CSE 373: Binary heaps

Michael Lee

Monday, Feb 5, 2018

Course overview

The course so far...

- Reviewing manipulating arrays and nodes
- Algorithm analysis
- Dictionaries (tree-based and hash-based)

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- Dictionaries (tree-based and hash-based)

Coming up next:

- Divide-and-conquer, sorting
- Graphs
- Misc topics (P vs NP, more?)

Timeline

When are we getting project grades/our midterm back?

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Tuesday or Wednesday

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  - You **MUST** find a new partner...
  - ...unless both partners email me and petition to stay together

Motivating question:

Suppose we have a collection of “items”.

We want to return whatever item has the ~~largest~~ smallest “priority”.

The Priority Queue ADT

Specifically, want to implement the **Priority Queue** ADT:

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A priority queue stores elements according to their "priority". It supports the following operations:

- **removeMin:** return the element with the *smallest* priority
- **peekMin:** find (but do not return) the *smallest* element
- **insert:** add a new element to the priority queue

The Priority Queue ADT

An alternative definition: instead of yielding the element with the largest priority, yield the one with the *largest* priority:

The Priority Queue ADT, alternative definition

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A priority queue stores elements according to their "priority". It supports the following operations:

- **removeMax:** return the element with the *largest* priority
- **peekMax:** find (but do not return) the *largest* element
- **insert:** add a new element to the priority queue

The way we implement both is almost identical – we just tweak how we compare elements

In this class, we will focus on implementing a "min" priority queue

Fill in this table with the worst-case runtimes:

Idea	remove~~Max~~ Min	peek~~Max~~ Min	insert
Unsorted array list	O(n)	O(n)	O(1)
Unsorted linked list			
Sorted array list	O(1)	O(1)	O(n)
Sorted linked list	O(1)	O(1)	O(n)
Binary tree	O(n)	O(n)	O(n)
AVL tree	O(log(n))		

Initial implementation ideas

Fill in this table with the worst-case runtimes:

Idea	removeMax	peekMax	insert
Unsorted array list	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
Unsorted linked list	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
Sorted array list			
Sorted linked list			
Binary tree			
AVL tree			

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Binary tree			
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Idea	**removeMax**	**peekMax**	**insert**
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Sorted array list			
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Binary tree	$\Theta(n)$	$\Theta(n)$	$\Theta(\log(n))$
AVL tree			

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Binary tree			
AVL tree	$\Theta(\log(n))$	$\Theta(\log(n))$	$\Theta(\log(n))$

Initial implementation ideas

We want something optimized both frequent inserts and removes.

An AVL tree (or some tree-ish thing) seems good enough... right?

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Today: learn how to implement a *binary heap*.

`peekMin` is $\mathcal{O}(1)$, and `insert` and `remove` are still $\mathcal{O}(\log(n))$ in the worst case.

However, `insert` is $\mathcal{O}(1)$ in the average case!

Binary heap invariants

Idea: adapt the tree-based method

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Insight: in a tree, finding the min is expensive! Rather then having it to the left, have it on the top!

Binary heap invariants

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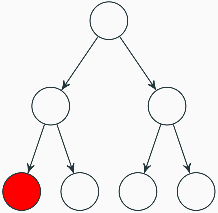
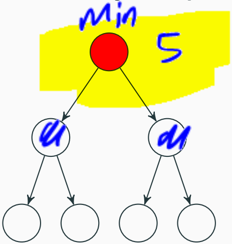
A BST or AVL tree



A binary heap



Binary heap invariants

We now need to change our invariants...

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A binary heap has three invariants:

- **Num children:** Every node has at most 2 children

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Binary heap invariants

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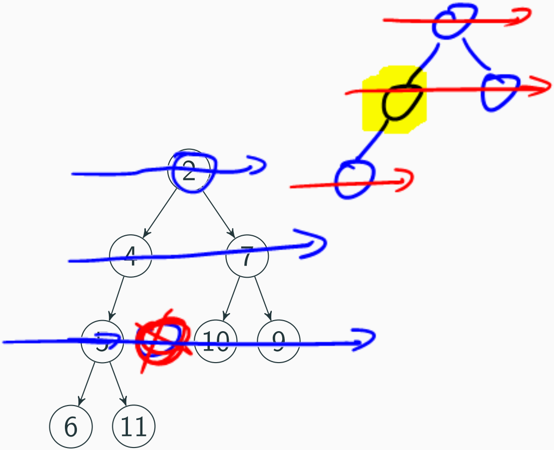
Binary heap invariants

A binary heap has three invariants:

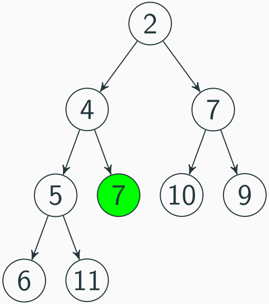
- **Num children:** Every node has at most 2 children
- **Heap:** Every node is smaller then its children
- **Structure:** Every heap is a "complete" tree – it has no "gaps"

A broken heap



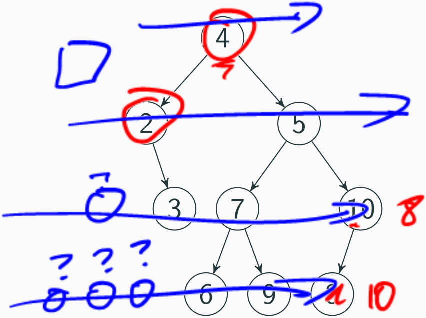
Example of a heap

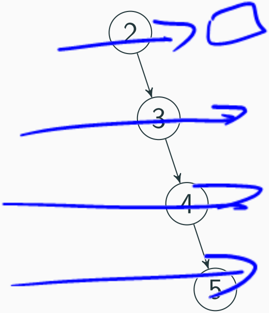
A fixed heap

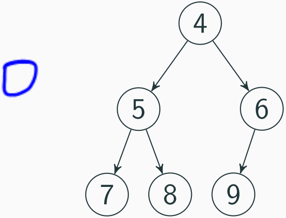


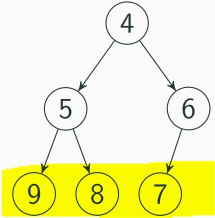
The heap invariant

Are these all heaps?



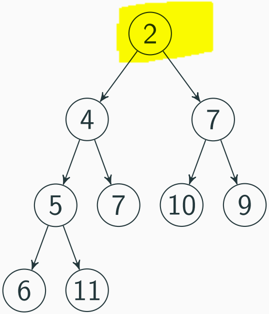






Implementing peekMin

How do we implement `peekMin`?



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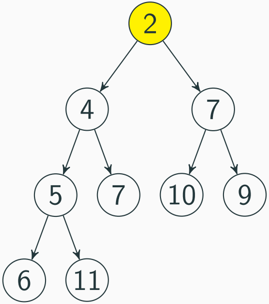
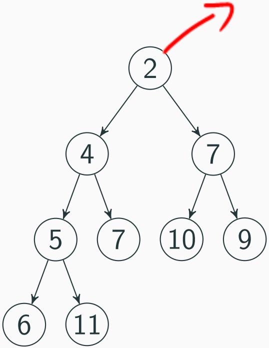


Easy: just return the root. Runtime: $\Theta(1)$.

Implementing removeMin

What about `removeMin`?



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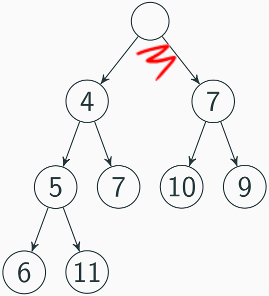
What about `removeMin`?

Step 1: Just remove it!

2



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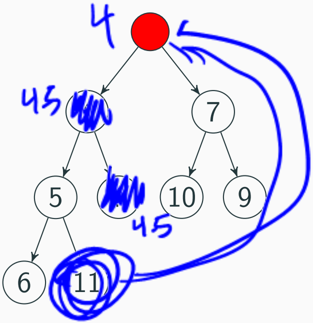
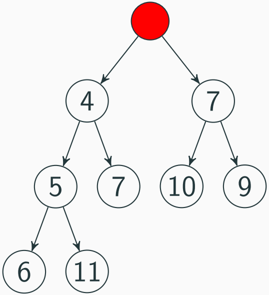


Problem: Structure invariant is broken – the tree has a gap!

Implementing removeMin

How do we fix the gap?



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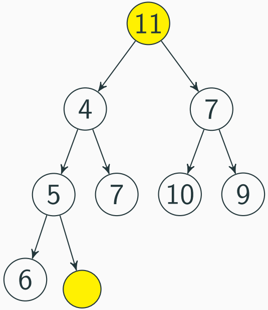
How do we fix the gap?

Step 2: Plug the gap by moving the last element to the top!



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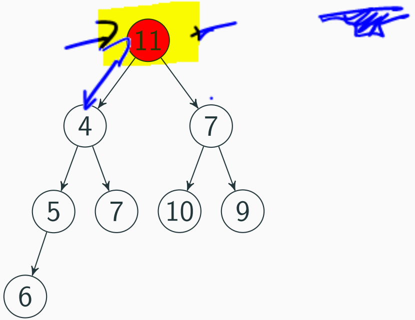
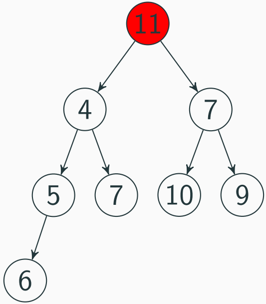


Problem: Heap invariant is broken – 11 is not smaller then 4 or 7!

Implementing removeMin

How do we fix the heap invariant?



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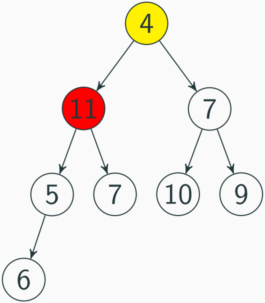

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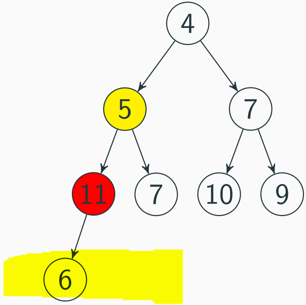

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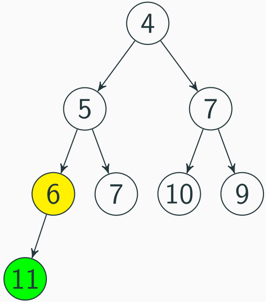
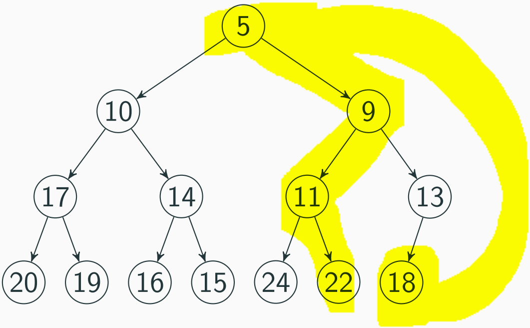


And we’re done!

Practice

Practice: What happens if we call `removeMin`?



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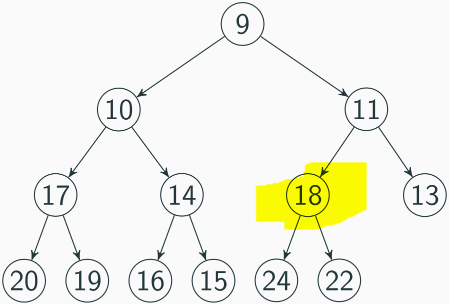

Analyzing removeMin

The percolateDown algorithm

```
percolateDown(node) {
    while (node.data is bigger then children) {
        swap data with smaller child
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This ends up being:

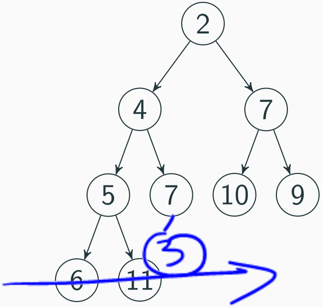
n

[redacted] $+ 1 + \log(n) \cdot 1$

...which is in [redacted] $\Theta(n)$

Implementing insert

What about `insert`? Suppose we insert `3` – what happens?



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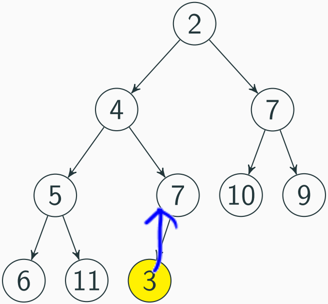

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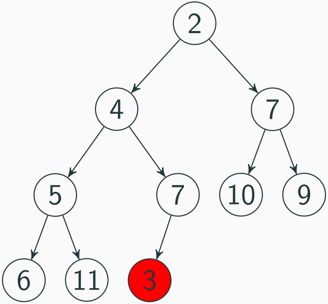
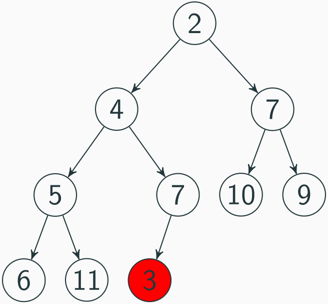


Problem: heap invariant broken! 7 is not smaller then 3!

Implementing insert

How do we fix the heap invariant?



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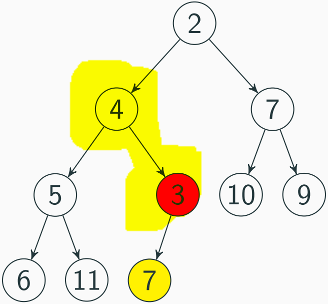
How do we fix the heap invariant?

Step 2: "percolate up" – keep swapping node with parent until heap invariant is fixed



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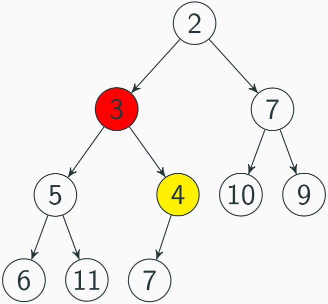

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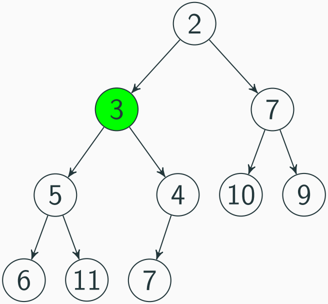

All ok!

Practice

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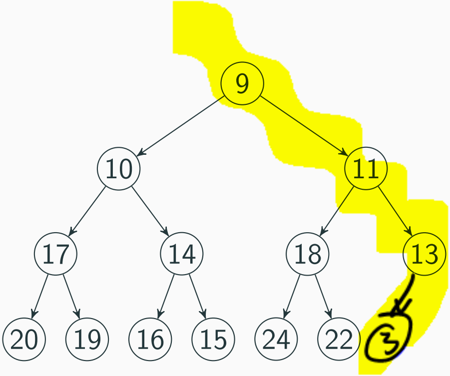

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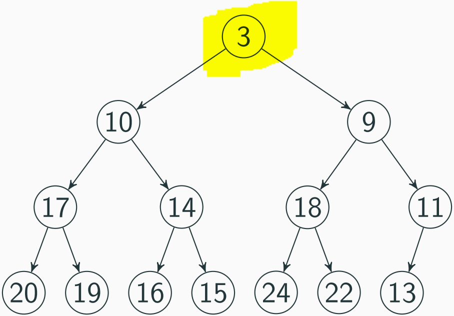
After inserting 3:



Analyzing insert

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This ends up being:

$$n + 1 + \log(n) \cdot 1$$

...which is in $\Theta(n)$

Analyzing removeMin, part 2

Problem: But wait! I promised worst-case $\Theta(\log(n))$ insert and average-case $\Theta(1)$ insert.

This algorithm is ~~[illegible]~~ in both the worst and average case!

$\Theta(n)$

Analyzing removeMin, part 2

Problem: But wait! I promised worst-case $\Theta(\log(n))$ insert and average-case $\Theta(1)$ insert.

This algorithm is ~~[illegible]~~ in both the worst and average case!
$\Theta(n)$

Why: Finding and modifying the last node is slow: requires traversal!

Can we speed it up?

Analyzing removeMin, part 2

Remember this slide?

Idea	removeMax	peekMax	insert
Unsorted array list	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
Unsorted linked list	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
Sorted array list	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$
Sorted linked list	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$
Binary tree	$\Theta(n)$	$\Theta(n)$	$\Theta(\log(n))$
AVL tree	$\Theta(\log(n))$	$\Theta(\log(n))$	$\Theta(\log(n))$

Analyzing removeMin, part 2

Observation:

- Arrays let us find and append to the end quickly
- Trees let us have nice $\log(n)$ traversal behavior

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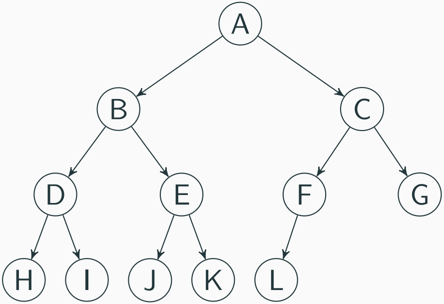
The trick: Why pick one or the other? Let's do both!

The array-based representation of binary heaps

Take a tree:



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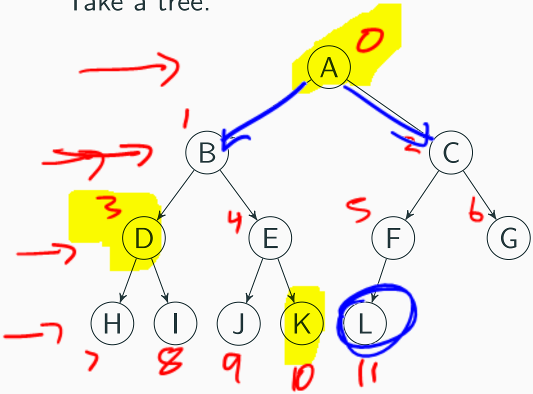

And fill an array in the **level-order** of the tree:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A	B	C	D	E	F	G	H	I	J	K	L			

The array-based representation of binary heaps

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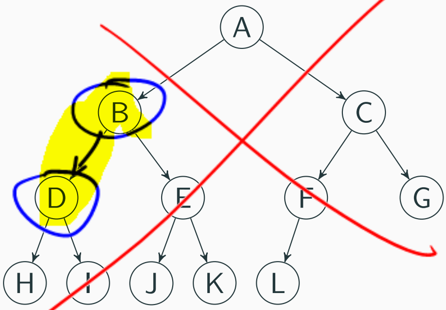

How do we find parent?

$$\text{parent}(i) = \left\lfloor \frac{i-1}{2} \right\rfloor$$

The left child?

$$\text{leftChild}(i) = 2i+1$$

The right child?

$$\text{leftChild}(i) = 2i+2$$

$2 \cdot 1 + 1 = 3$

And fill an array in the **level-order** of the tree:

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If our tree is represented using an array, what's the time needed to find the last node now?

$\Theta(1)$: just use `this.array[this.size - 1]`.

...assuming array has no 'gaps'. (Hey, it looks like the structure invariant was useful after all)

Re-analyzing insert

How does this change runtime of `insert`?

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Runtime of insert:

$$\underbrace{\texttt{findLastNodeTime}}_{1} + \text{addNodeToLastTime} + \underbrace{\text{numSwaps}}_{\log(n)} \times \text{swapTime}$$

...which is:

$$1 + 1 + \text{numSwaps} \times 1$$

$O(\log(n))$

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Observation: when percolating, we usually need to percolate up a few times! So, numSwaps ≈ 1 in the average case, and numSwaps $\approx$ height $= \log(n)$ in the worst case!

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...which is:

$$1 + 1 + \text{numSwaps} \times 1$$

Observation: unfortunately, in practice, usually must percolate all the way down. So numSwaps $\approx$ height $\approx \log(n)$ on average.