CSE 373: Asymptotic Analysis

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Wednesday, Jan 10, 2018

Warmup

Warmup: construct a *mathematical function* modeling the worst-case runtime of this method. Your model should be written in terms of q, the provided input integer.

Assume each println takes some constant c time to run.

```
public void mystery(int q) {
    for (int i = 0; i < q; i++) {
        for (int j = 0; j < q * q; j++) {
            System.out.println("Hello");
        }

        for (int j = 0; j < 10; j++) {
            System.out.println("World");
        }
    }
}
```

c, cq^2, c, $10c$, $q(cq^2 + 10c)$

$T(q) = q(cq^2 + 10c)$

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```

Answer: $T(q) = q(cq^2 + 10c) = cq^3 + 10cq$

Last time

Two step process:

1. **Model** what we care about as a mathematical function
2. **Analyze** that function using asymptotic analysis

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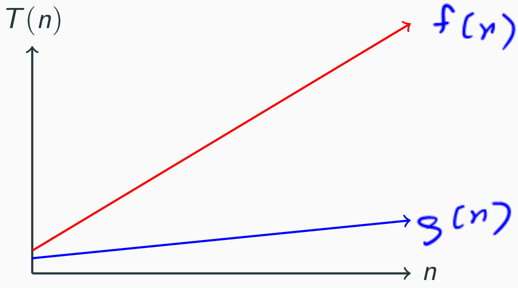
1. **Model** what we care about as a mathematical function
2. **Analyze** that function using asymptotic analysis
   Specifically: have a way to **compare** two functions
   Even more specifically: define a "less then or equal to" operator for functions

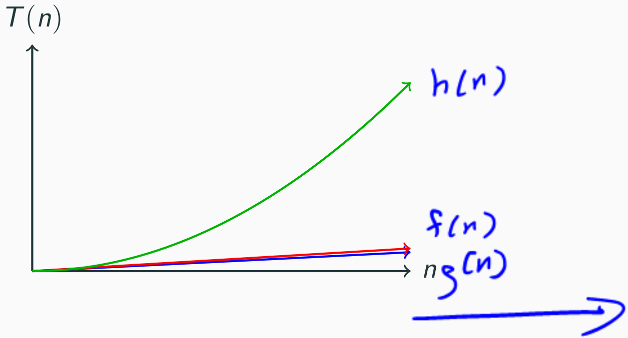
Analysis: comparing functions

Question: Should we treat these two functions the same?



Analysis: comparing functions

What about now?



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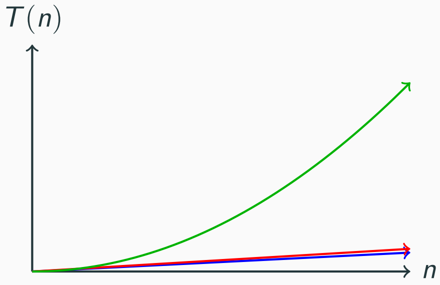
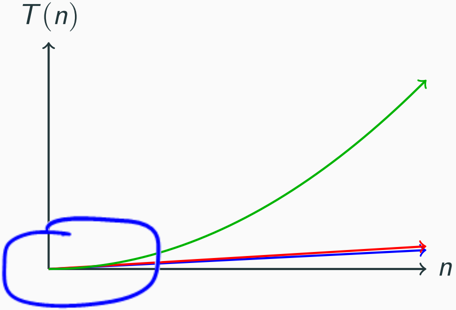


Intuition: our quadratic function is **dominating** the linear ones

Analysis: comparing functions

What about now?

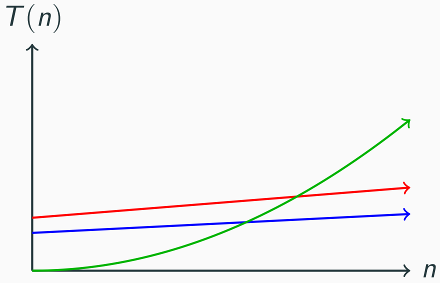


Intuition: our quadratic function is **dominating** the linear ones

Intuition: our linear functions (eventually) look the same

Analysis: comparing functions

Let's zoom in...



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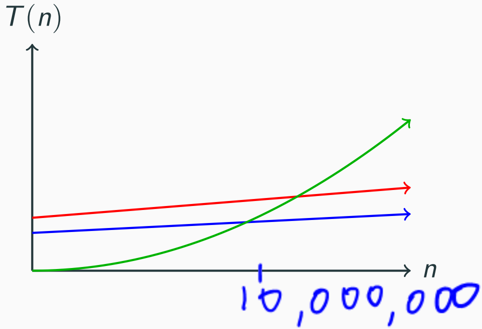

Intuition: quadratic function **eventually** dominates the linear ones

Analysis: comparing functions

Our goal:

- We want a way to say n^2 **eventually dominates** n

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- We want a way to say n^2 **eventually dominates** n
- We want a way to treat n and $4n$ the same way
  Intuition:
  - Model made simplifying assumptions about constant factors
  - Can usually improve constant-factor differences by being clever

Analysis: comparing functions

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- We want a way to say n^2 **eventually dominates** n
- We want a way to treat n and $4n$ the same way
  Intuition:
  - Model made simplifying assumptions about constant factors
  - Can usually improve constant-factor differences by being clever
- We want a way to do this rigorously!

True or false?

- Is n "less then or equal to" $5n + 3$? True
- Is $5n + 3$ "less then or equal to" n? True
- Is $5n + 3$ "less then or equal to" 1? False
- Is $5n + 3$ "less then or equal to" n^2? True
- Is $n^2 + 3n + 2$ "less then or equal to" n^3? True
- Is n^3 "less then or equal to" $n^2 + 3n + 2$? False

Function comparison: exercise

True or false?

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Analysis: comparing functions

Our goal:

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Let's formalize this...

Idea 1

A function $f(n)$ is "less then or equal to" $g(n)$ when $f(n) \leq g(n)$ is true for all values of $n \geq 0$.

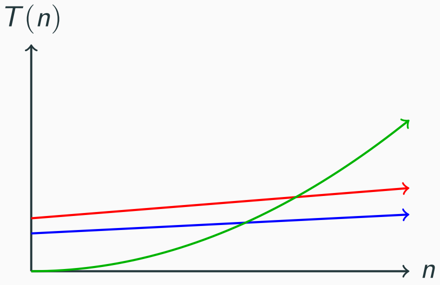
Does this work?

Let's formalize this...

Idea 1

A function $f(n)$ is "less then or equal to" $g(n)$ when $f(n) \leq g(n)$ is true for all values of $n \geq 0$.

Does this work? Remember this?



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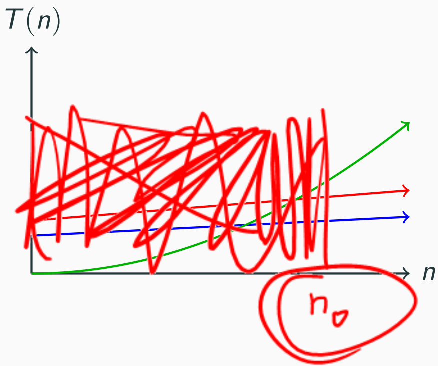


Problem: incorrectly handles the quadratic function!

Let’s formalize this...

Idea 2

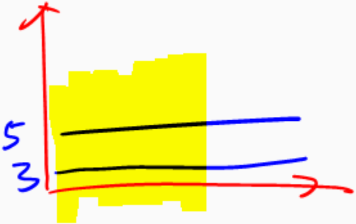
A function $f(n)$ is “less then or equal to” $g(n)$ when $f(n) \leq g(n)$ is true for all values of $n \geq n_0$.

$f(n) \leq c \cdot g(n)$

...where $n_0 > 0$ is some constant value.

Does it work now?

"5" $\leq$ c·"3"



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Does it work now?

We previously said we want to treat n and $4n$ as being the "same". Do we?

Problem: No, we don't!

Let's formalize this...

Idea 3

A function $f(n)$ is "less then or equal to" $g(n)$ when $f(n) \leq c \cdot g(n)$ is true for all values of $n \geq n_0$.

...where $n_0 > 0$ is some constant value.

...where $c > 0$ is some constant value.

Does it work now?

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Does it work now?

Yes!

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A function $f(n)$ is **dominated by** $g(n)$ when...

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The formal definition (not necessary to know this):

Formal definition: Dominated by

A function $f(n)$ is **dominated by** $g(n)$ when

$$\exists(c > 0, n_0 > 0). \forall(n \geq n_0). (f(n) \leq cg(n))$$

...is true.

Exercise

Demonstrate that $5n^2 + 3n + 6$ is dominated by n^3 by finding a c and n_0 that satisfy the above definition.

$c = 14$ $\quad n_0 = 1$

$5n^2 + 3n + 6 \leq 5n^2 + 3n^2 + 6n^2$ when $n \geq 1$

$5n^2 + 3n^2 + 6n^2 = 14n^2$

$5n^2 + 3n + 6 \leq 14n^2$ for $n \geq 1$

$14n^2 \leq c \cdot n^3$ ($14 \cdot n^3$) for $c = 14$, $n \geq 1$

$n = 0.5$

$n^2 = 0.5 \times 0.5$

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Exercise

Demonstrate that $5n^2 + 3n + 6$ is dominated by n^3 by finding a c and n_0 that satisfy the above definition.

Idea: pick $c = 10000$ and $n_0 = 10000$. (It probably works ¯_(ツ)_/¯)

Exercise

Demonstrate that $5n^2 + 3n + 6$ is dominated by n^3 by finding a c and n_0 that satisfy the above definition.

Idea: pick $c = 10000$ and $n_0 = 10000$. (It probably works ¯_(ツ)_/¯)

Better idea: show that $5n^2 + 3n + 6$ is dominated by an easier function to analyze. E.g. note that:

$$\begin{aligned} 5n^2 + 3n + 6 &\leq 5n^2 + 3n^2 + 6n^2 && \text{for all } n \geq 1 \\ &= 14n^2 \\ &\leq 14n^3 \end{aligned}$$

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Demonstrate that $5n^2 + 3n + 6$ is dominated by n^3 by finding a c and n_0 that satisfy the above definition.

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So, what value of c makes $14n^3 \leq cn^3$ true (when $n \geq 1$)?

One possible choice: $n_0 = 1$ and $c = 14$.

Exercise

Demonstrate that $5n^2 + 3n + 6$ is dominated by n^3 by finding a c and n_0 that satisfy the above definition.

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So, what value of c makes $14n^3 \leq cn^3$ true (when $n \geq 1$)?

One possible choice: $n_0 = 1$ and $c = 14$.

So, since we know $5n^2 + 3n + 6 \leq 14n^3$ for $n \geq n_0$ and also know $14n^3 \leq cn^3$, we conclude $5n^2 + 3n + 6 \leq cn^3$.

Exercise

Demonstrate that $2n^3 - 3 + 9n^2 + \sqrt{n}$ is dominated by n^3 (again by finding a c and n_0).

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Demonstrate that $2n^3 - 3 + 9n^2 + \sqrt{n}$ is dominated by n^3 (again by finding a c and n_0).

Do the same thing. Note that:

$$\begin{aligned} 2n^3 - 3 + 9n^2 + \sqrt{n} &\leq 2n^3 + 9n^2 + n && \text{for all } n \geq 1 \\ &\leq 2n^3 + 9n^3 + n^3 \\ &= 12n^3 \end{aligned}$$

So, one possible choice of n_0 and c is $n_0 = 1$ and $c = 12$.

Families of functions

Observation:

- n, $5n + 3$, $100n$, etc... all dominate each other
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$\mathcal{O}(f(n))$ is the "family" or "set" of **all** functions that are **dominated by** $f(n)$

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Yes! By convention, we pick the "simplest" way of writing this and refer to this "family" as $\mathcal{O}(n)$.

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A question: Do the following two sentences mean the same thing?

- $f(n)$ is dominated by $g(n)$
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An aside: some people write this as $f(n) = \mathcal{O}(g(n))$

This is **wrong** (but common, so we reluctantly accept this)

A few more questions

True or false:

- $5n + 3 \in \mathcal{O}(n)$
- $n \in \mathcal{O}(5n + 3)$
- $5n + 3 = \mathcal{O}(n)$
- $\mathcal{O}(5n + 3) = \mathcal{O}(n)$
- $\mathcal{O}(n^2) = \mathcal{O}(n)$
- $n^2 \in \mathcal{O}(1)$
- $n^2 \in \mathcal{O}(n)$
- $n^2 \in \mathcal{O}(n^2)$
- $n^2 \in \mathcal{O}(n^3)$
- $n^2 \in \mathcal{O}(n^{100})$

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True or false:

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- $n \in \mathcal{O}(5n + 3)$ — True
- $5n + 3 = \mathcal{O}(n)$ — True (by convention)
- $\mathcal{O}(5n + 3) = \mathcal{O}(n)$ — True
- $\mathcal{O}(n^2) = \mathcal{O}(n)$ — False
- $n^2 \in \mathcal{O}(1)$ — False
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- $n^2 \in \mathcal{O}(n^3)$ — True
- $n^2 \in \mathcal{O}(n^{100})$ — True

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Definition: Big-Ω

$\Omega(f(n))$ is the family of all functions that **dominates** $f(n)$.

A few more questions...

True or false?

- $4n^2 \in \Omega(1)$
- $4n^2 \in \Omega(n)$
- $4n^2 \in \Omega\left(n^2\right)$
- $4n^2 \in \Omega\left(n^3\right)$
- $4n^2 \in \Omega\left(n^4\right)$

- $4n^2 \in \mathcal{O}(1)$
- $4n^2 \in \mathcal{O}(n)$
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We say $f(n) \in \Theta(g(n))$ when both:

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- $f(n) \in \Omega(g(n))$

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...are true.

Note: in industry, it's common for many people to ask for the big-$\mathcal{O}$ when they really want the big-Θ!

Takeaways

Important things to know:

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- The precise definitions of:
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  - “Dominates” and big-Ω
  - Big-Θ

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Important things to know:

- Intuition behind the definitions of "dominated by" and big-$\mathcal{O}$
- The precise definitions of:
  - "Dominated by" and big-$\mathcal{O}$
  - "Dominates" and big-Ω
  - Big-Θ
- How to demonstrate that one function is dominated by another by finding c and n_0 and applying the correct definition