

CSE 373: AVL trees

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Friday, Jan 19, 2018

Warmup:

- ▶ What is an *invariant*?
- ▶ What are the AVL tree invariants, exactly?

Discuss with your neighbor.

AVL Trees: Invariants

Core idea: add extra **invariant** to BSTs that enforce balance.

AVL Tree Invariants

An AVL tree has the following invariants:

▶ **The “structure” invariant:**

All nodes have 0, 1, or 2 children.

▶ **The “BST” invariant:**

For all nodes, all keys in the *left* subtree are smaller;
all keys in the *right* subtree are larger

▶ **The “balance” invariant:**

For all nodes, $\text{abs}(\text{height}(\text{left})) - \text{height}(\text{right}) \leq 1$.

Interlude: Exploring the balance invariant

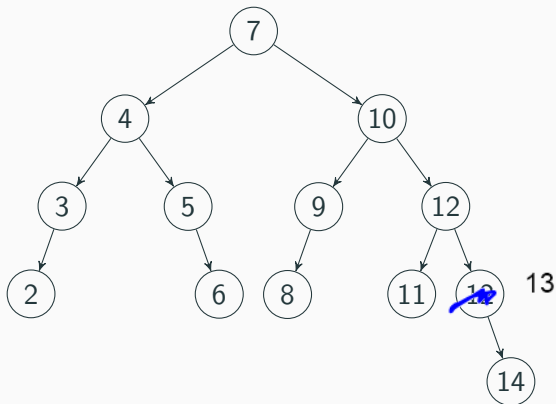
Question: why $\text{abs}(\text{height}(\text{left})) - \text{height}(\text{right}) \leq 1$?

Why not $\text{height}(\text{left}) = \text{height}(\text{right})$?

What happens if we insert two elements. What happens?

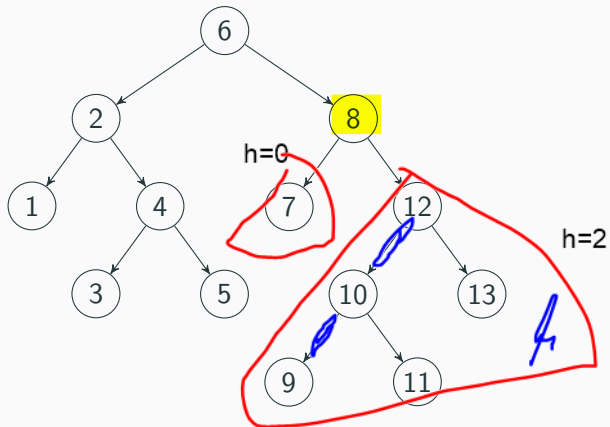
AVL tree invariants review

Question: is this a valid AVL tree?



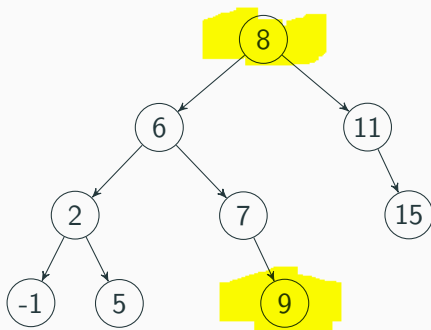
AVL tree invariants review

Question: is this also an AVL tree?



AVL tree invariants review

Question: ...and what about now?



Implementing an AVL dictionary

How do we implement an AVL dictionary?

- ▶ **get:** Same as BST!

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- ▶ **containsKey:** Same as BST!

Implementing an AVL dictionary

How do we implement an AVL dictionary?

- ▶ **get:** Same as BST!
- ▶ **containsKey:** Same as BST!
- ▶ **put:** ???
- ▶ **remove:** ???

A basic example

Suppose we insert 1, 2, and 3. What happens?

`insert(1)`

①

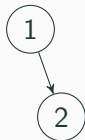
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insert(2)



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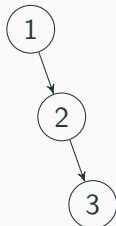
insert(1)



insert(2)



insert(3)



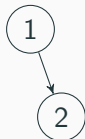
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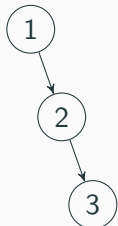
insert(1)



insert(2)



insert(3)



What do we do now? Hint: there's only one possible solution.

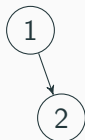
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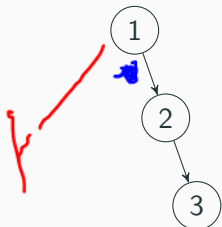
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insert(2)

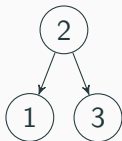


insert(3)



What do we do now? Hint: there's only one possible solution.

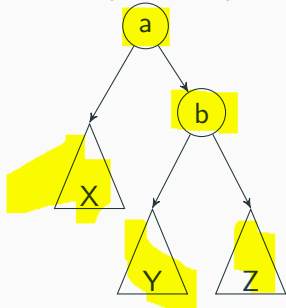
Rotate.



AVL rotation

An algorithm for “insert”/“put”, in pictures:

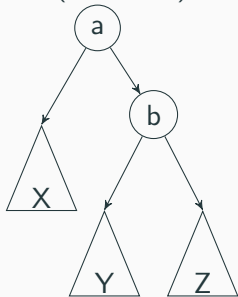
Original tree
(Balanced)



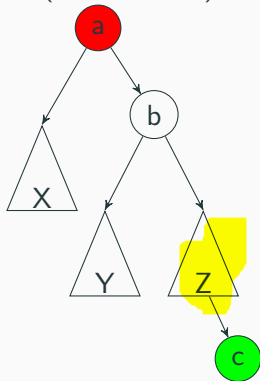
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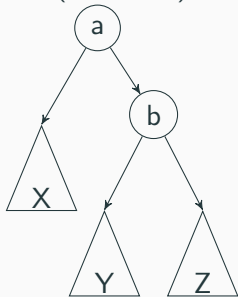
Insert “c”
(Unbalanced!)



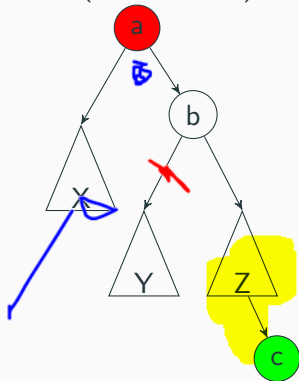
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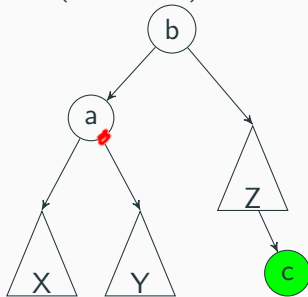
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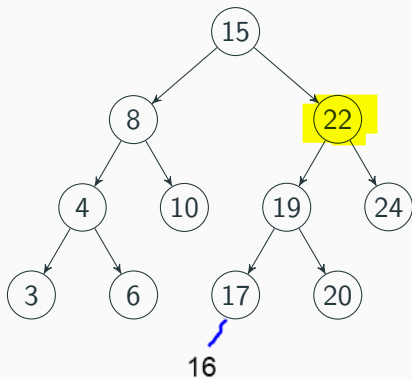


Rotate left
(Balanced!)



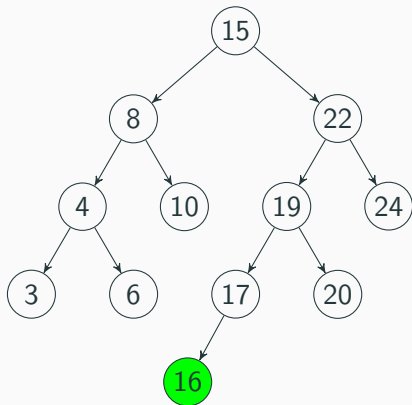
Practice

Practice: insert 16, and fix the tree:



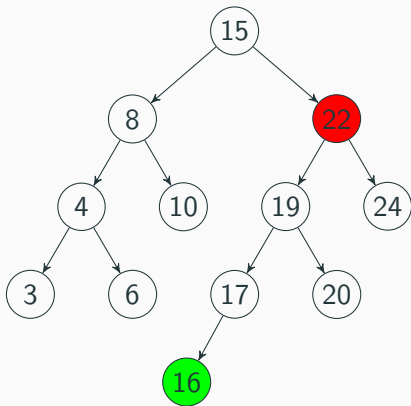
Practice

Step 1: insert 16



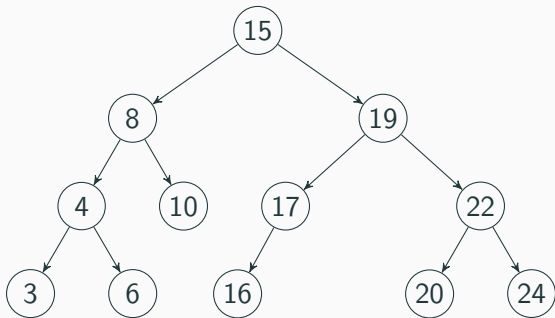
Practice

Step 2: Start from the inserted node and move back up to the root. Find the first unbalanced subtree.



Practice

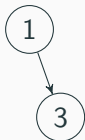
Step 3: Rotate left or right to fix. (Here, we rotate right).



A second case...

Now, try this. Insert 1, 3, then 2. What's the issue?

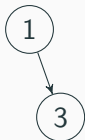
insert 1 and 3



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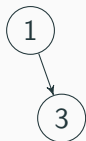
insert 2



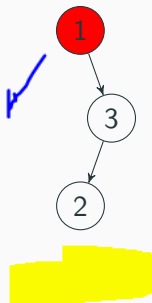
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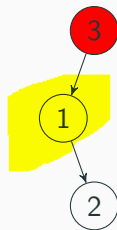
insert 1 and 3



insert 2



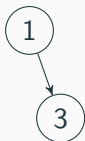
rotate left



A second case...

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insert 1 and 3



insert 2



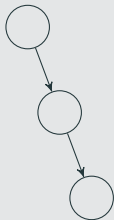
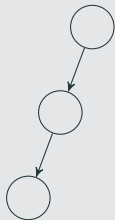
rotate left



Tree is still unbalanced!

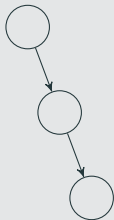
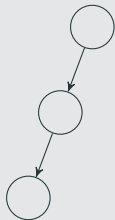
The two AVL cases

The "line" case



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The "line" case



The "kink" case



Handling the “kink” case

Insight: Handling the kink case is hard. Can we somehow convert the kink case into the line case?

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Solution: Yes, use two rotations!

Let's try again

A second attempt...

insert 1, 3, 2
(unbalanced!)



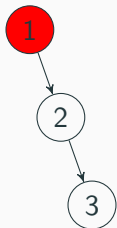
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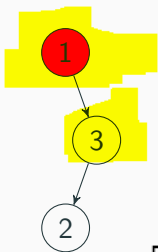
double-rotate:
convert to line



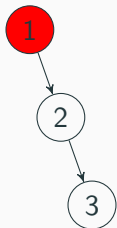
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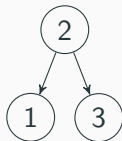
insert 1, 3, 2
(unbalanced!)



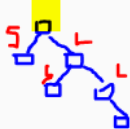
double-rotate:
convert to line



double-rotate:
fix tree



Ex: line

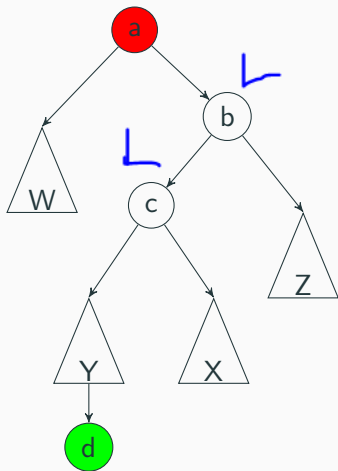


Ex: kink



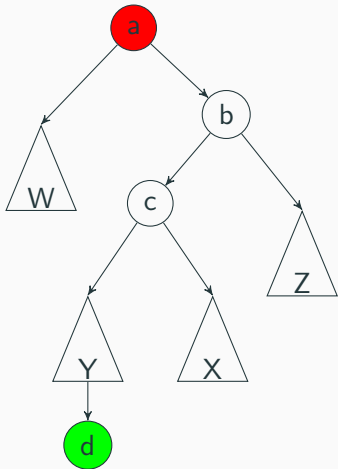
The kink case: rotation 1

Initial tree
(Unbalanced)

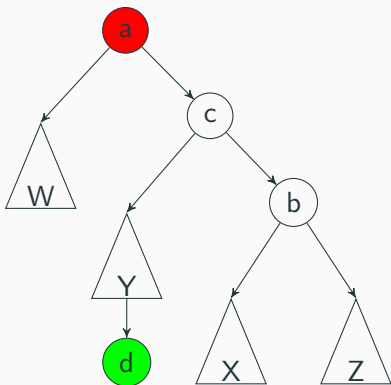


The kink case: rotation 1

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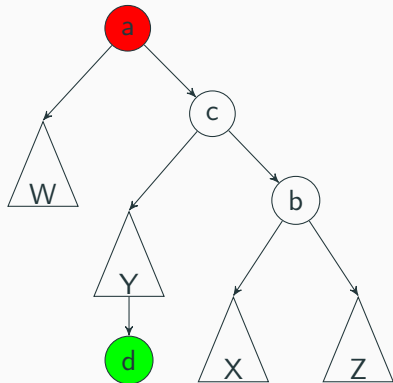


Fix the inner "b" subtree:



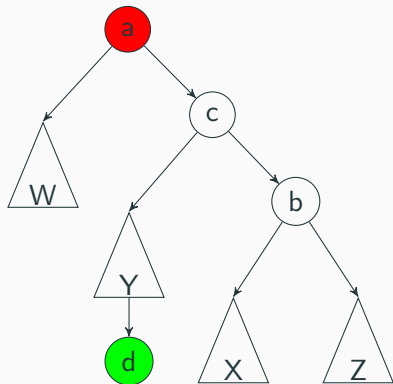
The kink case: rotation 2

After fixing the "b" subtree

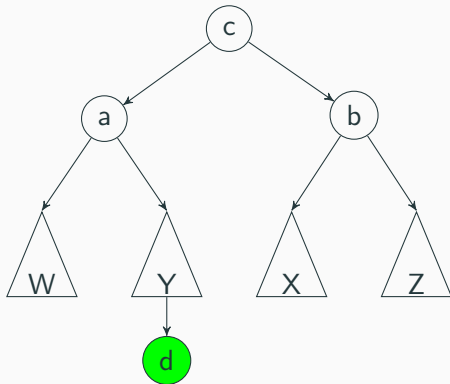


The kink case: rotation 2

After fixing the "b" subtree



Fix the outer "a" subtree:



Try inserting a, b, e, c, d into an AVL tree.

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insert a



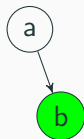
Practice

Try inserting a, b, e, c, d into an AVL tree.

insert a



insert b



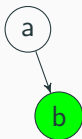
Practice

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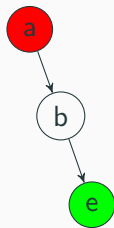
insert a



insert b

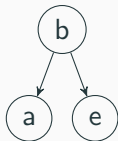


insert e



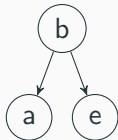
Practice

rotate left on a

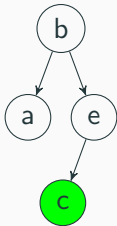


Practice

rotate left on a

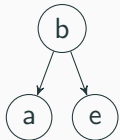


insert c

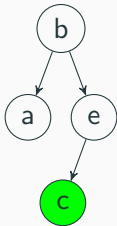


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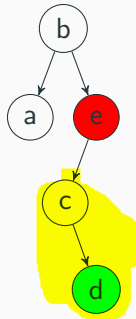
rotate left on a



insert c

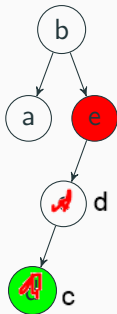


insert d



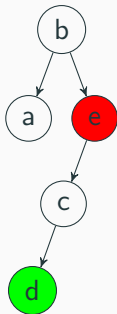
Practice

double rotation on e,
part 1

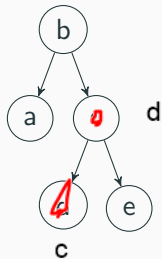


Practice

double rotation on e,
part 1



double rotation on e,
part 2



In summary...

Implementing AVL operations

- ▶ **get:** Same as BST!
- ▶ **containsKey:** Same as BST!
- ▶ **put:** Do BST insert, move up tree, perform single or double rotations to balance tree
- ▶ **remove:** Either lazy-delete or use similar method to insert

A note on implementation

We sometimes need to rotate left, rotate right, double-rotate left, or double-rotate right.

Do we need to implement 4 methods?

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We sometimes need to rotate left, rotate right, double-rotate left, or double-rotate right.

Do we need to implement 4 methods?

No: can reduce redundancy by having an *array* of children instead of using left or right fields. This lets us refer to children by index so we only have to write two methods: rotate, and double-rotate.

(E.g. we can have “rotate” accept two ints: the index to the “bigger” subtree, and the index to the “smaller” subtree)

And now, for a completely unrelated topic...

Analyzing ArrayList add

Exercise: model the worst-case runtime of ArrayList's add method in terms of n , the number of items inside the list:

```
public void add(T item) {  
    if (array is full) {  
        resize and copy  
    }  
    this.array[this.size] = item;  
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$$\text{Answer: } T(n) = \begin{cases} c & \text{when the array is not full} \\ n + c & \text{when the array is full} \end{cases}$$

So, in the **WORST** possible case, what's the runtime? .

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So, in the **WORST** possible case, what's the runtime? $\Theta(n)$.

Analyzing ArrayList add

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Core idea: cost of resizing is *amortized* over the subsequent calls

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Metaphors:

- ▶ When you pay rent, that large cost is *amortized* over the following month
- ▶ When you buy an expensive machine, that large cost is *amortized* and pays itself back over the next several years

Analyzing ArrayList's add

Our recurrence: $T(n) = \begin{cases} c & \text{when the array is not full} \\ n + c & \text{when the array is full} \end{cases}$

Scenario:

Let's suppose the array initially has size k . Let's also suppose the array initially is at capacity.

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- ▶ How much work do we need to do to resize once then fill back up to capacity?

- ▶ What is the *average* amount of work done?

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Scenario:

Let's suppose the array initially has size k . Let's also suppose the array initially is at capacity.

- ▶ How much work do we need to do to resize once then fill back up to capacity?

$$1 \cdot (k + c) + (k - 1) \cdot c = k + ck.$$

Note: since array was full, $n = k$ in first resize

- ▶ What is the *average* amount of work done?

$$\frac{k + ck}{k} = 1 + c$$

Analyzing ArrayList's add variations

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$$\frac{k + 100c}{100} = \frac{k}{100} + c$$

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So, add would be in $\Theta(n)$.

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Now, what if instead of resizing by doubling, we triple?

- ▶ Assuming we're full, how much work do we do in total to resize once then fill back up to capacity?

$$1 \cdot (k + c) + (2k - 1) \cdot c = k + 2kc$$

- ▶ What is the *average* amount of work done?

$$\frac{k + 2kc}{2k} = \frac{1}{2} + c$$

Analyzing ArrayList's add variations

Now, what if instead of resizing by doubling, we triple?

- ▶ Assuming we're full, how much work do we do in total to resize once then fill back up to capacity?

$$1 \cdot (k + c) + (2k - 1) \cdot c = k + 2kc$$

- ▶ What is the *average* amount of work done?

$$\frac{k + 2kc}{2k} = \frac{1}{2} + c$$

So, add would be in $\Theta(1)$.

Amortized analysis

This is called *amortized analysis*. The technique we discussed:

► **Aggregate analysis:**

Show a series of n operations has an upper-bound of $T(n)$. The average cost is then $\frac{T(n)}{n}$.

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Other common techniques (not covered in this class):

- ▶ **The accounting method:**

Assign each operation an “amortized cost”, which may differ from actual cost. If amortized cost $>$ actual cost, incur credit. Credit is later used to pay for operations where amortized cost $<$ actual cost.

- ▶ **The potential method:**

The data structure has “potential energy”, different operations alter that energy.

Hooray, physics metaphors?