Quickcheck 03: Solutions

Name:

Consider the following recursive function. You may assume that the input will be a multiple of 3.

```java
public int test(int n) {
    if (n <= 6) {
        return 2;
    } else {
        int curr = 0;
        for (int i = 0; i < n * n; i++) {
            curr += 1;
        }
        return curr + test(n - 3);
    }
}
```

(a) Write a recurrence modeling the *worst-case runtime* of `test`.

Solution:

$$T(n) = \begin{cases} 1 & \text{When } n \leq 6 \\ n^2 + T(n-3) & \text{Otherwise} \end{cases}$$

(b) Unfold the recurrence into a summation (for $n \geq 6$).

Solution:

$$1 + \sum_{i=3}^{n/3} (3i)^2$$

Modeling this recurrence correctly is slightly challenging because we want to decrease n in increments of 3.

To do this, what we do is set the summation bounds to go up to $n/3$ instead of n, and multiply i on the inside by 3, simulating changing i in those increments.

We then also set the lower summation bound to be 3 instead of 0 or 1. That way, our summation will only consider numbers in the range 9 to n – if we set $i = 2$ or lower, our summation would double-cound $n \leq 6$, which should be handled by the base case.

Note: our model only works if n is a multiple of 3.

(c) Simplify the summation into a closed form (for $n \geq 6$).

Solution:

$$
\begin{aligned}
1 + \sum_{i=3}^{n/3} (3i)^2 &= 1 + \sum_{i=0}^{n/3} (3i)^2 - \sum_{i=0}^{2} (3i)^2 && \text{Adjusting summation bounds} \\
&= 1 + 9 \sum_{i=0}^{n/3} i^2 - \sum_{i=0}^{2} (3i)^2 && \text{Pulling out a constant} \\
&= 1 + 9 \sum_{i=0}^{n/3} i^2 - (0 + 9 + 36) && \text{Evaluating the summation} \\
&= 9 \frac{\frac{n}{3}\left(\frac{n}{3}+1\right)\left(\frac{2n}{3}+1\right)}{6} - 44 && \text{Sum of squares}
\end{aligned}
$$

A "closed form", within the context of this class, is just any expression that does not contain a summation or is recursive. This means we can stop here without needing to further simplify the expression.

That said, if you wanted to continue simplifying, we could:

$$
\begin{aligned}
9 \frac{\frac{n}{3}\left(\frac{n}{3}+1\right)\left(\frac{2n}{3}+1\right)}{6} - 44 &= \frac{9}{6}\left(\frac{n}{3}\left(\frac{n}{3}+1\right)\left(\frac{2n}{3}+1\right)\right) - 44 \\
&= \frac{1}{2}\left(n\left(\frac{n}{3}+1\right)\left(\frac{2n}{3}+1\right)\right) - 44 \\
&= \frac{1}{2}\left(n\left(\frac{2}{9}n^2 + n + 1\right)\right) - 44 \\
&= \frac{1}{9}n^3 + \frac{1}{2}n^2 + \frac{1}{2}n - 44
\end{aligned}
$$