**Warmup**: construct a *mathematical function* modeling the worst-case runtime of this method. Your model should be written in terms of $q$, the provided input integer.

Assume each println takes some constant $c$ time to run.

```java
class Mystery {
    public void mystery(int q) {
        for (int i = 0; i < q; i++) {
            for (int j = 0; j < q * q; j++) {
                System.out.println("Hello");
            }
        }
        for (int j = 0; j < 10; j++) {
            System.out.println("World");
        }
    }
}
```

$$T(q) = q (cq^2 + 10c)$$
**Warmup:** construct a mathematical function modeling the worst-case runtime of this method. Your model should be written in terms of $q$, the provided input integer.

Assume each println takes some constant $c$ time to run.

```java
public void mystery(int q) {
    for (int i = 0; i < q; i++) {
        for (int j = 0; j < q * q; j++) {
            System.out.println("Hello");
        }
        for (int j = 0; j < 10; j++) {
            System.out.println("World");
        }
    }
}
```

Answer: $T(q) = q(cq^2 + 10c) = cq^3 + 10cq$
Last time

Two step process:

1. **Model** what we care about as a mathematical function
2. **Analyze** that function using asymptotic analysis
Last time

Two step process:

1. **Model** what we care about as a mathematical function
2. **Analyze** that function using asymptotic analysis
   Specifically: have a way to **compare** two functions
Two step process:

1. **Model** what we care about as a mathematical function
2. **Analyze** that function using asymptotic analysis
   - Specifically: have a way to **compare** two functions
   - Even more specifically: define a “less then or equal to” operator for functions
Analysis: comparing functions

Question: Should we treat these two functions the same?
Analysis: comparing functions

What about now?

$T(n)$

Intuition: our quadratic function is dominating the linear ones.

Intuition: our linear functions (eventually) look the same.
Analysis: comparing functions

What about now?

Intuition: our quadratic function is **dominating** the linear ones
Analysis: comparing functions

What about now?

$T(n)$

Intuition: our quadratic function is **dominating** the linear ones

Intuition: our linear functions (eventually) look the same
Let’s zoom in...

\[ T(n) \]

\[ n \]
Analysis: comparing functions

Let’s zoom in...

Intuition: quadratic function \textit{eventually} dominates the linear ones
Our goal:

- We want a way to say $n^2$ eventually dominates $n$
Our goal:

- We want a way to say \( n^2 \) eventually dominates \( n \)
- We want a way to treat \( n \) and \( 4n \) the same way

Intuition:

- Model made simplifying assumptions about constant factors
- Can usually improve constant-factor differences by being clever
Analysis: comparing functions

Our goal:

- We want a way to say $n^2$ eventually dominates $n$
- We want a way to treat $n$ and $4n$ the same way

Intuition:

- Model made simplifying assumptions about constant factors
- Can usually improve constant-factor differences by being clever

- We want a way to do this rigorously!
Function comparison: exercise

True or false?

▷ Is $n$ “less then or equal to” $5n + 3$?
  - True

▷ Is $5n + 3$ “less then or equal to” $n$?
  - True

▷ Is $5n + 3$ “less then or equal to” $1$?
  - False

▷ Is $5n + 3$ “less then or equal to” $n^2$?
  - True

▷ Is $n^2 + 3n + 2$ “less then or equal to” $n^3$?
  - True

▷ Is $n^3$ “less then or equal to” $n^2 + 3n + 2$?
  - False
Function comparison: exercise

True or false?

- Is $n$ “less then or equal to” $5n + 3$? True
- Is $5n + 3$ “less then or equal to” $n$? True
- Is $5n + 3$ “less then or equal to” $1$? False
- Is $5n + 3$ “less then or equal to” $n^2$? True
- Is $n^2 + 3n + 2$ “less then or equal to” $n^3$? True
- Is $n^3$ “less then or equal to” $n^2 + 3n + 2$? False
Our goal:

- We want a way to say $n^2$ eventually dominates $n$
- We want a way to treat $n$ and $4n$ the same way
Let’s formalize this...

<table>
<thead>
<tr>
<th>Idea 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>A function $f(n)$ is “less then or equal to” $g(n)$ when $f(n) \leq g(n)$ is true for all values of $n \geq 0$.</td>
</tr>
</tbody>
</table>

Does this work?
Let’s formalize this...

Idea 1

A function $f(n)$ is “less then or equal to” $g(n)$ when $f(n) \leq g(n)$ is true for all values of $n \geq 0$.

Does this work? Remember this?
Let’s formalize this...

**Idea 1**

A function $f(n)$ is “less then or equal to” $g(n)$ when $f(n) \leq g(n)$ is true for all values of $n \geq 0$.

Does this work? Remember this?

**Problem:** incorrectly handles the quadratic function!
Let’s formalize this...

**Idea 2**

A function $f(n)$ is “less then or equal to” $g(n)$ when $f(n) \leq g(n)$ is true for all values of $n \geq n_0$.

...where $n_0 > 0$ is some constant value.

Does it work now?
Let’s formalize this...

Idea 2

A function $f(n)$ is “less then or equal to” $g(n)$ when $f(n) \leq g(n)$ is true for all values of $n \geq n_0$.

...where $n_0 > 0$ is some constant value.

Does it work now?

We previously said we want to treat $n$ and $4n$ as being the “same”. Do we?
Let’s formalize this...

Idea 2

A function $f(n)$ is “less then or equal to” $g(n)$ when $f(n) \leq g(n)$ is true for all values of $n \geq n_0$.

...where $n_0 > 0$ is some constant value.

Does it work now?

We previously said we want to treat $n$ and $4n$ as being the “same”.
Do we?

**Problem:** No, we don’t!
Let’s formalize this...

**Idea 3**

A function $f(n)$ is “less then or equal to” $g(n)$ when $f(n) \leq c \cdot g(n)$ is true for all values of $n \geq n_0$.

...where $n_0 > 0$ is some constant value.

...where $c > 0$ is some constant value.

Does it work now?
Let’s formalize this...

<table>
<thead>
<tr>
<th>Idea 3</th>
</tr>
</thead>
</table>
| A function \( f(n) \) is “less then or equal to” \( g(n) \) when \( f(n) \leq c \cdot g(n) \) is true for all values of \( n \geq n_0 \).

...where \( n_0 > 0 \) is some constant value.

...where \( c > 0 \) is some constant value.

Does it work now?

Yes!
**Definition: Dominated by**

A function $f(n)$ is **dominated by** $g(n)$ when...

<table>
<thead>
<tr>
<th>Definition: Dominated by</th>
</tr>
</thead>
<tbody>
<tr>
<td>A function $f(n)$ is dominated by $g(n)$ when...</td>
</tr>
</tbody>
</table>
Definition: Dominated by

A function $f(n)$ is **dominated by** $g(n)$ when...

- There exists two constants $c > 0$ and $n_0 > 0$...

The formal definition (not necessary to know this):

Formal definition: Dominated by

A function $f(n)$ is dominated by $g(n)$ when

$\exists (c > 0, n_0 > 0)$. $\forall (n \geq n_0)$. $(f(n) \leq cg(n))$ is true.
Definition: Dominated by

A function $f(n)$ is **dominated by** $g(n)$ when...

- There exists two constants $c > 0$ and $n_0 > 0$...
- Such that for all values of $n \geq n_0$...
A function $f(n)$ is **dominated by** $g(n)$ when...

- There exists two constants $c > 0$ and $n_0 > 0$...
- Such that for all values of $n \geq n_0$...
- $f(n) \leq c \cdot g(n)$ is true
A function \( f(n) \) is dominated by \( g(n) \) when...

- There exists two constants \( c > 0 \) and \( n_0 > 0 \)...
- Such that for all values of \( n \geq n_0 \)...
- \( f(n) \leq c \cdot g(n) \) is true

The formal definition (not necessary to know this):

A function \( f(n) \) is dominated by \( g(n) \) when

\[
\exists (c > 0, n_0 > 0). \forall (n \geq n_0). (f(n) \leq cg(n))
\]

...is true.
Exercise

Demonstrate that \(5n^2 + 3n + 6\) is dominated by \(n^3\) by finding a \(c\) and \(n_0\) that satisfy the above definition.

\[
c = 14
\]

\[
n_0 = 1
\]

\[
5n^2 + 3n + 6 \leq 5n^2 + 3n^2 + 6n^2 = 14n^2
\]

\[
5n^2 + 3n + 6 \leq 14n^2 \text{ for } n \geq 1
\]

\[
14n^2 \leq c \cdot n^3 \text{ for } c = 14 \text{ and } n \geq 1
\]

\[
n = 0.5
\]

\[
n^2 = 0.5 \times 0.5
\]

\[
n^3 = 0.5 \times 0.5 \times 0.5
\]
Exercise

Demonstrate that $5n^2 + 3n + 6$ is dominated by $n^3$ by finding a $c$ and $n_0$ that satisfy the above definition.

**Idea:** pick $c = 10000$ and $n_0 = 10000$. (It probably works ¯\_(ツ)_/¯)
Demonstrate that $5n^2 + 3n + 6$ is dominated by $n^3$ by finding a $c$ and $n_0$ that satisfy the above definition.

Idea: pick $c = 10000$ and $n_0 = 10000$. (It probably works \_\_(‘UV’)_/\_)

Better idea: show that $5n^2 + 3n + 6$ is dominated by an easier function to analyze. E.g. note that:

$$5n^2 + 3n + 6 \leq 5n^2 + 3n^2 + 6n^2 \quad \text{for all } n \geq 1$$

$$= 14n^2$$

$$\leq 14n^3$$
Demonstrate that $5n^2 + 3n + 6$ is dominated by $n^3$ by finding a $c$ and $n_0$ that satisfy the above definition.

**Idea:** pick $c = 10000$ and $n_0 = 10000$. (It probably works \_\_(\^\^)_/\_)

**Better idea:** show that $5n^2 + 3n + 6$ is dominated by an easier function to analyze. E.g. note that:

\[
5n^2 + 3n + 6 \leq 5n^2 + 3n^2 + 6n^2 = 14n^2 \leq 14n^3
\]

So, what value of $c$ makes $14n^3 \leq cn^3$ true (when $n \geq 1$)?

One possible choice: $n_0 = 1$ and $c = 14$. 
Demonstrate that $5n^2 + 3n + 6$ is dominated by $n^3$ by finding a $c$ and $n_0$ that satisfy the above definition.

**Idea:** pick $c = 10000$ and $n_0 = 10000$. (It probably works 🙄)(╯°□°)╯︵ ┻━┻)

**Better idea:** show that $5n^2 + 3n + 6$ is dominated by an easier function to analyze. E.g. note that:

\[
5n^2 + 3n + 6 \leq 5n^2 + 3n^2 + 6n^2 = 14n^2 \leq 14n^3
\]

for all $n \geq 1$

So, what value of $c$ makes $14n^3 \leq cn^3$ true (when $n \geq 1$)?

One possible choice: $n_0 = 1$ and $c = 14$.

So, since we know $5n^2 + 3n + 6 \leq 14n^3$ for $n \geq n_0$ and also know $14n^3 \leq cn^3$, we conclude $5n^2 + 3n + 6 \leq cn^3$. 
Exercise

Demonstrate that $2n^3 - 3 + 9n^2 + \sqrt{n}$ is dominated by $n^3$ (again by finding a $c$ and $n_0$).
Exercise

Demonstrate that $2n^3 - 3 + 9n^2 + \sqrt{n}$ is dominated by $n^3$ (again by finding a $c$ and $n_0$).

Do the same thing. Note that:

$$2n^3 - 3 + 9n^2 + \sqrt{n} \leq 2n^3 + 9n^2 + n \quad \text{for all } n \geq 1$$

$$\leq 2n^3 + 9n^3 + n^3$$

$$= 12n^3$$

So, one possible choice of $n_0$ and $c$ is $n_0 = 1$ and $c = 12$. 
Observation:

- $n$, $5n + 3$, $100n$, etc... all dominate each other
- These three functions are the “same”
Families of functions

Observation:

- $n$, $5n + 3$, $100n$, etc... all dominate each other
- These three functions are the “same”

Idea: can we give a name to this “family” of functions?
Families of functions

Observation:

- $n$, $5n + 3$, $100n$, etc... all dominate each other
- These three functions are the “same”

Idea: can we give a name to this “family” of functions?

**Definition: Big-$\mathcal{O}$**

$\mathcal{O}(f(n))$ is the “family” or “set” of all functions that are dominated by $f(n)$.
Families of functions

Observation:

▶ $n$, $5n + 3$, $100n$, etc... all dominate each other
▶ These three functions are the “same”

Idea: can we give a name to this “family” of functions?

**Definition: Big-$O$**

$O(f(n))$ is the “family” or “set” of all functions that are dominated by $f(n)$

Question: are $O(n)$, $O(5n + 3)$, and $O(100n)$ all the same thing?
Families of functions

Observation:

▶ $n, 5n + 3, 100n$, etc... all dominate each other
▶ These three functions are the “same”

Idea: can we give a name to this “family” of functions?

**Definition: Big-$O$**

$O(f(n))$ is the “family” or “set” of all functions that are dominated by $f(n)$

Question: are $O(n)$, $O(5n + 3)$, and $O(100n)$ all the same thing?

Yes! By convention, we pick the “simplest” way of writing this and refer to this “family” as $O(n)$. 
A question: Do the following two sentences mean the same thing?

- $f(n)$ is dominated by $g(n)$
- $f(n)$ is contained inside $O(g(n))$
A question: Do the following two sentences mean the same thing?

- $f(n)$ is dominated by $g(n)$
- $f(n)$ is contained inside $\mathcal{O}(g(n))$

Yes!

We can write this more concisely as $f(n) \in \mathcal{O}(g(n))$. 

\[\]
A question: Do the following two sentences mean the same thing?

- $f(n)$ is dominated by $g(n)$
- $f(n)$ is contained inside $O(g(n))$

Yes!

We can write this more concisely as $f(n) \in O(g(n))$.

An aside: some people write this as $f(n) = O(g(n))$

This is **wrong** (but common, so we reluctantly accept this)
A few more questions

True or false:

- $5n + 3 \in O(n)$
- $n \in O(5n + 3)$
- $5n + 3 = O(n)$
- $O(5n + 3) = O(n)$
- $O(n^2) = O(n)$
- $n^2 \in O(1)$
- $n^2 \in O(n)$
- $n^2 \in O(n^2)$
- $n^2 \in O(n^3)$
- $n^2 \in O(n^{100})$
A few more questions

True or false:

- $5n + 3 \in O(n)$ \hspace{1cm} True
- $n \in O(5n + 3)$ \hspace{1cm} True
- $5n + 3 = O(n)$ \hspace{1cm} True (by convention)
- $O(5n + 3) = O(n)$ \hspace{1cm} True
- $O(n^2) = O(n)$ \hspace{1cm} False
- $n^2 \in O(1)$ \hspace{1cm} False
- $n^2 \in O(n)$ \hspace{1cm} False
- $n^2 \in O(n^2)$ \hspace{1cm} True
- $n^2 \in O(n^3)$ \hspace{1cm} True
- $n^2 \in O(n^{100})$ \hspace{1cm} True
Definitions: Dominates

\[ f(n) \in \mathcal{O}(g(n)) \] is like saying \( f(n) \) is less then or equal to \( g(n) \).

Is there a way to say “greater then or equal to”?
Definitions: Dominates

$f(n) \in \mathcal{O}(g(n))$ is like saying “$f(n)$ is less then or equal to $g(n)$”.

Is there a way to say “greater then or equal to”? Yes!

**Definition: Dominates**

We say $f(n)$ **dominates** $g(n)$ when:

- There exists two constants $c > 0$ and $n_0 > 0$...
- Such that for all values of $n \geq n_0$...
- $f(n) \geq c \cdot g(n)$ is true
Definitions: Dominates

\( f(n) \in \mathcal{O}(g(n)) \) is like saying “\( f(n) \) is less then or equal to \( g(n) \)”.

Is there a way to say “greater then or equal to”? Yes!

**Definition: Dominates**

We say \( f(n) \) **dominates** \( g(n) \) when:

- There exists two constants \( c > 0 \) and \( n_0 > 0 \)...
- Such that for all values of \( n \geq n_0 \)...
- \( f(n) \geq c \cdot g(n) \) is true

**Definition: Big-Ω**

\( \Omega(f(n)) \) is the family of all functions that **dominates** \( f(n) \).
A few more questions...

True or false?

- $4n^2 \in \Omega(1)$  
  - True

- $4n^2 \in \Omega(n)$  
  - True

- $4n^2 \in \Omega(n^2)$  
  - False

- $4n^2 \in \Omega(n^3)$  
  - True

- $4n^2 \in \Omega(n^4)$  
  - True

- $4n^2 \in O(1)$  
  - False

- $4n^2 \in O(n)$  
  - False

- $4n^2 \in O(n^2)$  
  - True

- $4n^2 \in O(n^3)$  
  - True

- $4n^2 \in O(n^4)$  
  - True
A few more questions...

<table>
<thead>
<tr>
<th>True or false?</th>
<th>4n² ∈ Ω(1)</th>
<th>True</th>
<th>4n² ∈ O(1)</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4n² ∈ Ω(n)</td>
<td>True</td>
<td>4n² ∈ O(n)</td>
<td>False</td>
</tr>
<tr>
<td></td>
<td>4n² ∈ Ω(n²)</td>
<td>True</td>
<td>4n² ∈ O(n²)</td>
<td>True</td>
</tr>
<tr>
<td></td>
<td>4n² ∈ Ω(n³)</td>
<td>False</td>
<td>4n² ∈ O(n³)</td>
<td>True</td>
</tr>
<tr>
<td></td>
<td>4n² ∈ Ω(n⁴)</td>
<td>False</td>
<td>4n² ∈ O(n⁴)</td>
<td>True</td>
</tr>
</tbody>
</table>
Definition: Big-$\Theta$

We say $f(n) \in \Theta(g(n))$ when both:

1. $f(n) \in \mathcal{O}(g(n))$ and...
2. $f(n) \in \Omega(g(n))$

...are true.
Definition: Big-$\Theta$

We say $f(n) \in \Theta (g(n))$ when both:

- $f(n) \in \mathcal{O} (g(n))$ and...
- $f(n) \in \Omega (g(n))$

...are true.

Note: in industry, it’s common for many people to ask for the big-$\mathcal{O}$ when they really want the big-$\Theta$!
Important things to know:

- Intuition behind the definitions of “dominated by” and big-$O$
Important things to know:

- Intuition behind the definitions of “dominated by” and big-Ο
- The precise definitions of:
  - “Dominated by” and big-Ο
  - “Dominates” and big-Ω
  - Big-Θ
Important things to know:

- Intuition behind the definitions of “dominated by” and big-$O$
- The precise definitions of:
  - “Dominated by” and big-$O$
  - “Dominates” and big-$\Omega$
  - Big-$\Theta$
- How to demonstrate that one function is dominated by another by finding $c$ and $n_0$ and applying the correct definition