Lecture 3: How to measure efficiency

Data Structures and Algorithms

Announcements

- Course background survey due by Friday
- HW 1 is Due Friday
- Alex has Office Hours after class (2:30-4:30) CSE 006, will help with setup
  - If you have any questions about your setup please come to office hours so we can iron out all the wrinkles before the partnered projects begin next week.

- HW 2 Assigned on Friday – Partner selection forms due by 11:59pm Thursday

https://goo.gl/forms/rVrVUkFDdsqI8pkD2

Review: Sequential Search

sequential search: Locates a target value in an array / list by examining each element from start to finish.

- How many elements will it need to examine?
- Example: Searching the array below for the value 42:

index	0	1	2	3	4	5	6	7	8	9	**10**	11	12	13	14	15	16
value	-4	2	7	10	15	20	22	25	30	36	**42**	50	56	68	85	92	103

i

- What is the best case?
- What is the worst case?
- What is the complexity class?

Review: Binary Search

binary search: Locates a target value in a *sorted* array or list by successively eliminating half of the array from consideration.

- How many elements will it need to examine?
- Example: Searching the array below for the value 42:

index	0	1	2	3	4	5	6	7	8	9	**10**	11	12	13	14	15	16
value	-4	2	7	10	15	20	22	25	30	36	**42**	50	56	68	85	92	103
	min								mid								max

- What is the best case?
- What is the worst case?
- What is the complexity class?

Analyzing Binary Search

What is the pattern?

- At each iteration, we eliminate half of the remaining elements

How long does it take to finish?

- 1st iteration – N/2 elements remain
- 2nd iteration – N/4 elements remain
- Kth iteration - N/2^k elements remain
- Done when N/2^k = 1

index	0	1	2	3	4	5	6	7	8	9	**10**	11	12	13	14	15	16
value	-4	2	7	10	15	20	22	25	30	36	**42**	50	56	68	85	92	103

Analyzing Binary Search

$$\frac{N}{2^K} = 1$$

$$N = 2^K$$

$$\log_2 N = \log_2 N$$

$$\Rightarrow O(\log n)$$

We lost the base! This will be explained later.

Review: logarithms

$\log_b a = x$ means that x solves the problem $b^? = a$. This means that

$$\log_b (b^z) = x$$

asks for x to solve $b^? = b^z$, so $x = z$.

Analyzing Binary Search

Finishes when $N / 2^K = 1$

$N / 2^K = 1$

-> multiply right side by 2^K

$N = 2^K$

-> isolate K exponent with logarithm

$Log_2 N = k$

Is this exact?

- N can be things other than powers of 2
- If N is odd we can't technically use Log_2
- When we have an odd number of elements we select the larger half
- Within a fair rounding error

Asymptotic Analysis

asymptotic analysis: how the runtime of an algorithm grows as the data set grows

Approximations / Rules

- Basic operations take "constant" time
  - Assigning a variable
  - Accessing a field or array index
- Consecutive statements
  - Sum of time for each statement
- Function calls
  - Time of function's body
- Conditionals
  - Time of condition + maximum time of branch code
- Loops
  - Number of iterations x time for loop body

Modeling Case Study

Goal: return 'true' if a sorted array of ints contains duplicates

Solution 1: compare each pair of elements

```
public boolean hasDuplicate1(int[] array) {
   for (int i = 0; i < array.length; i++) {
      for (int j = 0; j < array.length; j++) {
         if (i != j && array[i] == array[j]) {
            return true;
         }
      }
   }
   return false;
}
```

Solution 2: compare each consecutive pair of elements

```
public boolean hasDuplicate2(int[] array) {
   for (int i = 0; i < array.length - 1; i++) {
      if (array[i] == array[i + 1]) {
            return true;
      }
   }
   return false;
}
```

Modeling Case Study: Solution 2

T(n) where n = array.length

-> work inside out

Solution 2: compare each consecutive pair of elements

```
public boolean hasDuplicate2(int[] array) {
   for (int i = 0; i < array.length - 1; i++) {
      if (array[i] == array[i + 1]) { +4
            return true; +1
      }
   }
   return false; +1
}
```

If statement = 5

x n - 1

T(n) = 5 (n-1) + 1

linear time complexity class O(n)

Modeling Case Study: Solution 1

Solution 1: compare each consecutive pair of elements

```
public boolean hasDuplicate1(int[] array) {
   for (int i = 0; i < array.length; i++) { x n
      for (int j = 0; j < array.length; j++) { x n
         if (i != j && array[i] == array[j]) { +5
            return true; +1
         }
      }
   }
   return false; +1
}
```

6

6n

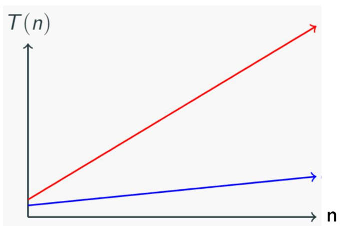
$6n^2$

$T(n) = 6\,n^2 + 1$

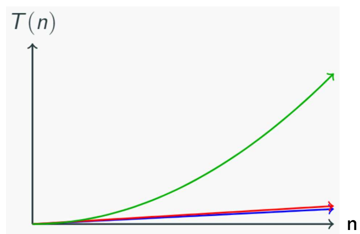
quadratic time complexity class $O(n^2)$

Comparing Functions

Function growth

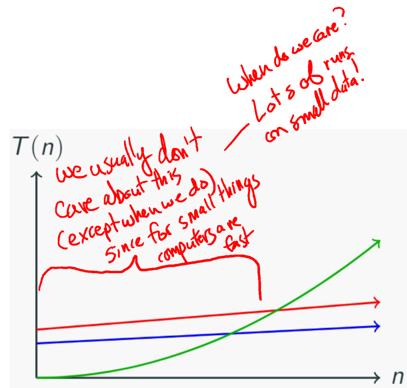


n and 4n look very different up close



n and 4n look the same over time

n^2 eventually dominates n



n^2 doesn't start off dominating the linear functions

It eventually takes over…

Function comparison: exercise

$f(n) = n \le g(n) = 5n + 3$? True – all linear functions are treated as equivalent

$f(n) = 5n + 3 \le g(n) = n$? True

$f(n) = 5n + 3 \le g(n) = 1$? False

$f(n) = 5n + 3 \le g(n) = n^2$? True – quadratic will always dominate linear

$f(n) = n^2 + 3n + 2 \le g(n) = n^3$? True

$f(n) = n^3 \le g(n) = n^2 + 3n + 2$? False

Definition: function domination

Definition: Domination

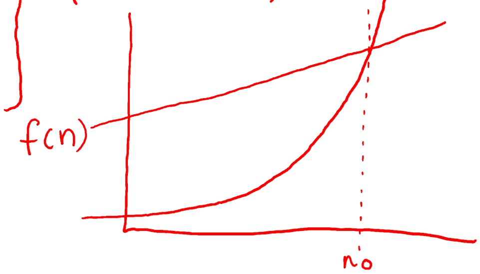
A function f(n) is dominated by g(n) when…

There exists two constants $c > 0$ and $n_0 > 0$

Such that for all values of $n \geq n_0$

$f(n) \leq c * g(n)$

i.e. g(n) is eventually always bigger than f(n): c·g(n)
(up to a constant factor)



Example:

Is f(n) = n dominated by g(n) = 5n + 3 ?

c = 1

$n_0 = 1$

Yes!

Exercise: Function Domination

Demonstrate that $5n^2 + 3n + 6$ is dominated by n^3 by finding a c and n_0 that satisfy the definition of domination

$5n^2 + 3n + 6 \leq 5n^2 + 3n^2 + 6n^2$ when $n \geq 1$

$5n^2 + 3n^2 + 6n^2 = 14n^2$

$5n^2 + 3n + 6 \leq 14n^2$ for $n \geq 1$

$14n^2 \leq c*n^3$ for c = ? n >= ?

$\frac{14}{n}$ -> c = 14 & n >= 1

Is there a (smaller) tighter upper bound? Yes! n^2, use $c=6$

The "smallest" $g(n)$ such that $f(n) \in O(g(n))$ is a tight bound!

Definition: Big O

If f(n) = n ≤ g(n) = 5n + 3 ≤ h(n) = 100n and

h(n) = 100n ≤ g(n) = 5n + 3 ≤ f(n) n

Really they are all the "same"

(for measuring upper bounds)
often worst case
(but this is a general mathematical categorization of functions)

Definition: Big O
O(f(n)) is the "family" or "set" of all functions that are dominated by f(n)

Question: are O(n), O(5n + 3) and O(100n) all the same?

True! By convention we pick simplest of the above -> O(n) ie "linear"

Definitions: Big Ω

"f(n) is greater than or equal to g(n)"

F(n) dominates g(n) when:

There exists two constants such that c > 0 and n0 > 0

Such that for all values n >= n0

F(n) >= c * g(n) is true

Definition: Big Ω
Ω(f(n)) is the family of all functions that dominates f(n)

∈ Element Of

f(n) is dominated by g(n)

Is that the same as

"f(n) is contained inside O(g(n))"

Yes!

f(n) ∈ g(n)

Examples

$4n^2 \in \Omega(1)$

true

$4n^2 \in \Omega(n)$

true

$4n^2 \in \Omega(n^2)$

true

$4n^2 \in \Omega(n^3)$

false

$4n^2 \in \Omega(n^4)$

false

$4n^2 \in O(1)$

false

$4n^2 \in O(n)$

false

$4n^2 \in O(n^2)$

true

$4n^2 \in O(n^3)$

true

$4n^2 \in O(n^4)$

true

Definition: Big O

O(f(n)) is the "family" or "set" of all functions that are dominated by f(n)

Definition: Big Ω

Ω(f(n)) is the family of all functions that dominates f(n)

Definitions: Big Θ

These are the "tight" bounds— the function is "squeezed" on both sides.

We say $f(n) \in \Theta(g(n))$ when both

$f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$ are true

Which is only when $f(n) = g(n)$

Definition: Big Θ
$\Theta(f(n))$ is the family of functions that are equivalent to $f(n)$

Industry uses "Big Θ" and "Big O" interchangeably

Summary

$O(f(n)) \leq f(n) == \Theta(f(n)) \leq \Omega(f(n))$

	Dominated by	Dominates
	$f(n) \in O(g(n))$	$f(n) \in \Omega(g(n))$
$f(n)$		
$O(1)$		
$O(\log n)$		
$O(n)$		
$O(n^2)$		
$O(n^3)$		

Justifying the "Rules"

Approximations / Rules

- Basic operations take "constant" time
  - Assigning a variable
  - Accessing a field or array index
- Consecutive statements
  - Sum of time for each statement
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upper bound, so take largest constant.

Some finite # of statements, take largest, mult. by # of stmts

overhead of a fnctn call is const, so dominated by fn running time

worst case: similar to all ops taking the same time

All of these rules are, up to a constant, upper-bound approximations
(they only can increase your estimate, so you definitely have an upper bound.)

A Slightly Harder example

```
public void mystery(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n * n; j++) {
            System.out.println("Hello");
        }
        for (int j = 0; j < 10; j++) {
            System.out.println("world");
        }
    }
}
```

+1 n^2

+1 10

n

Remember: work outside in

Solution: $T(n) = n(n^2 + 10) = n^3 + 10n$

Modeling Complex Loops

```
for (int i = 0; i < n; i++) {
   for (int j = 0; j < i; j++) {
      System.out.println("Hello!");
   }
}
```

$\cancel{+1 \quad n \quad n \quad f(n) = n^2}$

Keep an eye on loop bounds!

Modeling Complex Loops

```
for (int i = 0; i < n; i++) {
   for (int j = 0; j < i; j++) {
      System.out.println("Hello!");
   }
}
```

$+c$ $\quad 0 + 1 + 2 + 3 + \ldots + n-1 \quad n$

Summation

1 + 2 + 3 + 4 +… + n = $\sum_{i=1}^{n} i$

Definition: Summation

$\sum_{i=a}^{b} f(i)$ = f(a) + f(a + 1) + f(a + 2) + … + f(b-2) + f(b-1) + f(b)

T(n) = $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} c$

Simplifying Summations

```
for (int i = 0; i < n; i++) {
   for (int j = 0; j < i; j++) {
      System.out.println("Hello!");
   }
}
```

$$\left.\left.\texttt{System.out.println("Hello!");}\ +c\right\}\ 0+1+2+3+\ldots+n-1\right\}\ n$$

$\Rightarrow$ (0c + 1c + 2c + 3c + … + i-1c)
+ (0c + 1c + 2c + 3c + … + i-1c)
+ (0c + 1c + 2c + 3c + … + i-1c)
+ repeat n times

T(n) = $\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} c$ = $\sum_{i=0}^{n-1} ci$ Summation of a constant

= $c\sum_{i=0}^{n-1} i$ Factoring out a constant

= $c\frac{n(n-1)}{2}$ Gauss's Identity

= $\frac{c}{2}n^2 - \frac{c}{2}n$ O(n^2)

Function Modeling: Recursion

```
public int factorial(int n) {
   if (n == 0 || n == 1) {  +3
      return 1; +1
   } else {
      return n * factorial(n - 1);  +????
}
```

+c

Function Modeling: Recursion

```
public int factorial(int n) {
   if (n == 0 || n == 1) {
      return 1;
   } else {
      return n * factorial(n - 1);
}
```

$+c_1$

$+T(n-1)$

$+c_2$

$$T(n) = \begin{cases} C_1 & \text{when } n = 0 \text{ or } 1 \\ C_2 + T(n-1) & \text{otherwise} \end{cases}$$

Definition: Recurrence

Mathematical equivalent of an if/else statement
f(n) =

Unfolding Method

$$T(n) = \begin{cases} C_1 & \text{when } n = 0 \text{ or } 1 \\ C_2 + T(n-1) & \text{otherwise} \end{cases}$$

$T(3) = C_2 + T(3-1) \quad = C_2 + (C_2 + T(2-1)) \quad = C_2 + (C_2 + (C_1)) \qquad = 2C_2 + C_1$

$$T(n) = C_1 + \sum_{i=0}^{n-1} C_2$$

Summation of a constant

$T(n) = C_1 + (n-1)C_2$