CSE 373: AVL trees

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Friday, Jan 19, 2018

Warmup:

- What is an *invariant*?
- What are the AVL tree invariants, exactly?

Discuss with your neighbor.

AVL Trees: Invariants

Core idea: add extra **invariant** to BSTs that enforce balance.

AVL Tree Invariants

An AVL tree has the following invariants:

- **The "structure" invariant:**
  All nodes have 0, 1, or 2 children.
- **The "BST" invariant:**
  For all nodes, all keys in the *left* subtree are smaller;
  all keys in the *right* subtree are larger
- **The "balance" invariant:**
  For all nodes, $\text{abs}\,(\text{height}\,(\text{left})) - \text{height}\,(\text{height}\,(\text{right})) \leq 1$.

AVL = **A**delson-**V**elsky and **L**andis

Interlude: Exploring the balance invariant

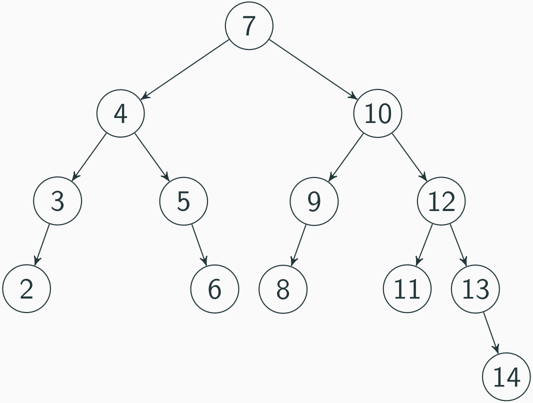
Question: why $\text{abs}\,(\text{height}\,(\text{left})) - \text{height}\,(\text{height}\,(\text{right})) \leq 1$?

Why not $\text{height}\,(\text{left}) = \text{height}\,(\text{right})$?

What happens if we insert two elements. What happens?

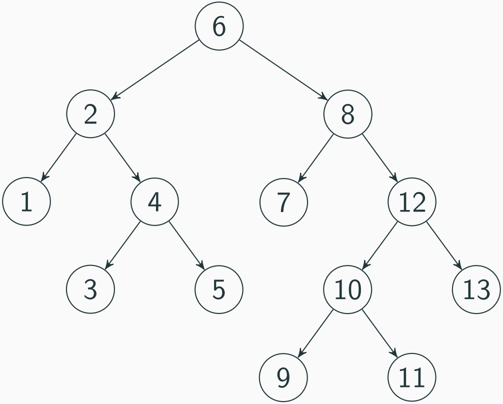
AVL tree invariants review

Question: is this a valid AVL tree?



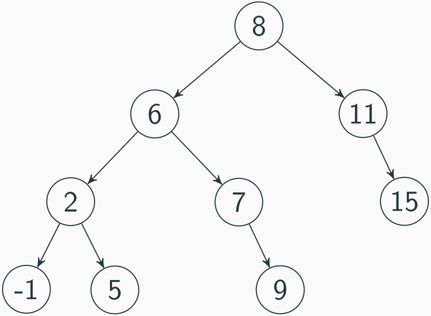
AVL tree invariants review

Question: is this also an AVL tree?



AVL tree invariants review

Question: ...and what about now?



Implementing an AVL dictionary

How do we implement an AVL dictionary?

- **get:** Same as BST!

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Implementing an AVL dictionary

How do we implement an AVL dictionary?

- **get:** Same as BST!
- **containsKey:** Same as BST!
- **put:** ???
- **remove:** ???

A basic example

Suppose we insert 1, 2, and 3. What happens?

```
insert(1)
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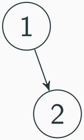

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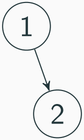
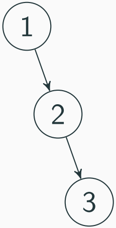


`insert(3)`



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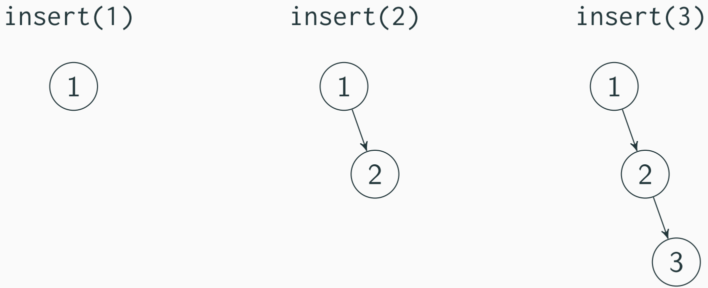

What do we do now? Hint: there's only one possible solution.

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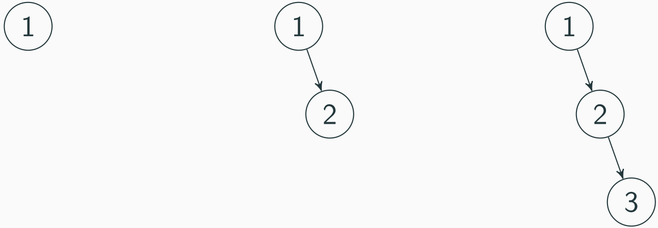

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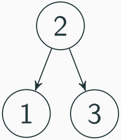
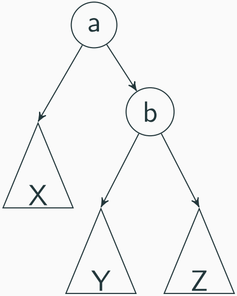
Rotate.



AVL rotation

An algorithm for "insert"/"put", in pictures:

Original tree
(Balanced)



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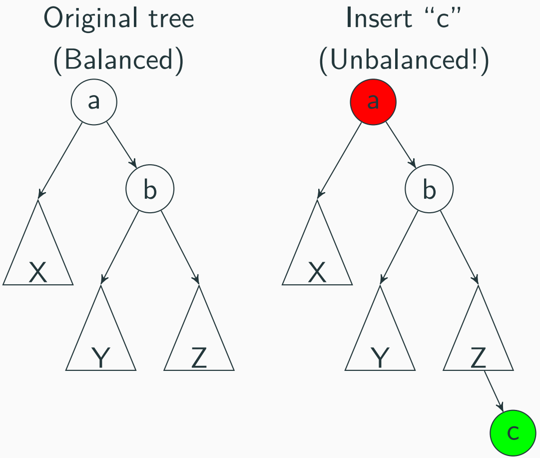

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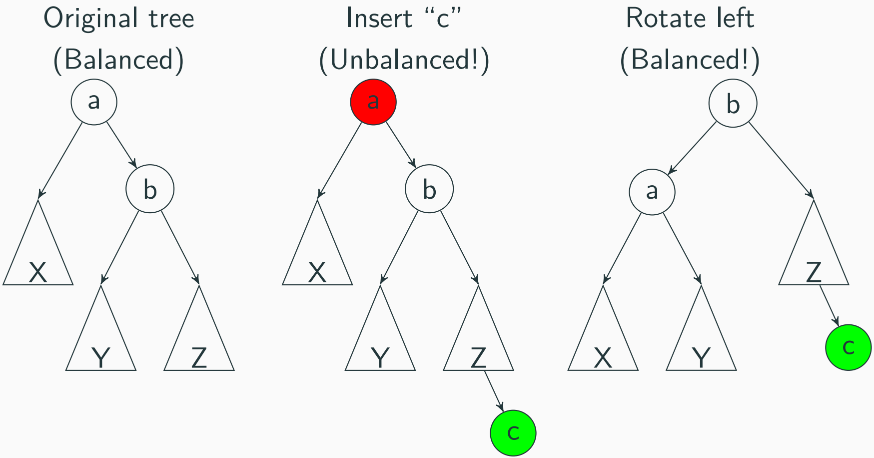
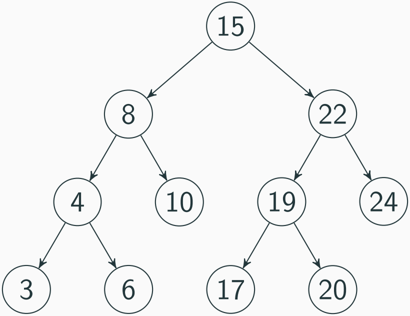


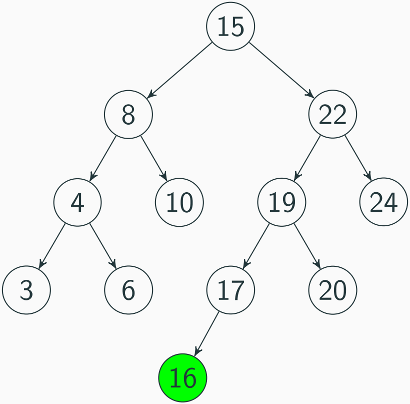
Practice

Practice: insert 16, and fix the tree:



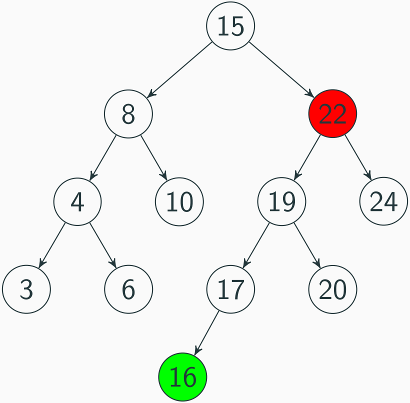
Practice

Step 1: insert 16



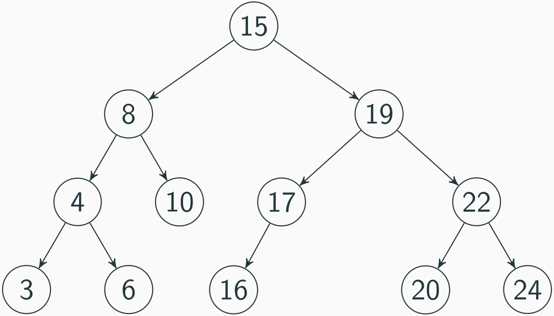
Practice

Step 2: Start from the inserted node and move back up to the root. Find the first unbalanced subtree.



Practice

Step 3: Rotate left or right to fix. (Here, we rotate right).



A second case...

Now, try this. Insert 1, 3, then 2. What's the issue?

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insert 1 and 3
```

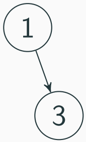

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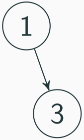
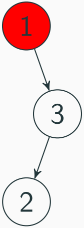


`insert 2`



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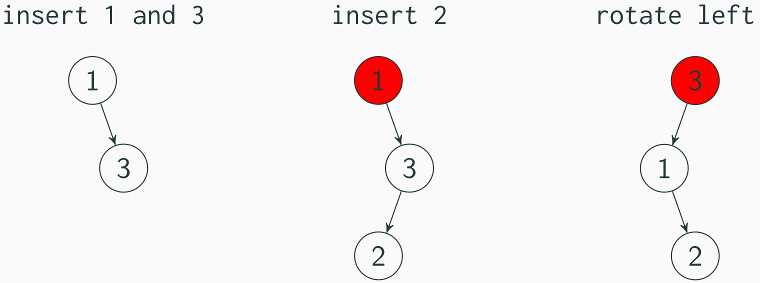

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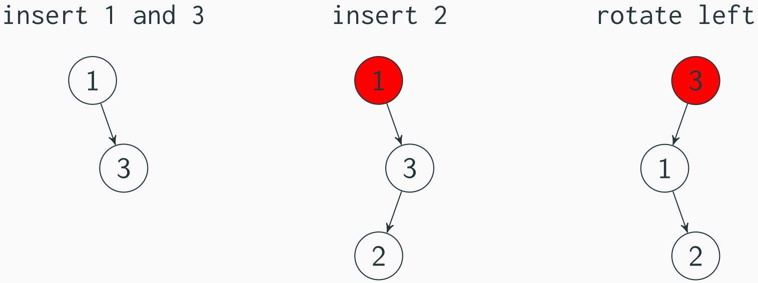
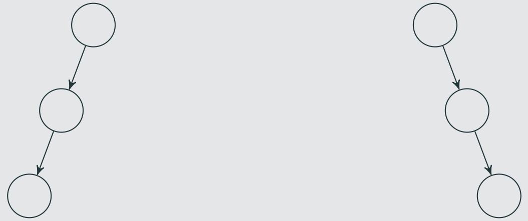


Tree is still unbalanced!

The two AVL cases

The “line” case



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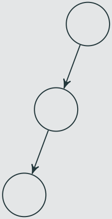
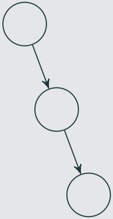
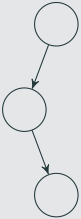
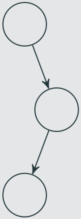




The “kink” case





Handling the "kink" case

Insight: Handling the kink case is hard. Can we somehow convert the kink case into the line case?

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Solution: Yes, use two rotations!

Let's try again

A second attempt...

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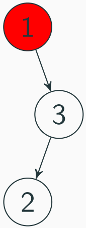
insert 1, 3, 2
(unbalanced!)
```



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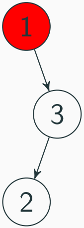
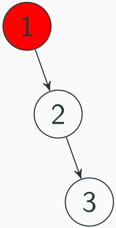


`double-rotate:`
`convert to line`



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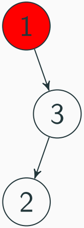

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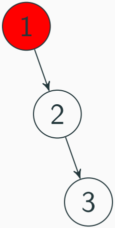

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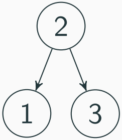
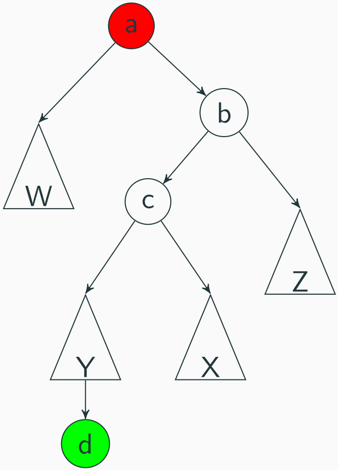
`fix tree`



The kink case: rotation 1

Initial tree
(Unbalanced)



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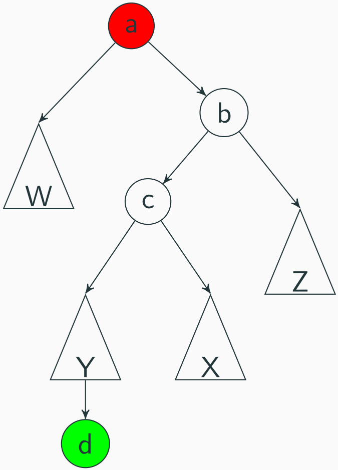
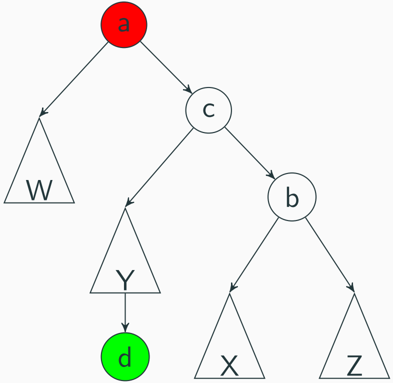
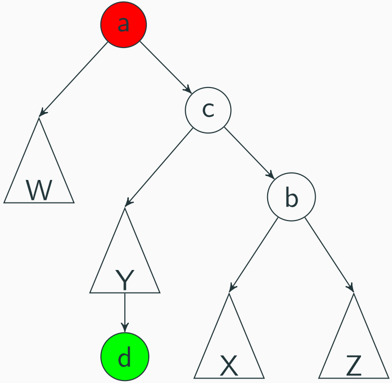


Fix the inner "b" subtree:



The kink case: rotation 2

After fixing the “b” subtree



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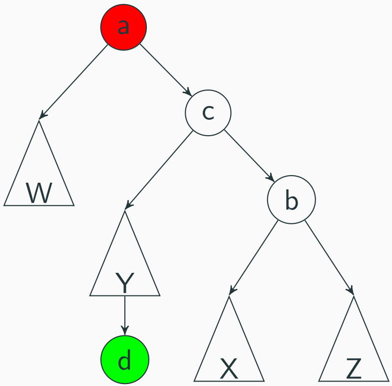
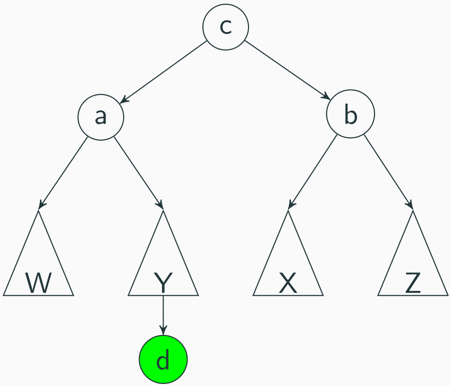


Fix the outer “a” subtree:



Practice

Try inserting a, b, e, c, d into an AVL tree.

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```
insert a
```

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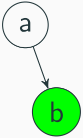
`insert a`

`insert b`



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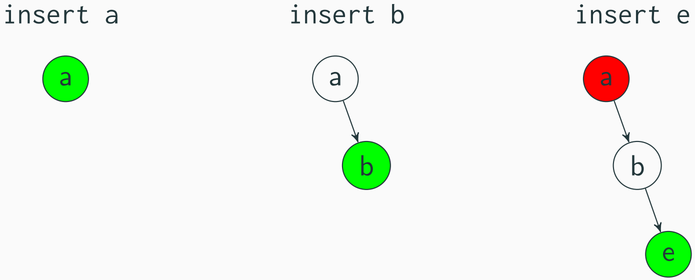

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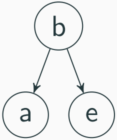
`rotate left on a`



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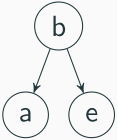
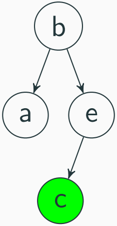


`insert c`



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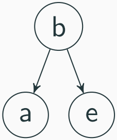

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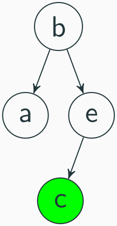
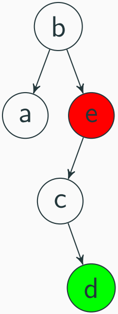
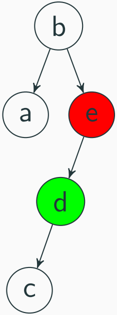


`insert d`



Practice

```
double rotation on e,
       part 1
```



Practice

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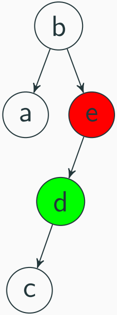
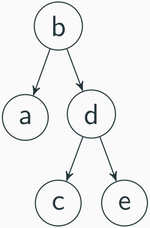


double rotation on e,
part 2



In summary...

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Implementing AVL operations

- **get:** Same as BST!
- **containsKey:** Same as BST!
- **put:** Do BST insert, move up tree, perform single or double rotations to balance tree
- **remove:** Either lazy-delete or use similar method to insert

A note on implementation

We sometimes need to rotate left, rotate right, double-rotate left, or double-rotate right.

Do we need to implement 4 methods?

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Do we need to implement 4 methods?

No: can reduce redundancy by having an *array* of children instead of using left or right fields. This lets us refer to children by index so we only have to write two methods: rotate, and double-rotate.

(E.g. we can have “rotate” accept two ints: the index to the “bigger” subtree, and the index to the “smaller” subtree)

Analyzing Arraylist add

And now, for a completely unrelated topic...

Analyzing ArrayList add

Exercise: model the worst-case runtime of ArrayList's add method in terms of n, the number of items inside the list:

```
public void add(T item) {
    if (array is full) {
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Answer: $T(n) = \begin{cases} c & \text{when the array is not full} \\ n + c & \text{when the array is full} \end{cases}$

So, in the **WORST** possible case, what's the runtime? .

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So, in the **WORST** possible case, what's the runtime? $\Theta(n)$.

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Core idea: cost of resizing is *amortized* over the subsequent calls

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Core idea: cost of resizing is *amortized* over the subsequent calls

Metaphors:

- When you pay rent, that large cost is *amortized* over the following month
- When you buy an expensive machine, that large cost is *amortized* and pays itself back over the next several years

Analyzing ArrayList's add

Our recurrence: $T(n) = \begin{cases} c & \text{when the array is not full} \\ n + c & \text{when the array is full} \end{cases}$

Scenario:

Let's suppose the array initially has size k. Let's also suppose the array initially is at capacity.

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- How much work do we need to do to resize once then fill back up to capacity?
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- How much work do we need to do to resize once then fill back up to capacity?

  $1 \cdot (k + c) + (k - 1) \cdot c = k + ck$.
  Note: since array was full, $n = k$ in first resize

- What is the *average* amount of work done?

  $$\frac{k + ck}{k} = 1 + c$$

Analyzing ArrayList's add variations

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  $1 \cdot (k + c) + 99 \cdot c = k + 100c$

- What is the *average* amount of work done?

  $\frac{k + 100c}{100} = \frac{k}{100} + c$

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- Assuming we're full, how much work do we do in total to resize once then fill back up to capacity?

  $1 \cdot (k + c) + 99 \cdot c = k + 100c$

- What is the *average* amount of work done?

  $$\frac{k + 100c}{100} = \frac{k}{100} + c$$

  What is k? k is the value of n each time we resize. If we plot this, we'll get a step-wise function that grows linearly!

Analyzing ArrayList's add variations

Now, what if instead of resizing by doubling, what if we increased the capacity by 100 each time?

- Assuming we're full, how much work do we do in total to resize once then fill back up to capacity?

  $1 \cdot (k + c) + 99 \cdot c = k + 100c$

- What is the *average* amount of work done?

  $$\frac{k + 100c}{100} = \frac{k}{100} + c$$

  What is k? k is the value of n each time we resize. If we plot this, we'll get a step-wise function that grows linearly!

  So, add would be in $\Theta(n)$.

Analyzing ArrayList's add variations

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Analyzing ArrayList's add variations

Now, what if instead of resizing by doubling, we triple?

- Assuming we're full, how much work do we do in total to resize once then fill back up to capacity?

  $1 \cdot (k + c) + (2k - 1) \cdot c = k + 2kc$

- What is the *average* amount of work done?

  $\frac{k + 2kc}{2k} = \frac{1}{2} + c$

Analyzing ArrayList's add variations

Now, what if instead of resizing by doubling, we triple?

- Assuming we're full, how much work do we do in total to resize once then fill back up to capacity?

  $1 \cdot (k + c) + (2k - 1) \cdot c = k + 2kc$

- What is the *average* amount of work done?

  $\frac{k + 2kc}{2k} = \frac{1}{2} + c$

  So, add would be in $\Theta(1)$.

Amortized analysis

This is called *amortized analysis*. The technique we discussed:

- **Aggregate analysis:**
  Show a series of n operations has an upper-bound of $T(n)$. The average cost is then $\frac{T(n)}{n}$.

Amortized analysis

This is called *amortized analysis*. The technique we discussed:

- **Aggregate analysis:**

  Show a series of n operations has an upper-bound of $T(n)$. The average cost is then $\frac{T(n)}{n}$.

Other common techniques (not covered in this class):

- **The accounting method:**

  Assign each operation an "amortized cost", which may differ from actual cost. If amortized cost $>$ actual cost, incur credit. Credit is later used to pay for operations where amortized cost $<$ actual cost.
- **The potential method:**

  The data structure has "potential energy", different operations alter that energy.

  Hooray, physics metaphors?