CSE 373: More sorts, tree method, the master method

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Wednesday, Feb 7, 2018

Technique: Divide-and-Conquer

Divide-and-conquer is a useful technique for solving many kinds of problems. It consists of the following steps:

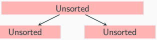
1. *Divide* your work up into smaller pieces (recursively)
2. *Conquer* the individual pieces (as base cases)
3. *Combine* the results together (recursively)

Example template

```
algorithm(input) {
    if (small enough) {
        CONQUER, solve, and return input
    } else {
        DIVIDE input into multiple pieces
        RECURSE on each piece
        COMBINE and return results
    }
}
```

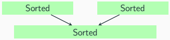

Merge sort: Core pieces

Divide: Split array roughly into half



Conquer: Return array when length ≤ 1

Combine: Combine two sorted arrays using `merge`



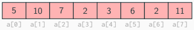
Merge sort: Summary

Core idea: split array in half, sort each half, merge back together. If the array has size 0 or 1, just return it unchanged.

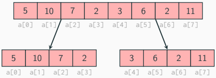
Pseudocode

```
sort(input) {
    if (input.length < 2) {
        return input;
    } else {
        smallerHalf = sort(input[0, ..., mid]);
        largerHalf = sort(input[mid + 1, ...]);
        return merge(smallerHalf, largerHalf);
    }
}
```

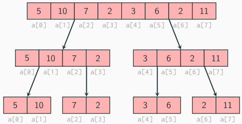

Merge sort: Example



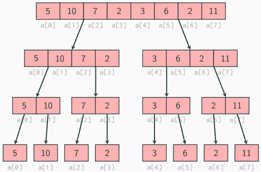
Merge sort: Example



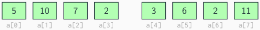
Merge sort: Example



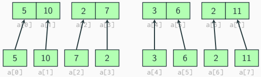
Merge sort: Example



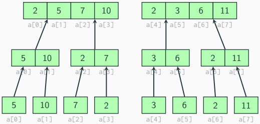
Merge sort: Example



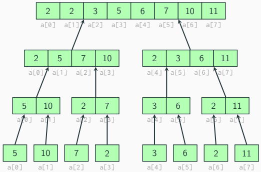
Merge sort: Example



Merge sort: Example



Merge sort: Example



Merge sort: Analysis

Pseudocode

```
sort(input) {
    if (input.length < 2) {
        return input;
    } else {
        smallerHalf = sort(input[0, ..., mid]);
        largerHalf = sort(input[mid + 1, ...]);
        return merge(smallerHalf, largerHalf);
    }
}
```

Best case runtime?

Worst case runtime?

Merge sort: Analysis

Best and worst case

We always subdivide the array in half on each recursive call, and merge takes $\mathcal{O}(n)$ time to run. So, the best and worst case runtime is the same:

$$T(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ 2T(n/2) + n & \text{otherwise} \end{cases}$$

Spoiler alert: this is $\Theta(n \log(n))$

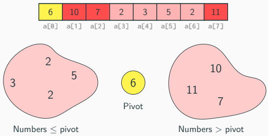
Merge sort: Analysis

Stability and In-place

If we implement the `merge` function correctly, merge sort will be **stable**.

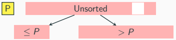
However, `merge` must construct a new array to contain the output, so merge sort is **not in-place**.

Quick sort: Divide step



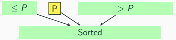
Quick sort: Core pieces

Divide: Pick a pivot, partition into groups



Conquer: Return array when length ≤ 1

Combine: Combine sorted portions and the pivot



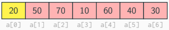
Quick sort: Summary

Core idea: Pick some item from the array and call it the **pivot**. Put all items **smaller** in the pivot into one group and all items **larger** in the other and recursively sort. If the array has size 0 or 1, just return it unchanged.

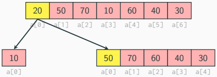
Pseudocode

```
sort(input) {
    if (input.length < 2) {
        return input;
    } else {
        pivot = getPivot(input);
        smallerHalf = sort(getSmaller(pivot, input));
        largerHalf = sort(getBigger(pivot, input));
        return smallerHalf + pivot + largerHalf;
    }
}
```

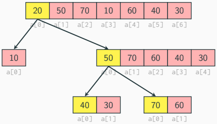
Quick sort: Example



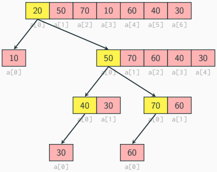
Quick sort: Example



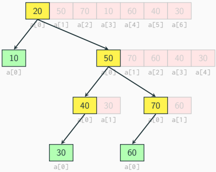
Quick sort: Example



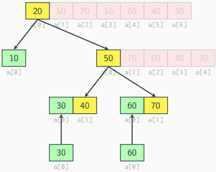
Quick sort: Example



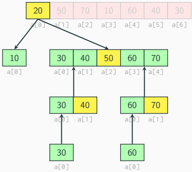
Quick sort: Example



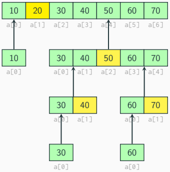
Quick sort: Example



Quick sort: Example



Quick sort: Example



Quick sort: Analysis

Pseudocode

```
sort(input) {
    if (input.length < 2) {
        return input;
    } else {
        pivot = getPivot(input);
        smallerHalf = sort(getSmaller(pivot, input));
        largerHalf = sort(getBigger(pivot, input));
        return smallerHalf + pivot + largerHalf;
    }
}
```

Best case runtime?

Worst case runtime?

Quick sort: Analysis

Best case analysis

In the **best** case, we always pick the **median** element.

$$T(n) = \begin{cases} 2T(n/2) + n & \text{if } n > 1 \\ 1 & \text{otherwise} \end{cases}$$

(Spoiler alert: this is $\Theta(n \log(n))$)

Quick sort: Analysis

Worst case analysis

In the **worst** case, we always end up picking the **minimum** or **maximum** element.

$$T(n) = \begin{cases} T(n-1) + n & \text{if } n > 1 \\ 1 & \text{otherwise} \end{cases}$$

So, the worst-case runtime is $\Theta\left(n^2\right)$.

Quick sort: Analysis

Best case analysis

In the **best** case, we always pick the **median** element, so the best-case runtime is $\Theta(n \log(n))$.

Worst case analysis

In the **worst** case, we always end up picking the **minimum** or **maximum** element, so, the worst-case runtime is $\Theta\left(n^2\right)$.

Average case runtime

Usually, we'll pick a **random** element, which makes the runtime $\Theta(n \log(n))$.

Quick sort: Analysis

Stability

Quick sort is **not stable** – our partition step ends up disregarding and sometimes ignoring the existing relative ordering of duplicate elements.

In-place?

Quick sort is **in-place** – see next few slides for details!

Quick sort: Unresolved questions

How do we pick a pivot?

- Worst case? Pick the minimum or the maximum. The work will shrink by only 1 on each recursive call.
- Ideally? Pick the median. The work will split in half on each recursive call.

How do we partition?

Quick sort: Picking a pivot

How do we find the median?

- Idea: pick the first item in the array
  - Problem: what if the array is already sorted?
  - (Real world data often is partially sorted)
  - But hey, it's speedy ($\mathcal{O}(1)$)
- Idea: try finding it by looping through the array
  - Problem: hard to implement, and expensive ($\mathcal{O}(n)$)

These seem like bad ideas :(

Quick sort: Picking a pivot

Other ideas:

- Idea: pick a random element
  - On average, guaranteed to do well – no easy worst case
  - Random number generation can sometimes be expensive/fraught with peril
- Idea: pick the median of first, middle, and last
  - Adversary could still construct malicious input
  - ...but works well in practice, and is efficient

These seem like good ideas :)

Quick sort: Unresolved questions

How do we pick a pivot?

How do we partition?

Quick sort: Partitioning (using median-of-three pivot)

Find the `lo`, med, and `hi`

8	1	4	9	0	3	5	2	7	6
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Quick sort: Partitioning (using median-of-three pivot)

Find the lo, med, and hi

8	1	4	9	0	3	5	2	7	6
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Find the median of the three and swap with front

8	1	4	9	0	3	5	2	7	6
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Quick sort: Partitioning (using median-of-three pivot)

Find the lo, med, and hi

8	1	4	9	0	3	5	2	7	6
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Find the median of the three and swap with front

8	1	4	9	0	3	5	2	7	6
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Final result: pivot is now at index 0

6	1	4	9	0	3	5	2	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

6	1	4	9	0	3	5	2	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Partitioning:

6	1	4	9	0	3	5	2	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
	↑								↑
	low								high
	$1 \leq 6$								$8 > 6$

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

6	1	4	9	0	3	5	2	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Partitioning:

6	1	4	9	0	3	5	2	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
		↑						↑	
		low						high	
		$4 \leq 6$						$7 > 6$	

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

6	1	4	9	0	3	5	2	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Partitioning:

6	1	4	9	0	3	5	2	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
			↑				↑		
			low				high		
			$9 \leq 6$				$2 > 6$		

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

6	1	4	9	0	3	5	2	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Partitioning:

6	1	4	9	0	3	5	2	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
			↑		SWAP		↑		
			low				high		
			$9 \leq 6$				$2 > 6$		

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	9	0	3	5	2	7	8

Partitioning:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	2	0	3	5	9	7	8

low $2 \leq 6$ high $9 > 6$

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	9	0	3	5	2	7	8

Partitioning:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	2	0	3	5	9	7	8

low $0 \leq 6$ high $5 > 6$

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	9	0	3	5	2	7	8

Partitioning:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	2	0	3	5	9	7	8

low $3 \leq 6$ high $5 > 6$

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	9	0	3	5	2	7	8

Partitioning:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	2	0	3	5	9	7	8

low $5 \leq 6$ high $5 > 6$

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	9	0	3	5	2	7	8

Partitioning:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	2	0	3	5	9	7	8

high $5 > 6$ low $9 \leq 6$

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	9	0	3	5	2	7	8

Partitioning:

a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]
6	1	4	2	0	3	5	9	7	8

SWAP high $5 > 6$ low $9 \leq 6$

Quick sort: Partitioning (using median-of-three pivot)

Array after moving pivot:

6	1	4	9	0	3	5	2	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

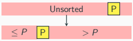
Partitioning:

5	1	4	2	0	3	6	9	7	8
a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]	a[8]	a[9]

Unsorted ≤ 6 Unsorted > 6

Quick sort: Core pieces revisited

Divide: Pick a pivot, partition in-place into groups



Conquer: When subarray is length ≤ 1, do nothing

Combine: Do nothing; already done!

Analyzing recurrences, part 2

So, merge sort and quick sort are both:

$$T(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ 2T(n/2) + n & \text{otherwise} \end{cases}$$

I claim $T(n) \in \Theta(n \log(n))$. How can we show this?

Analyzing recurrences, part 2

We could try unfolding, but it's annoying:

$$\begin{aligned} T(n) &= n + 2T\left(\frac{n}{2}\right) \\ &= n + 2\left(\frac{n}{2} + 2T\left(\frac{n}{4}\right)\right) \\ &= n + 2\left(\frac{n}{2} + 2T\left(\frac{n}{4}\right)\right) \\ &= n + 2\left(\frac{n}{2} + 2T\left(\frac{n}{4} + 2T(\frac{n}{8})\right)\right) \\ &= n + n + 4T\left(\frac{n}{4} + 2T\left(\frac{n}{8}\right)\right) \\ &= n + n + n + 8T\left(\frac{n}{8}\right) \\ &= \underbrace{n + n + \cdots + n}_{\text{about } \log(n) \text{ times}} + n \\ &= n \log(n) \end{aligned}$$

The tree method: overview

Core idea:

1. Draw what the work looks like visually, as a *tree*
2. Use the visualization to help us analyze the overall behavior
3. Either find the closed form, or construct a summation that we can simplify to get the closed form

The tree method: example

Step 1: Start with the function, let n be the input value

$$\boxed{T(n)}$$

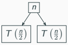
The tree method: example

Step 2: Replace with definition

$$\boxed{T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + n}$$

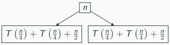
The tree method: example

Step 3: Stick each recursive call into a *subtree*



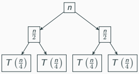
The tree method: example

Step 4: Replace with definition



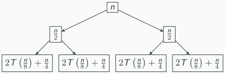
The tree method: example

Repeat step 3 (move recursive call to subtrees):



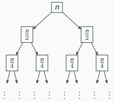
The tree method: example

Repeat step 4 (replace recursive call with definition):



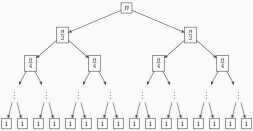
The tree method: example

Repeat...



The tree method: example

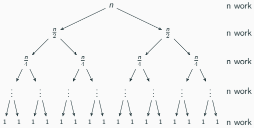
Final step: how much work does each base case do?



The tree method: analysis

Now, let's add everything up!

How much work is done per level?



Height is roughly $\log_2(n)$, so total work is about $n \log_2(n)$.

The tree method: practice

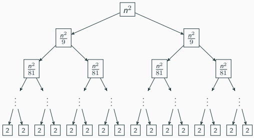
Consider the following recurrence:

$$S(n) = \begin{cases} 2 & \text{if } n \leq 1 \\ 2S\left(n/3\right) + n^2 & \text{otherwise} \end{cases}$$

Draw a tree to help you visualize the work done.

The tree method: practice

Final step: how much work does each base case do?



Now what?

The tree method: precise analysis

Problem: Need a rigorous way of getting a closed form

We want to answer a few core questions:

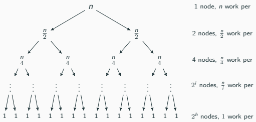
How much work does each recursive level do?

1. How many nodes are there on level i? ($i = 0$ is "root" level)
2. At some level i, how much work does a *single* node do? (Ignoring subtrees)
3. How many recursive levels are there?

How much work does the leaf level (base cases) do?

1. How much work does a single leaf node do?
2. How many leaf nodes are there?

The tree method: precise analysis

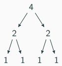


1. numNodes(i) $= 2^i$
2. workPerNode(n, i) $= \frac{n}{2^i}$
3. numLevels(n) $= ?$
4. workPerLeafNode(n) $= 1$
5. numLeafNodes(n) $= ?$

The tree method: precise analysis

How many levels are there, exactly? Is it $\log_2(n)$?

Let's try an example. Suppose we have $T(4)$. What happens?



Height is $\log_2(4) = 2$.

For this recursive function, num recursive levels is same as height.

Important: total levels, counting base case, is height $+ 1$.

Important: for other recursive functions, where base case doesn't happen at $n \leq 1$, num recursive levels might be different then height.

The tree method: precise analysis

We discovered:

1. numNodes(i) $= 2^i$
2. workPerNode(n, i) $= \frac{n}{2^i}$
3. numLevels(n) $= \log_2(n)$
4. workPerLeafNode(n) $= 1$
5. numLeafNodes(n) $= 2^{\text{numLevels}(n)} = 2^{\log_2(n)} = n$

Our formulas:

$$\text{recursiveWork} = \sum_{i=0}^{\text{numLevels(n)}} \text{numNodes}(i) \cdot \text{workPerNode}(n, i)$$

$$\text{baseCaseWork} = \text{numLeafNodes}(n) \cdot \text{workPerLeafNode}(n)$$

$$\text{totalWork} = \text{recursiveWork} + \text{baseCaseWork}$$

The tree method: precise analysis

Solve for recursive case:

$$\begin{aligned} \text{recursiveWork} &= \sum_{i=0}^{\log_2(n)} 2^i \cdot \frac{n}{2^i} \\ &= \sum_{i=0}^{\log_2(n)} n \\ &= n \log_2(n) \end{aligned}$$

Solve for base case:

$$\begin{aligned} \text{baseCaseWork} &= \text{numLeafNodes}(n) \cdot \text{workDonePerLeafNode}(n) \\ &= n \cdot 1 = n \end{aligned}$$

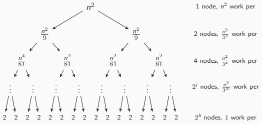
So exact closed form is $n \log_2(n) + n$.

The tree method: practice

Practice: Let's go back to our old recurrence...

$$S(n) = \begin{cases} 2 & \text{if } n \leq 1 \\ 2S(n/3) + n^2 & \text{otherwise} \end{cases}$$

The tree method: practice



1. numNodes(i) $= 2^i$
2. workPerNode(n, i) $= \frac{n^2}{9^i}$
3. numLevels(n) $= \log_3(n)$
4. workPerLeafNode(n) $= 2$
5. numLeafNodes(n) $= 2^{\text{numLevels}(n)} = 2^{\log_3(n)} = n^{\log_3(2)}$

The tree method: practice

1. numNodes(i) $= 2^i$
2. workPerNode(n, i) $= \frac{n^2}{9^i}$
3. numLevels(n) $= \log_3(n)$
4. workPerLeafNode(n) $= 2$
5. numLeafNodes(n) $= 2^{\text{numLevels}(n)} = 2^{\log_3(n)} = n^{\log_3(2)}$

Combine into a single expression representing the total runtime.

$$\begin{aligned} \text{totalWork} &= \left(\sum_{i=0}^{\log_3(n)} 2^i \cdot \frac{n^2}{9^i} \right) + 2n^{\log_3(2)} \\ &= n^2 \sum_{i=0}^{\log_3(n)} \frac{2^i}{9^i} + 2n^{\log_3(2)} \\ &= n^2 \sum_{i=0}^{\log_3(n)} \left(\frac{2}{9} \right)^i + 2n^{\log_3(2)} \end{aligned}$$

The finite geometric series

We have: $n^2 \sum_{i=0}^{\log_3(n)} \left(\frac{2}{9}\right)^i + 2n^{\log_3(2)}$

The finite geometric series identity: $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$

Plug and chug:

$$\begin{aligned}
\text{totalWork} &= n^2 \sum_{i=0}^{\log_3(n)} \left(\frac{2}{9}\right)^i + 2n^{\log_3(2)} \\
&= n^2 \sum_{i=0}^{\log_3(n)+1-1} \left(\frac{2}{9}\right)^i + 2n^{\log_3(2)} \\
&= n^2 \frac{1-\left(\frac{2}{9}\right)^{\log_3(n)+1}}{1-\frac{2}{9}} + 2n^{\log_3(2)}
\end{aligned}$$

Applying the finite geometric series

With a bunch of effort...

$$\begin{aligned}
\text{totalWork} &= n^2 \frac{1-\left(\frac{2}{9}\right)^{\log_3(n)+1}}{1-\frac{2}{9}} + 2n^{\log_3(2)} \\
&= \frac{9}{7}n^2\left(1-\frac{2}{9}\left(\frac{2}{9}\right)^{\log_3(n)}\right) + 2n^{\log_3(2)} \\
&= \frac{9}{7}n^2 - \frac{2}{7}n^2\left(\frac{2}{9}\right)^{\log_3(n)} + 2n^{\log_3(2)} \\
&= \frac{9}{7}n^2 - \frac{2}{7}n^2 n^{\log_3(2/9)} + 2n^{\log_3(2)} \\
&= \frac{9}{7}n^2 - \frac{2}{7}n^2 n^{\log_3(2)-2} + 2n^{\log_3(2)} \\
&= \frac{9}{7}n^2 - \frac{2}{7}n^{\log_3(2)} + 2n^{\log_3(2)} \\
&= \frac{9}{7}n^2 + \frac{12}{7}n^{\log_3(2)}
\end{aligned}$$

The master theorem

Is there an easier way?

If we want to find an exact closed form, no. Must use either the unfolding technique or the tree technique.

If we want to find a big-Θ bound, yes.

The master theorem

The master theorem

Suppose we have a recurrence of the following form:

$$T(n) = \begin{cases} d & \text{if } n = 1 \\ aT\left(\frac{n}{b}\right) + n^c & \text{otherwise} \end{cases}$$

Then...

- If $\log_b(a) < c$, then $T(n) \in \Theta\left(n^c\right)$
- If $\log_b(a) = c$, then $T(n) \in \Theta\left(n^c \log(n)\right)$
- If $\log_b(a) > c$, then $T(n) \in \Theta\left(n^{\log_b(a)}\right)$

The master theorem

Given:

$$T(n) = \begin{cases} d \\ aT\left(\frac{n}{b}\right) + n^c \end{cases}$$

Then...

If $\log_b(a) < c$, then $T(n) \in \Theta\left(n^c\right)$

If $\log_b(a) = c$, then $T(n) \in \Theta\left(n^c \log(n)\right)$

If $\log_b(a) > c$, then $T(n) \in \Theta\left(n^{\log_b(a)}\right)$

Sanity check: try checking merge sort.

We have $a = 2$, $b = 2$, and $c = 1$. We know $\log_b(a) = \log_2(2) = 1 = c$, therefore merge sort is $\Theta\left(n\log(n)\right)$.

Sanity check: try checking $S(n) = 2S(n/3) + n^2$.

We have $a = 2$, $b = 3$, and $c = 2$. We know $\log_3(2) \leq 1 < 2 = c$, therefore $S(n) \in \Theta\left(n^2\right)$.

The master theorem: intuition

Intuition, the $\log_b(a) < c$ case:

1. We do work more rapidly then we divide.
2. So, more of the work happens near the "top", which means that the n^c term dominates.

The master theorem: intuition

Intuition, the $\log_b(a) > c$ case:

1. We divide more rapidly then we do work.
2. So, most of the work happens near the "bottom", which means the work done in the leaves dominates.
3. Note: Work in leaves is about $d \cdot a^{\text{height}} = d \cdot a^{\log_b(n)} = d \cdot n^{\log_b(a)}$.

The master theorem: intuition

Intuition, the $\log_b(a) = c$ case:

1. Work is done roughly equally throughout tree.
2. Each level does about the same amount of work, so we approximate by just multiplying work done on first level by the height: $n^c \log_b(n)$.