Section 03: Solutions

1. Analysis

For each of the following code blocks, what is the worst-case runtime? Give a big-Θ bound.

(a)
```
public IList<String> repeat(DoubleLinkedList<String> list, int n) {
    IList<String> result = new DoubleLinkedList<String>();
    for(String str : list) {
        for(int i = 0; i < n; i++) {
            result.add(str);
        }
    }
    return result;
}
```

Solution:

The runtime is $\Theta(nm)$, where m is the length of the input list and n is equal to the int `n` parameter.

One thing to note here is that unlike many of the methods we've analyzed before, we can't quite describe the runtime of this algorithm using just a single variable: we need two, one for each loop.

The other thing to remember is that in Java, foreach loops are converted into a while loop using iterators, which will influence the final runtime of our algorithm.

(b)
```
public void foo(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 5; j < i; j++) {
            System.out.println("Hello!");
        }

        for (int j = i; j >= 0; j -= 2) {
            System.out.println("Hello!");
        }
    }
}
```

Solution:

$\Theta(n^2)$.

(c)
```
public int num(int n){
    if (n < 10) {
        return n;
    } else if (n < 1000) {
        return num(n - 2);
    } else {
        return num(n / 2);
    }
}
```

Solution:

The answer is $\Theta(\log(n))$.

One thing to note is that the second case effectively has no impact on the runtime. That second case occurs only for $n < 1000$ – when discussing asymptotic analysis, we only care what happens with the runtime as n grows large.

(d)
```
public int foo(int n) {
    if (n <= 0) {
        return 3;
    }
    int x = foo(n - 1);
    System.out.println("hello");
    x += foo(n - 1);
    return x;
}
```

Solution:

The answer is $\Theta(2^n)$.

In order to determine that this is exponential, let's start by considering the following recurrence:

$$T(n) = \begin{cases} 1 & \text{If } n = 0 \\ 2T(n-1) + 1 & \text{Otherwise} \end{cases}$$

While we could unfold this to get an exact closed form, we can approximate the final asymptotic behavior by taking a step back and thinking on a higher level what this is doing.

Basically, what happens is we take the work done by $T(n-1)$ and multiply it by 2. If we ignore the $+1$ constant work done in the recursive case, the net effect is that we multiply 2 approximately n times. This simplifies to 2^n.

2. Recurrences

For each of the following recurrences, use the unfolding method to first convert the recurrence into a summation. Then, find a big-Θ bound on the function in terms of n. Assume all division operations are integer division.

(a) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 3 & \text{otherwise} \end{cases}$

Solution:

The summation is $1 + \sum_{i=2}^{\log(n)+1} 3$. The big-$\Theta$ bound is $\Theta(\log(n))$.

Something you may notice is that depending on what exactly n is, the expression $\log(n) + 1$ may not evaluate to an integer. In that case, what does it mean to have $\log(n)+1$ as the upper limit of a summation?

What exactly this mean differs based on convention, but for the purposes of this class, we'll assume that i varies starting at 2 up to the largest possible integer that is $\leq \log(n) + 1$. We could write this more explicitly using floors: $1 + \sum_{i=2}^{\lfloor \log(n)+1 \rfloor} 3$.

(b) $T(n) = \begin{cases} 1 & \text{if } n = 0 \\ T(n-1) + 2 & \text{otherwise} \end{cases}$

Solution:

The summation is $1 + \sum_{i=1}^{n} 2$. The big-Θ bound is $\Theta(n)$.

(c) $T(n) = \begin{cases} 1 & \text{if } n = 0 \\ T(n/3) + 4 & \text{otherwise} \end{cases}$

Solution:

The summation is $1 + \sum_{i=1}^{\log_3(n)+1} 4$. The big-$\Theta$ bound is $\Theta(n)$.

(d) $T(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2T(n/3) + n & \text{otherwise} \end{cases}$

Solution:

In order to determine what this expression looks like as a summation, it helps to first partially unroll it:

$$\begin{aligned} T(n) &= n + 2T\left(\frac{n}{3}\right) \\ &= n + 2\left(\frac{n}{3} + 2T\left(\frac{n}{9}\right)\right) \\ &= n + 2\left(\frac{n}{3} + 2\left(\frac{n}{9} + 2T\left(\frac{n}{27}\right)\right)\right) \\ &= n + 2\left(\frac{n}{3} + 2\left(\frac{n}{9} + 2\left(\frac{n}{27} + 2T\left(\frac{n}{81}\right)\right)\right)\right) \end{aligned}$$

We then multiply in the 2 on the outside:

$$\begin{aligned} T(n) &= n + 2\left(\frac{n}{3} + 2\left(\frac{n}{9} + 2\left(\frac{n}{27} + 2T\left(\frac{n}{81}\right)\right)\right)\right) \\ &= n + 2\frac{n}{3} + 2^2\left(\frac{n}{9} + 2\left(\frac{n}{27} + 2T\left(\frac{n}{81}\right)\right)\right) \\ &= n + 2\frac{n}{3} + 2^2\frac{n}{9} + 2^3\left(\frac{n}{27} + 2T\left(\frac{n}{81}\right)\right) \\ &= n + 2\frac{n}{3} + 2^2\frac{n}{9} + 2^3\frac{n}{27} + 2^4T\left(\frac{n}{81}\right) \end{aligned}$$

We can start to see the pattern now: our summation is roughly of the form $\sum_{i=?}^{?} 2^i \frac{n}{3^i}$.

What about the base case? It's not just 1, we need to multiply it by some power of 2 to account for the accumulating multiples.

We put all the pieces together and finish: $1 \cdot 2^{\lfloor \log_3(n)+1 \rfloor} + \sum_{i=0}^{\log_3(n)} \frac{2^i}{3^i} n$

To compute the Θ bound, we observe that the large constant, despite being large, is still ultimately a constant. We can also simplify the summation by pulling out the n (since it doesn't vary on i). The remaining summation must simplify to some integer. So, we conclude $\Theta(n)$.

(e) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n-1) + 1 & \text{otherwise} \end{cases}$

Solution:

Using a similar process, we get the following expression: $2^{n-1} + \sum_{i=2}^{n} 2^{i-2}$.

This ends up being in $\Theta(2^i)$.

(f) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + 100 & \text{otherwise} \end{cases}$

Solution:

We first get $2^{\lfloor \log(n) \rfloor} + \sum_{i=2}^{\log(n)+1} 100 \cdot 2^{i-2}$.

Therefore, we have $\Theta(\log(n))$.

3. Modeling recursive functions

Consider the following method.

```java
public static int f(int n) {
    if (n == 0) {
        return 0;
    }

    int result = 0;
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < i; j++) {
            result += j;
        }
    }
    return 5 * f(n / 2) + 3 * result + 2 * f(n / 2);
}
```

(a) Find a recurrence $T(n)$ modeling the worst-case runtime of `f(n)`.

Solution:

$$T(n) = \begin{cases} 1 & \text{When } n = 0 \\ \frac{n(n-1)}{2} + 2T(n/2) & \text{Otherwise} \end{cases}$$

(b) Find a recurrence $W(n)$ modeling the *integer output* of `f(n)`.

Solution:

$$W(n) = \begin{cases} 0 & \text{When } n = 0 \\ \frac{3n(n-1)}{2} + 7T(n/2) & \text{Otherwise} \end{cases}$$

4. Modeling recursive functions 2

```
public static int g(n) {
    if (n <= 1) {
        return 1000;
    }
    if (g(n / 3) > 5) {
        for (int i = 0; i < n; i++) {
            System.out.println("Hello");
        }
        return 5 * g(n / 3);
    } else {
        for (int i = 0; i < n * n; i++) {
            System.out.println("World");
        }
        return 4 * g(n / 3);
    }
}
```

(a) Find a recurrence $S(n)$ modeling the worst-case runtime of `g(n)`.

Solution:

$$S(n) = \begin{cases} 1 & \text{When } n \leq 1 \\ 2S(n/3) + n & \text{Otherwise} \end{cases}$$

Important: note that the if statement contains a recursive call that must be evaluated for $n > 1$.

(b) Find a recurrence $X(n)$ modeling the *integer output* of `g(n)`.

Solution:

$$X(n) = \begin{cases} 1000 & \text{When } n \leq 1 \\ 5T(n/3) & \text{Otherwise} \end{cases}$$

5. Modeling recursive functions 3

Consider the following set of recursive methods.

```
public int test(int n) {
    IDictionary<Integer, Integer> dict = new AvlDictionary<>();
    populate(n, dict);
    int counter = 0;
    for (int i = 0; i < n; i++) {
        counter += dict.get(i);
    }
    return counter;
}

private void populate(int k, IDictionary<Integer, Integer> dict) {
    if (k == 0) {
        dict.put(0, k);
    } else {
        for (int i = 0; i < k; i++) {
            dict.put(i, i);
        }
        populate(k / 2, dict);
    }
}
```

(a) Write a mathematical function representing the *worst-case runtime* of `test`.

You should write two functions, one for the runtime of `test` and one for the runtime of `populate`.

Solution:

The runtime of the `populate` method is:

$$P(k) = \begin{cases} \log(N) & \text{When } k = 0 \\ k\log(k) + P(k/2) & \text{Otherwise} \end{cases}$$

Here, N is the maximum possible value of n

The runtime of the `test` method is then $R(n) = P(n) + n$.

(b) Write a mathematical function representing the *integer output* of `test`.

Solution:

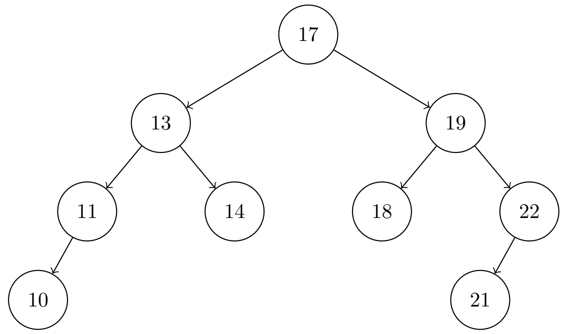
$$Y(n) = \frac{n(n-1)}{2}$$

6. AVL Trees

(a) Draw an AVL Tree as each of the following keys are added in the order given. Show intermediate steps.

$$\{13, 17, 14, 19, 22, 18, 11, 10, 21\}$$

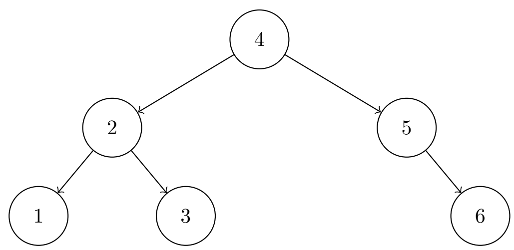
Solution:



(b) Draw an AVL Tree as each of the following keys are added in the order given. Show intermediate steps.

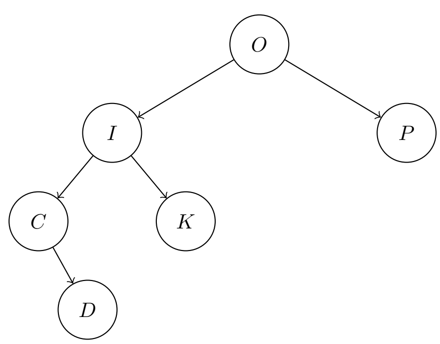
$$\{1, 2, 3, 4, 5, 6\}$$

Solution:



7. More AVL Trees

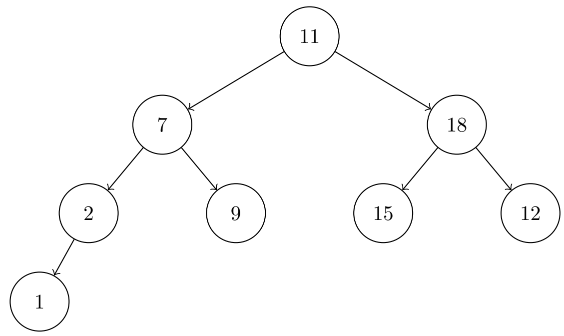
(a) Is this a valid AVL tree? Explain your answer.



Solution:

No, does not meet the balance property.

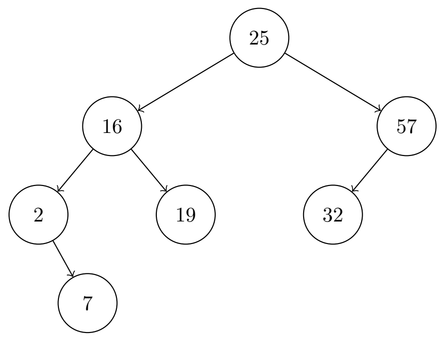
(b) Is this a valid AVL tree? Explain your answer.



Solution:

No, does not meet the BST property. 12 is not greater than 18.

(c) Is this a valid AVL tree? Explain your answer.



Solution:

Yes, it satisfies the balance and BST properties.

8. Algorithm Design

(a) Given a binary search tree, describe how you could convert it into an AVL tree with worst-case time $\mathcal{O}(n \log(n))$. What is the best case runtime of your algorithm?

Solution:

Since we already have a BST, we can do an in-order traversal on the tree to get a sorted array of nodes. We could now simply insert all of these nodes back into an AVL tree using rotations which would give us an $\mathcal{O}(n \log(n))$ runtime.

(b) Given an AVL tree, describe how would you do a level order tree traversal. What is the worst-case runtime of your algorithm?

Solution:

Since an AVL tree is just a balanced BST, we can use a queue to add each node we visit. As we dequeue each node, we will add it's children to the queue. We would get an $\mathcal{O}(n)$ runtime.