Quickcheck 07: Solutions

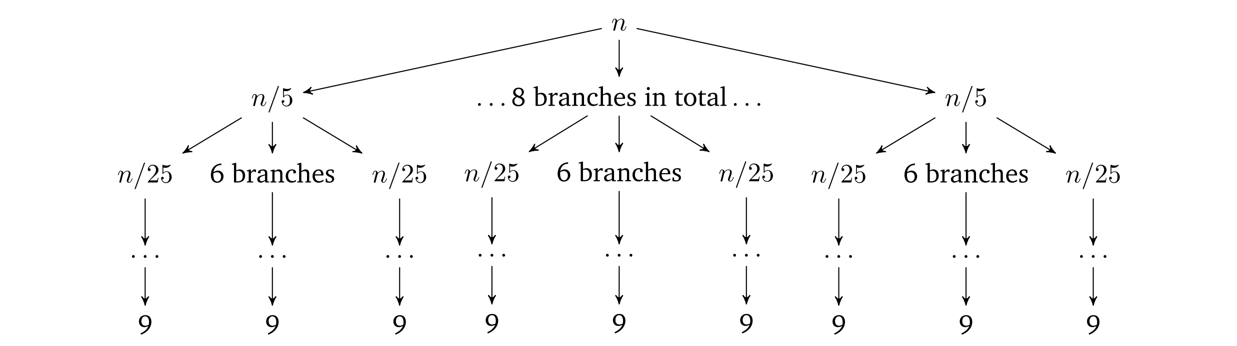
Name:

Consider the following recurrence:

$$T(n) = \begin{cases} 9 & \text{if } n = 1 \\ 8T(n/5) + n & \text{otherwise} \end{cases}$$

(a) Draw out a visualization of what this recurrence looks like as a tree.

Solution:



(b) How much work is done on level i?

Solution:

We do $8^i \cdot \frac{n}{5^i}$ work – there are 8^i nodes per level, each one does $\frac{n}{5^i}$ work.

(c) How many recursive levels are there in the tree?

Solution:

There are $\log_5(n)$ levels in total.

Note that this is different from the *total* number of levels in the tree – if we also count the base case, there are $log_5(n) + 1$ levels in total. But this question is only asking for the number of recursive levels.

(d) How much work is done at the leaf level?

Solution:

We do $9 \cdot 8^{\log_5(n)}$ work at the very bottom level.

We obtain this quantity by plugging in $i = \log_5(n)$ into 8^i – we let i be the last level. (Note that we do *not* plug in $i = \log_5(n) + 1$. This is because we count our levels starting from 0, not 1.)

(e) Construct a non-recursive expression equivalent to the recurrence. Your solution may use a summation.

Solution:

We combine our answers from above: $\left(\sum_{i=0}^{\log_5(n)-1} n \cdot \frac{8^i}{5^i}\right) + 9 \cdot 8^{\log_5(n)}$.

The question didn't ask us to find a closed form, but for the sake of completeness, here's how we could. We first start by simplifying the summation a little. The n is independent from i, so we can pull that out of the summation. We can also rewrite $\frac{8^i}{5^i}$ into $\left(\frac{8}{5}\right)^i$. This gives us:

$$\left(n \sum_{i=0}^{\log_5(n)-1} \left(\frac{8}{5}\right)^i\right) + 9 \cdot 8^{\log_5(n)}$$

At this point, we can use the finite geometric series identity, which is defined as:

$$\sum_{i=0}^{x-1} r^i = \frac{r^x - 1}{r - 1}$$

Here, $x = \log_5(n)$, and $r = \frac{8}{5}$. So, we apply the identity and get:

$$n\left(\frac{\left(\frac{8}{5}\right)^{\log_5(n)} - 1}{\frac{8}{5} - 1}\right) + 9 \cdot 8^{\log_5(n)}$$

As this point, we're done. Our answer contains no summations or recurrences and so is a valid closed form.

But what if we want to simplify this expression? Again, for the sake of completeness, here's how we would do that. We'll first start by simplifying the complicated fraction to get:

$$\begin{aligned}
T(n) &= n\left(\frac{\left(\frac{8}{5}\right)^{\log_5(n)} - 1}{\frac{8}{5} - 1}\right) + 9 \cdot 8^{\log_5(n)} \\
&= n\left(\frac{\left(\frac{8}{5}\right)^{\log_5(n)} - 1}{\frac{3}{5}}\right) + 9 \cdot 8^{\log_5(n)} \\
&= n \cdot \frac{5}{3} \cdot \left(\left(\frac{8}{5}\right)^{\log_5(n)} - 1\right) + 9 \cdot 8^{\log_5(n)}
\end{aligned}$$

Next, we can simplify our exponents. In order to do so, we will use the "power of a log" identity, which states that $x^{\log_b(y)} = y^{\log_b(x)}$. We apply this identity to both powers to get:

$$n \cdot \frac{5}{3} \cdot \left(n^{\log_5(8/5)} - 1\right) + 9 \cdot n^{\log_5(8)}$$

We next want to clean up the $n^{\log_5(8/n)}$ part. To do so, we can use a few more log and exponent identities:

$$\begin{aligned} T(n) &= n \cdot \frac{5}{3} \cdot \left(n^{\log_5(8) - \log_5(5)} - 1\right) + 9 \cdot n^{\log_5(8)} \\ &= n \cdot \frac{5}{3} \cdot \left(n^{\log_5(8) - 1} - 1\right) + 9 \cdot n^{\log_5(8)} \\ &= n \cdot \frac{5}{3} \cdot \left(\frac{n^{\log_5(8)}}{n} - 1\right) + 9 \cdot n^{\log_5(8)} \end{aligned}$$

Finally, we distribute in and simplify:

$$\begin{aligned} T(n) &= n \cdot \frac{5}{3} \cdot \frac{n^{\log_5(8)}}{n} - n \cdot \frac{5}{3} + 9 \cdot n^{\log_5(8)} \\ &= \frac{5}{3} \cdot n^{\log_5(8)} - n \cdot \frac{5}{3} + 9 \cdot n^{\log_5(8)} \\ &= \frac{32}{3} \cdot n^{\log_5(8)} - n \cdot \frac{5}{3} \end{aligned}$$

This is our final answer.

(f) Use the master theorem to find the big-Θ bound for the recurrence.

Solution:

We know $a = 8$, $b = 5$, and $c = 1$. Note that $\log_b(a) = \log_5(8) > 1 = c$. So, we know $T(n) \in \Theta\left(n^{\log_b(a)}\right)$. So, we know $T(n) \in \Theta\left(n^{\log_5(8)}\right)$.

This matches up with the (simplified) closed form we found up above. The term $\frac{32}{5}n^{\log_5(8)}$ is roughly equivalent to $\frac{32}{5}n^{1.292}$, which definitely grows faster then (and so drowns out) the $-\frac{5n}{3}$ term.