CSE 373: Binary heaps

Michael Lee

Monday, Feb 5, 2018

Course overview

The course so far...

- Reviewing manipulating arrays and nodes
- Algorithm analysis
- Dictionaries (tree-based and hash-based)

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- Algorithm analysis
- Dictionaries (tree-based and hash-based)

Coming up next:

- Divide-and-conquer, sorting
- Graphs
- Misc topics (P vs NP, more?)

Timeline

When are we getting project grades/our midterm back?

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Tuesday or Wednesday

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  - You **MUST** find a new partner...
  - ...unless both partners email me and petition to stay together

Today

Motivating question:

Suppose we have a collection of “items”.

We want to return whatever item has the smallest “priority”.

The Priority Queue ADT

Specifically, want to implement the **Priority Queue** ADT:

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A priority queue stores elements according to their "priority". It supports the following operations:

- **removeMin:** return the element with the *smallest* priority
- **peekMin:** find (but do not return) the *smallest* element
- **insert:** add a new element to the priority queue

The Priority Queue ADT

An alternative definition: instead of yielding the element with the largest priority, yield the one with the *largest* priority:

The Priority Queue ADT, alternative definition

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A priority queue stores elements according to their "priority". It supports the following operations:

- **removeMax:** return the element with the *largest* priority
- **peekMax:** find (but do not return) the *largest* element
- **insert:** add a new element to the priority queue

The way we implement both is almost identical – we just tweak how we compare elements

In this class, we will focus on implementing a "min" priority queue

Initial implementation ideas

Fill in this table with the worst-case runtimes:

Idea	**removeMin**	**peekMin**	**insert**
Unsorted array list			
Unsorted linked list			
Sorted array list			
Sorted linked list			
Binary tree			
AVL tree			

Initial implementation ideas

Fill in this table with the worst-case runtimes:

Idea	**removeMin**	**peekMin**	**insert**
Unsorted array list	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
Unsorted linked list	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
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AVL tree	$\Theta(\log(n))$	$\Theta(\log(n))$	$\Theta(\log(n))$

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We want something optimized both frequent inserts and removes.

An AVL tree (or some tree-ish thing) seems good enough... right?

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An AVL tree (or some tree-ish thing) seems good enough... right?

Today: learn how to implement a *binary heap*.

`peekMin` is $\mathcal{O}(1)$, and `insert` and `remove` are still $\mathcal{O}(\log(n))$ in the worst case.

However, `insert` is $\mathcal{O}(1)$ in the average case!

Binary heap invariants

Idea: adapt the tree-based method

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Insight: in a tree, finding the min is expensive! Rather then having it to the left, have it on the top!

Binary heap invariants

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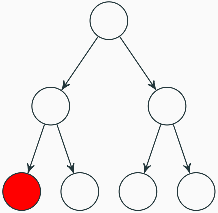
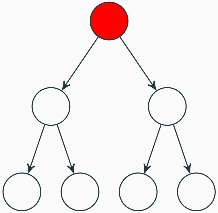
A BST or AVL tree



A binary heap



Binary heap invariants

We now need to change our invariants...

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A binary heap has three invariants:

- **Num children:** Every node has at most 2 children

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Binary heap invariants

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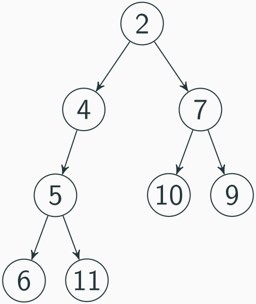
Binary heap invariants

A binary heap has three invariants:

- **Num children:** Every node has at most 2 children
- **Heap:** Every node is smaller then its children
- **Structure:** Every heap is a “complete” tree – it has no “gaps”

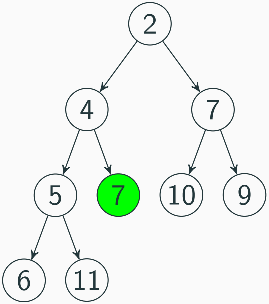
Example of a heap

A broken heap



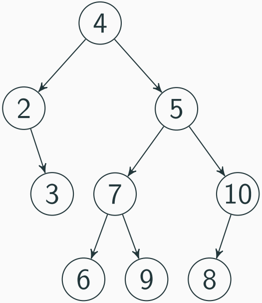
Example of a heap

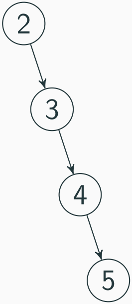
A fixed heap

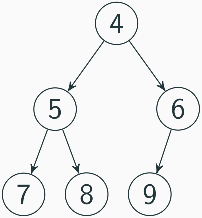


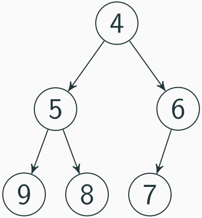
The heap invariant

Are these all heaps?



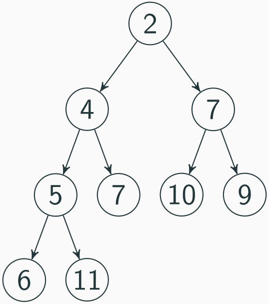






Implementing peekMin

How do we implement `peekMin`?



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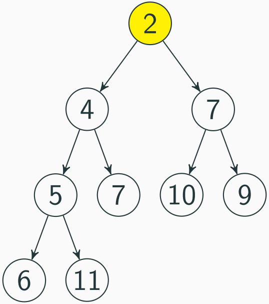
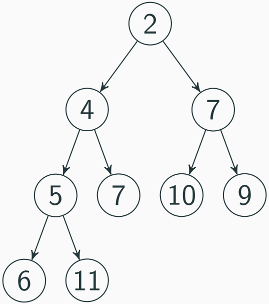


Easy: just return the root. Runtime: $\Theta(1)$.

Implementing removeMin

What about `removeMin`?



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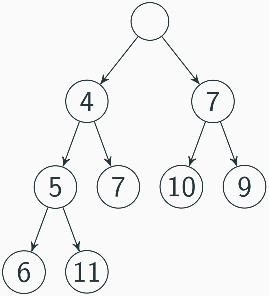
What about `removeMin`?

Step 1: Just remove it!



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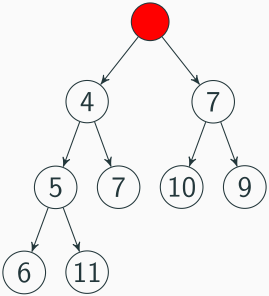
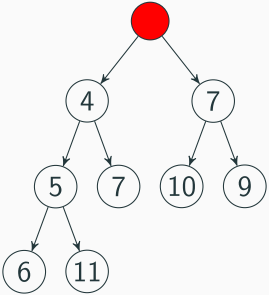


Problem: Structure invariant is broken – the tree has a gap!

Implementing removeMin

How do we fix the gap?



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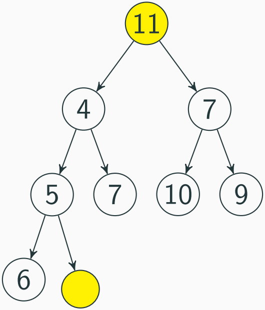
How do we fix the gap?

Step 2: Plug the gap by moving the last element to the top!



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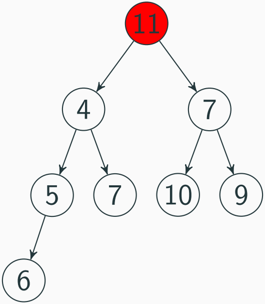
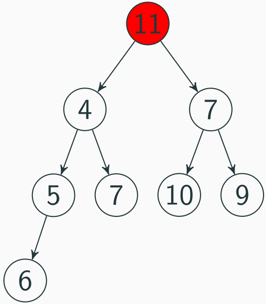


Problem: Heap invariant is broken – 11 is not smaller then 4 or 7!

Implementing removeMin

How do we fix the heap invariant?



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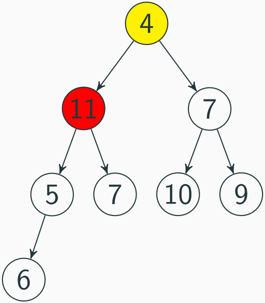

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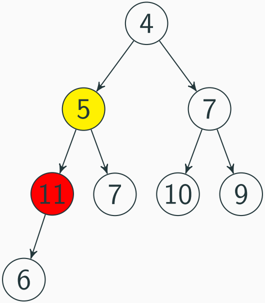

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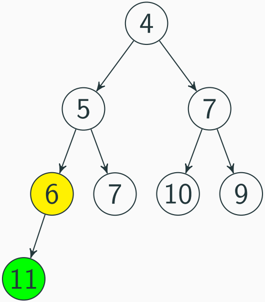
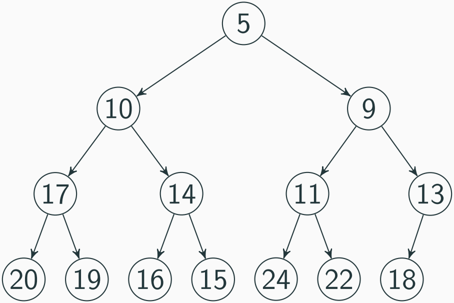


And we’re done!

Practice

Practice: What happens if we call `removeMin`?



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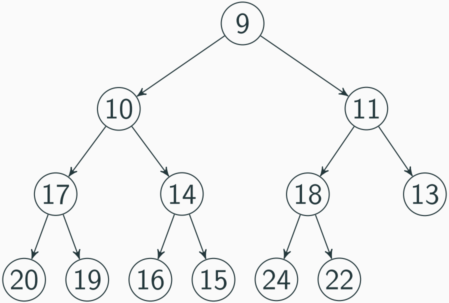

Analyzing removeMin

The percolateDown algorithm

```
percolateDown(node) {
    while (node.data is bigger then children) {
        swap data with smaller child
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This ends up being:

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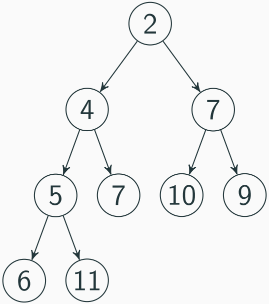
...which is in $\Theta(n)$.

Implementing insert

What about `insert`? Suppose we insert `3` – what happens?



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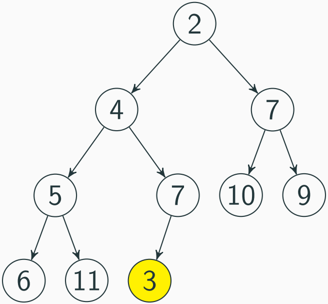

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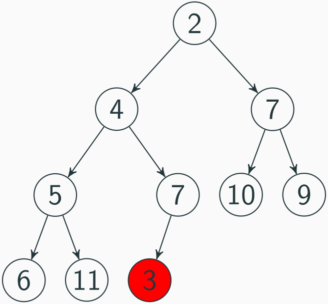
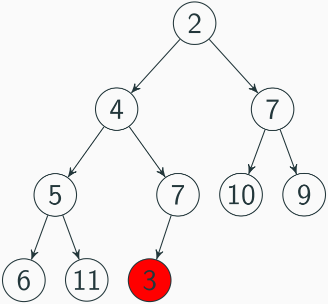


Problem: heap invariant broken! 7 is not smaller then 3!

Implementing insert

How do we fix the heap invariant?



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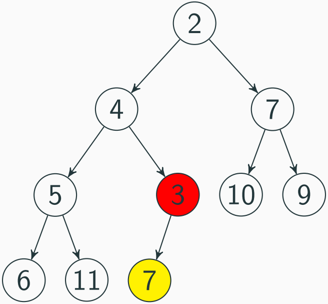
How do we fix the heap invariant?

Step 2: “percolate up” – keep swapping node with parent until heap invariant is fixed



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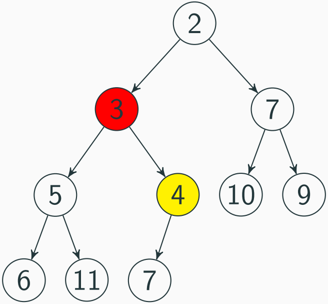

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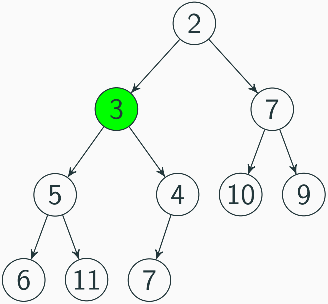

All ok!

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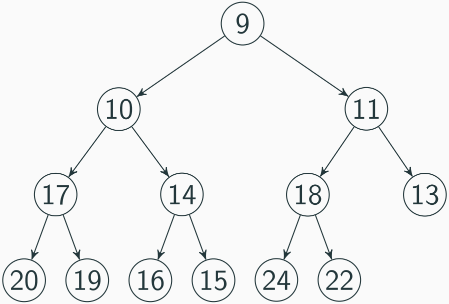

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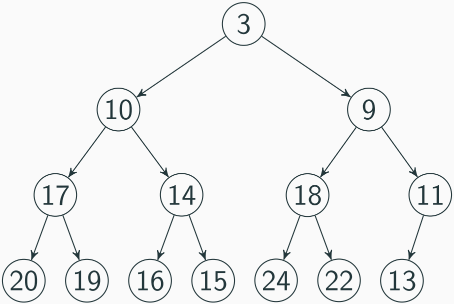
After inserting 3:



Analyzing insert

The `percolateUp` algorithm

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This ends up being:

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...which is in $\Theta(n)$.

Analyzing removeMin, part 2

Problem: But wait! I promised worst-case $\Theta(\log(n))$ insert and average-case $\Theta(1)$ insert.

This algorithm is $\Theta(\log(n))$ in both the worst and average case!

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This algorithm is $\Theta(\log(n))$ in both the worst and average case!

Why: Finding and modifying the last node is slow: requires traversal!

Can we speed it up?

Analyzing removeMin, part 2

Remember this slide?

Idea	**removeMax**	**peekMax**	**insert**
Unsorted array list	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
Unsorted linked list	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
Sorted array list	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$
Sorted linked list	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$
Binary tree	$\Theta(n)$	$\Theta(n)$	$\Theta(\log(n))$
AVL tree	$\Theta(\log(n))$	$\Theta(\log(n))$	$\Theta(\log(n))$

Analyzing removeMin, part 2

Observation:

- Arrays let us find and append to the end quickly
- Trees let us have nice $\log(n)$ traversal behavior

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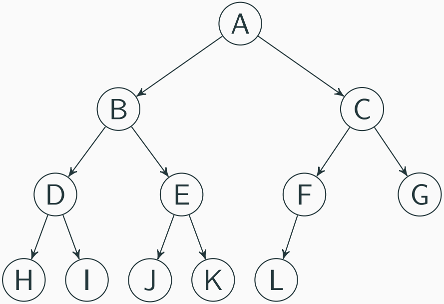
The trick: Why pick one or the other? Let's do both!

The array-based representation of binary heaps

Take a tree:



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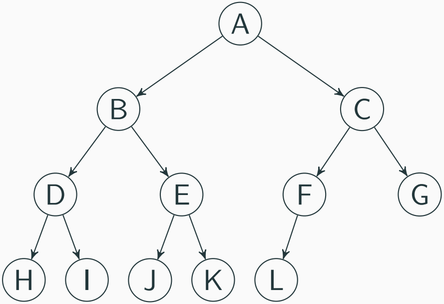

And fill an array in the **level-order** of the tree:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A	B	C	D	E	F	G	H	I	J	K	L			

The array-based representation of binary heaps

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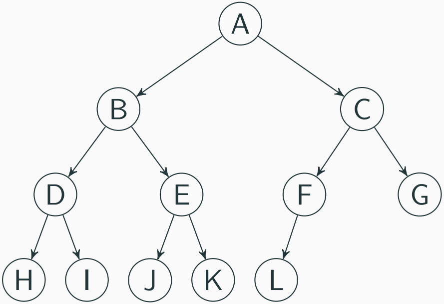

How do we find parent?

$$\text{parent}(i) = \left\lfloor \frac{i-1}{2} \right\rfloor$$

The left child?

$$\text{leftChild}(i) = 2i + 1$$

The right child?

$$\text{leftChild}(i) = 2i + 2$$

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$\Theta(1)$: just use `this.array[this.size - 1]`.

...assuming array has no 'gaps'. (Hey, it looks like the structure invariant was useful after all)

Re-analyzing insert

How does this change runtime of `insert`?

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Runtime of insert:

$$\texttt{findLastNodeTime} + \text{addNodeToLastTime} + \text{numSwaps} \times \text{swapTime}$$

...which is:

$$1 + 1 + \text{numSwaps} \times 1$$

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Observation: when percolating, we usually need to percolate up a few times! So, numSwaps ≈ 1 in the average case, and numSwaps $\approx$ height $= \log(n)$ in the worst case!

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Observation: unfortunately, in practice, usually must percolate all the way down. So numSwaps $\approx$ height $\approx \log(n)$ on average.