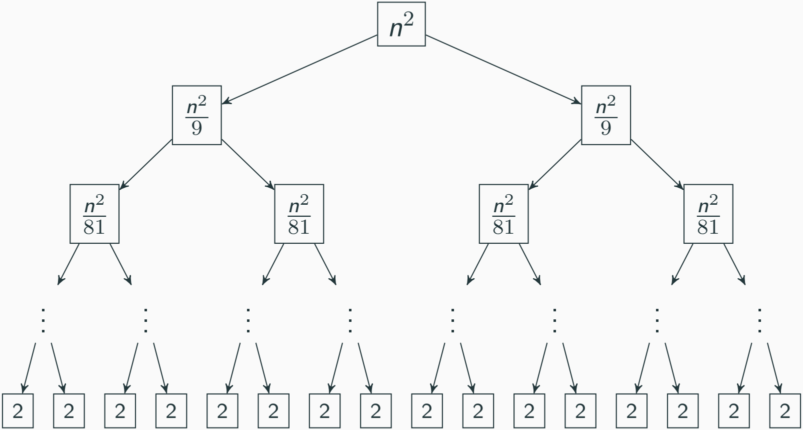

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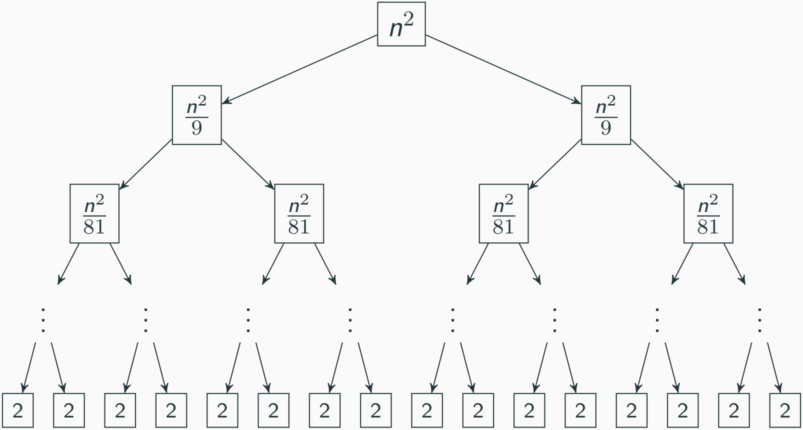

Now what?

The tree method: precise analysis

Problem: Need a rigorous way of getting a closed form

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We want to answer a few core questions:

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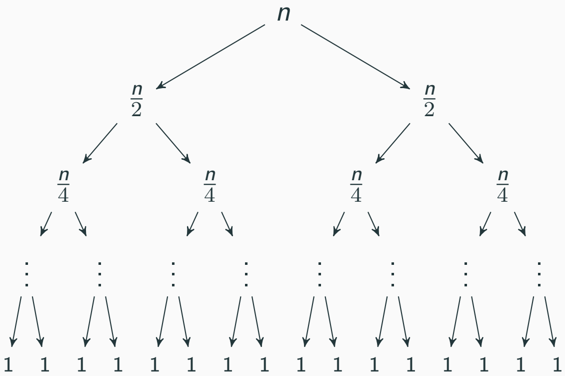

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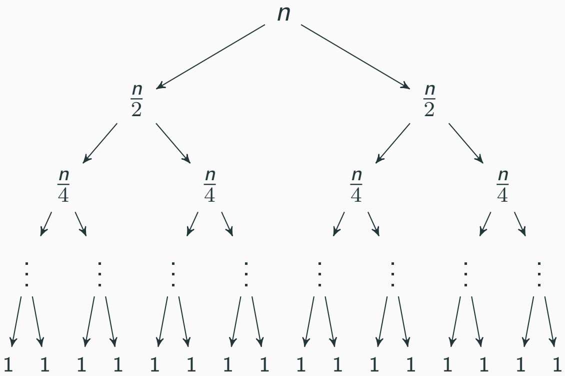
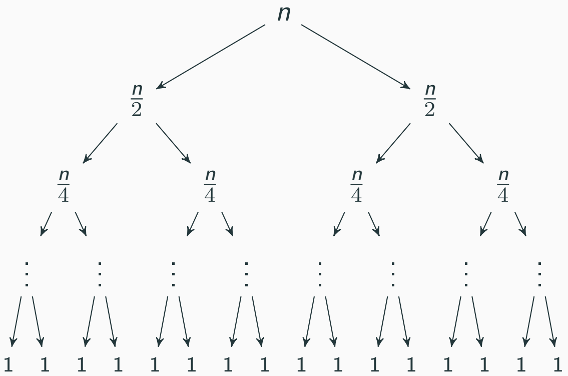


1. numNodes(i) $= ?$
2. workPerNode(n, i) $= ?$
3. numLevels(n) $= ?$
4. workPerLeafNode(n) $= ?$
5. numLeafNodes(n) $= ?$

The tree method: precise analysis



1 node, n work per

2 nodes, $\frac{n}{2}$ work per

4 nodes, $\frac{n}{4}$ work per

2^i nodes, $\frac{n}{i}$ work per

2^h nodes, 1 work per

1. $\text{numNodes}(i) = ?$
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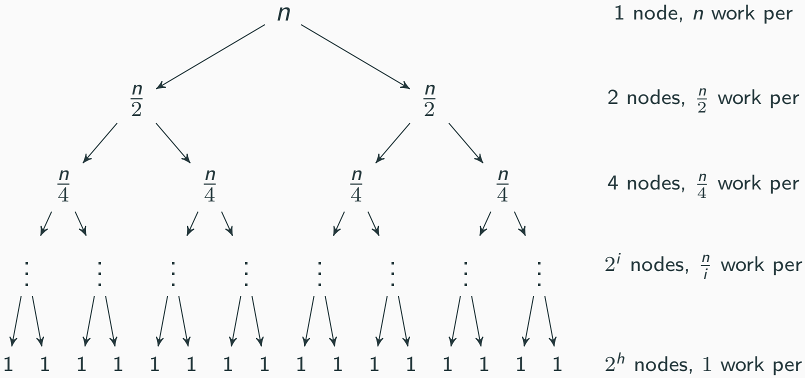

1. numNodes(i) $= 2^i$
2. workPerNode(n, i) $= \frac{n}{2^i}$
3. numLevels(n) $= ?$
4. workPerLeafNode(n) $= 1$
5. numLeafNodes(n) $= ?$

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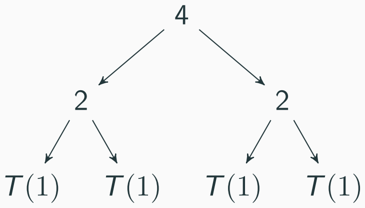

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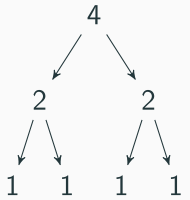

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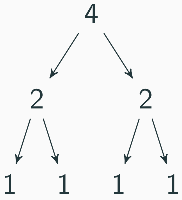

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For this recursive function, num recursive levels is same as height.

The tree method: precise analysis

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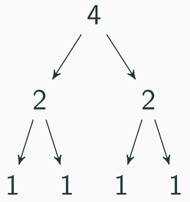

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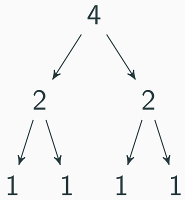

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Important: for other recursive functions, where base case doesn't happen at $n \leq 1$, num recursive levels might be different then

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We discovered:

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$$\text{totalWork} = \text{recursiveWork} + \text{baseCaseWork}$$

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Solve for recursive case:

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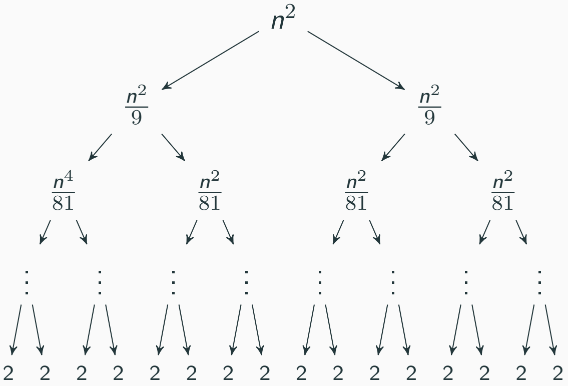
So exact closed form is $n \log_2(n) + n$.

The tree method: practice

Practice: Let's go back to our old recurrence...

$$S(n) = \begin{cases} 2 & \text{if } n \leq 1 \\ 2S\left(n/3\right) + n^2 & \text{otherwise} \end{cases}$$

The tree method: practice



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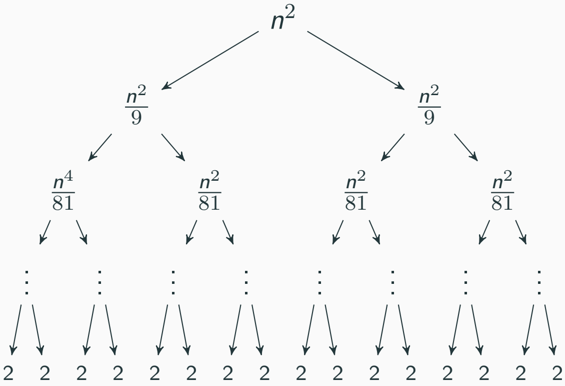

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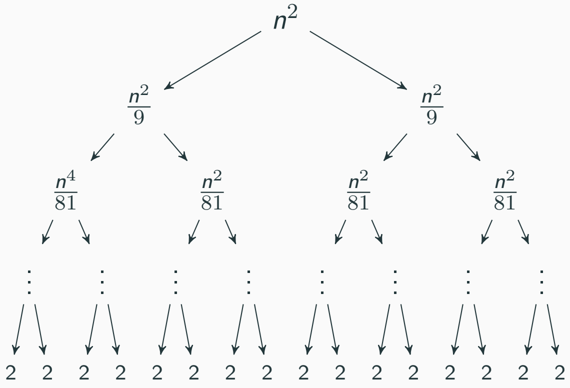

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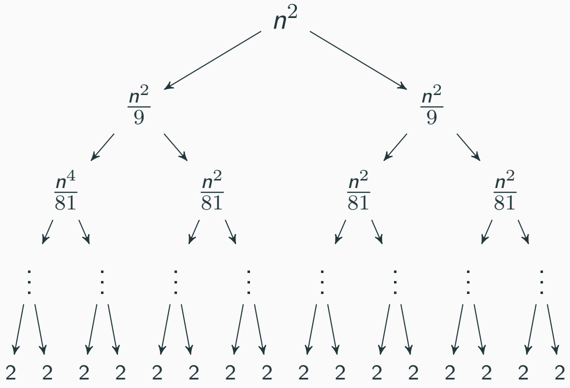

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The finite geometric series

We have: $n^2 \sum_{i=0}^{\log_3(n)} \left(\frac{2}{9}\right)^i + 2n^{\log_3(2)}$

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Plug and chug:

$$\begin{aligned}
\text{totalWork} &= n^2 \sum_{i=0}^{\log_3(n)} \left(\frac{2}{9}\right)^i + 2n^{\log_3(2)} \\
&= n^2 \sum_{i=0}^{\log_3(n)+1-1} \left(\frac{2}{9}\right)^i + 2n^{\log_3(2)} \\
&= n^2 \frac{1 - \left(\frac{2}{9}\right)^{\log_3(n)+1}}{1 - \frac{2}{9}} + 2n^{\log_3(2)}
\end{aligned}$$

Applying the finite geometric series

With a bunch of effort...

$$
\begin{aligned}
\text{totalWork} &= n^2 \frac{1 - \left(\frac{2}{9}\right)^{\log_3(n)+1}}{1 - \frac{2}{9}} + 2n^{\log_3(2)} \\
&= \frac{9}{7} n^2 \left(1 - \frac{2}{9}\left(\frac{2}{9}\right)^{\log_3(n)}\right) + 2n^{\log_3(2)} \\
&= \frac{9}{7} n^2 - \frac{2}{7} n^2 \left(\frac{2}{9}\right)^{\log_3(n)} + 2n^{\log_3(2)} \\
&= \frac{9}{7} n^2 - \frac{2}{7} n^2 n^{\log_3(2/9)} + 2n^{\log_3(2)} \\
&= \frac{9}{7} n^2 - \frac{2}{7} n^2 n^{\log_3(2)-2} + 2n^{\log_3(2)} \\
&= \frac{9}{7} n^2 - \frac{2}{7} n^{\log_3(2)} + 2n^{\log_3(2)} \\
&= \frac{9}{7} n^2 + \frac{12}{7} n^{\log_3(2)}
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If we want to find a big-Θ bound, yes.

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$$T(n) = \begin{cases} d & \text{if } n = 1 \\ aT\left(\frac{n}{b}\right) + n^c & \text{otherwise} \end{cases}$$

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Then...

- If $\log_b(a) < c$, then $T(n) \in \Theta\left(n^c\right)$
- If $\log_b(a) = c$, then $T(n) \in \Theta\left(n^c \log(n)\right)$
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We have $a = 2$, $b = 3$, and $c = 2$. We know $\log_3(2) \leq 1 < 2 = c$, therefore $S(n) \in \Theta\left(n^2\right)$.

The master theorem: intuition

Intuition, the $\log_b(a) < c$ case:

1. We do work more rapidly then we divide.
2. So, more of the work happens near the "top", which means that the n^c term dominates.

The master theorem: intuition

Intuition, the $\log_b(a) > c$ case:

1. We divide more rapidly then we do work.
2. So, most of the work happens near the "bottom", which means the work done in the leaves dominates.
3. Note: Work in leaves is about $d \cdot a^{\text{height}} = d \cdot a^{\log_b(n)} = d \cdot n^{\log_b(a)}$.

The master theorem: intuition

Intuition, the $\log_b(a) = c$ case:

1. Work is done roughly equally throughout tree.
2. Each level does about the same amount of work, so we approximate by just multiplying work done on first level by the height: $n^c \log_b(n)$.