

# More Graph Algorithms

Data Structures and Algorithms

# Announcements

Calendar on the webpage has updated office hours for this week.

Last Time:

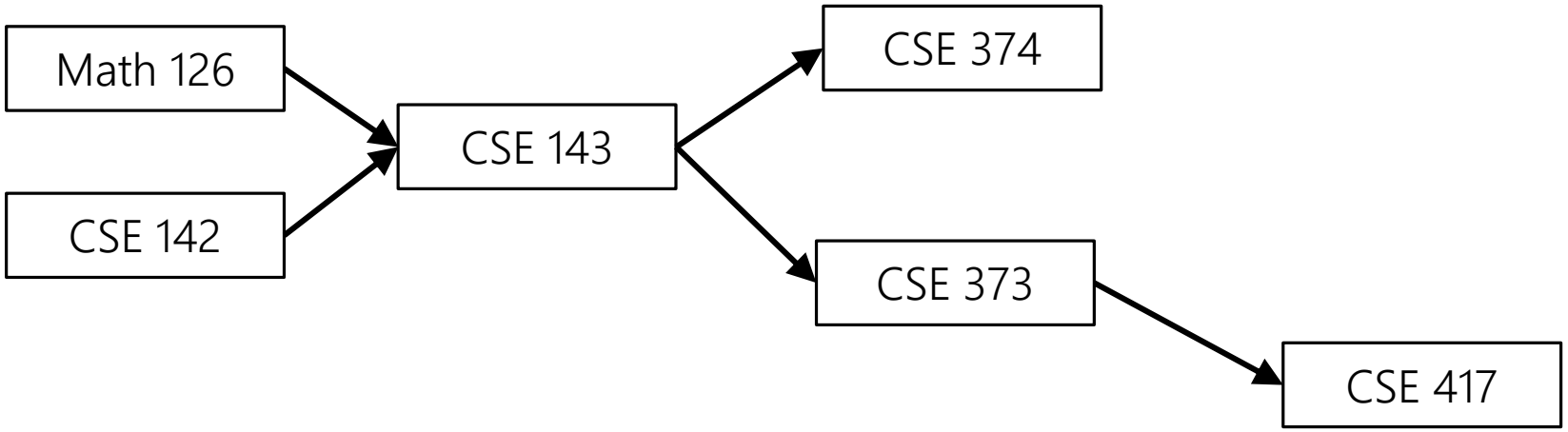
Dijkstra's Algorithm to find shortest paths.

Today:

Two more problems on directed graphs

# Problem 1: Ordering Dependencies

Today's (first) problem: Given a bunch of courses with prerequisites, find an order to take the courses in.



# Problem 1: Ordering Dependencies

Given a directed graph G, where we have an edge from u to v if u must happen before v.

We can only do things one at a time, can we find an order that **respects dependencies**?

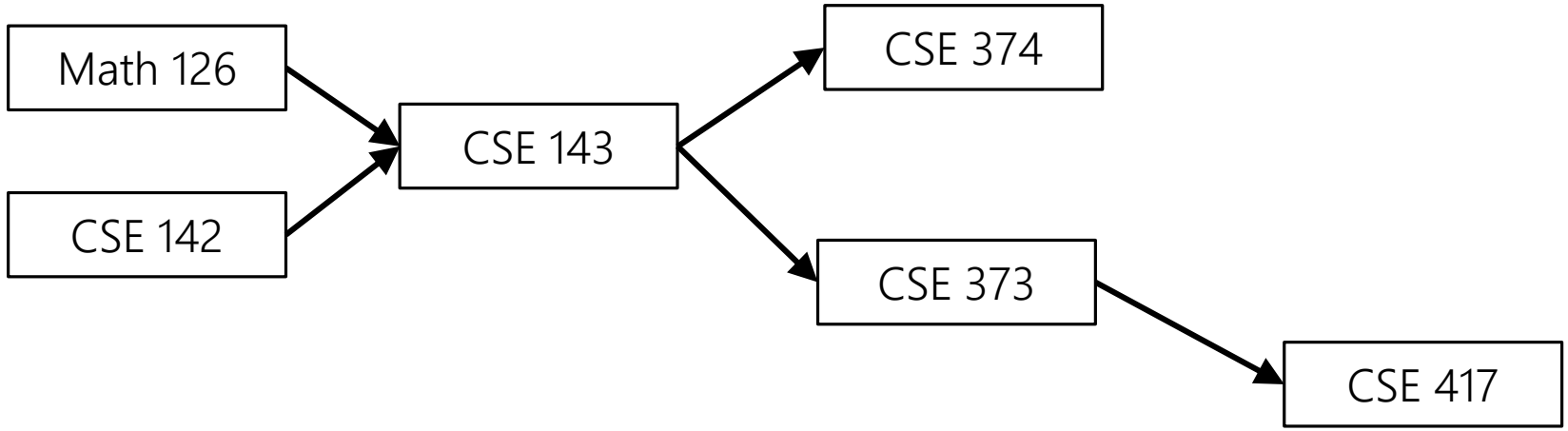
**Topological Sort (aka Topological Ordering)**

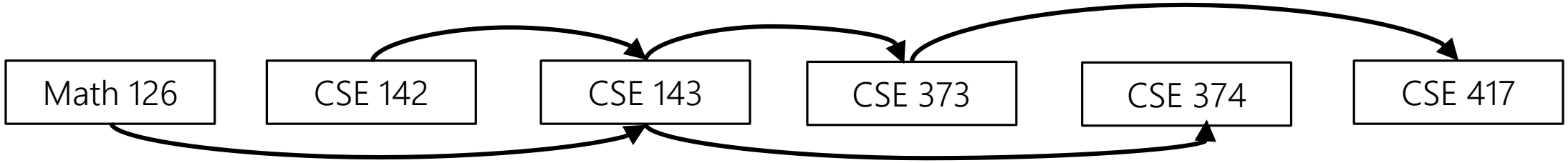
**Given:** a directed graph G
**Find:** an ordering of the vertices so all edges go from left to right.

Uses:
Compiling multiple files
Graduating.

# Topological Ordering

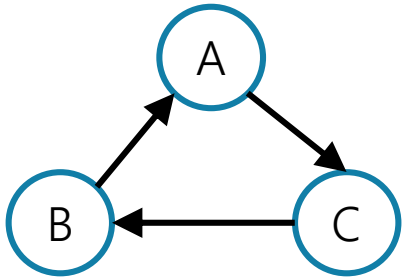
A course prerequisite chart and a possible topological ordering.



# Can we always order a graph?

Can you topologically order this graph?



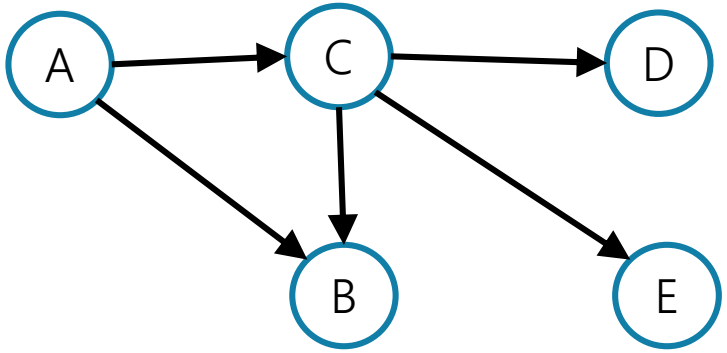

**Directed Acyclic Graph (DAG)**

A directed graph without any cycles.

A graph has a topological ordering if and only if it is a DAG.

# Ordering a DAG

Does this graph have a topological ordering? If so find one.




If a vertex doesn't have any edges going into it, we can add it to the ordering.
More generally, if the only incoming edges are from vertices already in the ordering, it's safe to add.

# How Do We Find a Topological Ordering?

```
TopologicalSort(Graph G, Vertex source)
   count how many incoming edges each vertex has
   Collection toProcess = new Collection()
   foreach(Vertex v in G){
      if(v.edgesRemaining == 0)
         toProcess.insert(v)
   }
   topOrder = new List()
   while(toProcess is not empty){
      u = toProcess.remove()
      topOrder.insert(u)
      foreach(edge (u,v) leaving u){
         v.edgesRemaining--
         if(v.edgesRemaining == 0)
            toProcess.insert(v)
      }
   }
```

# What's the running time?

```
TopologicalSort(Graph G, Vertex source)
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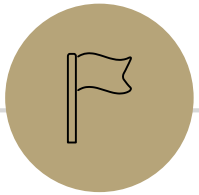
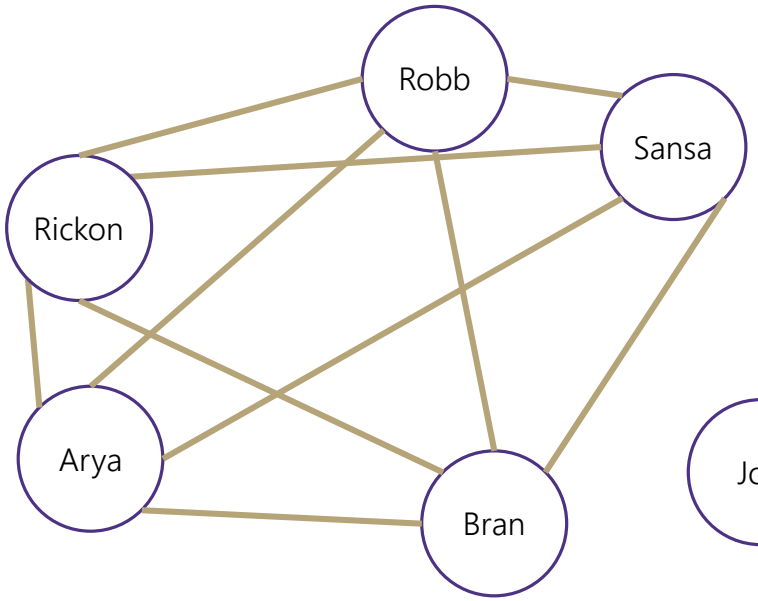


# Strongly Connected Components

# Connected [Undirected] Graphs

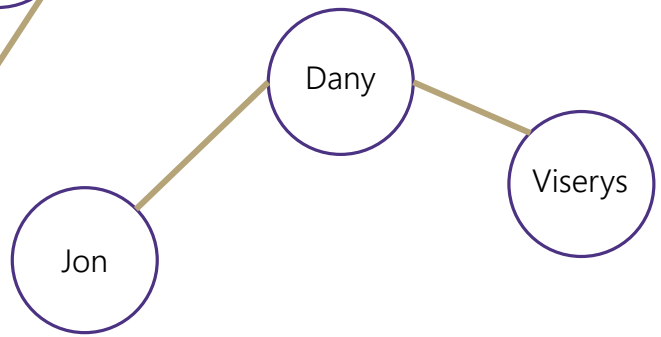
**Connected graph** – a graph where every vertex is connected to every other vertex via some path. It is not required for every vertex to have an edge to every other vertex

There exists some way to get from each vertex to every other vertex




**Connected Component** – a *subgraph* in which any two vertices are connected via some path, but is connected to no additional vertices in the *supergraph*

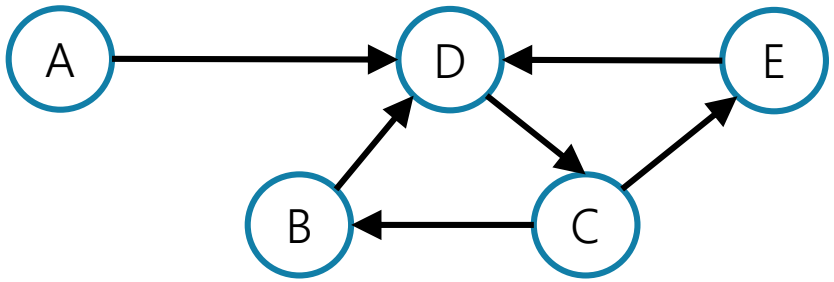
- There exists some way to get from each vertex within the connected component to every other vertex in the connected component
- A vertex with no edges is itself a connected component



# Strongly Connected Components

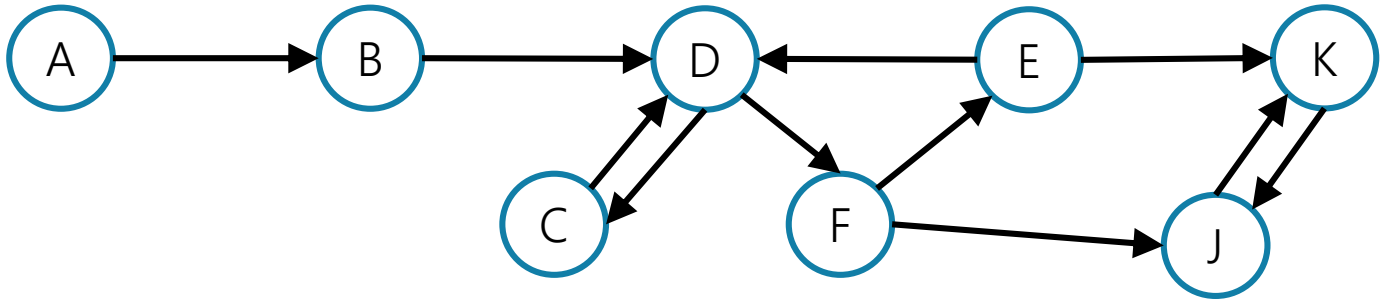
**Strongly Connected Component**

A subgraph C such that every pair of vertices in C is connected via some path **in both directions,** and there is no other vertex which is connected to every vertex of C in both directions.




Note: the direction of the edges matters!

# Strongly Connected Components Problem




{A}, {B}, {C,D,E,F}, {J,K}

**Strongly Connected Components Problem**

**Given**: A directed graph G
**Find**: The strongly connected components of G

# SCC Algorithm

Ok. How do we make a computer do this?

You could:

- run a [B/D]FS from every vertex,
- For each vertex record what other vertices it can get to
- and figure it out from there.

But you can do better. There's actually an O(|V|+|E|) algorithm!

I only want you to remember two things about the algorithm:

- It is an application of depth first search.
- It runs in linear time

The problem with running a [B/D]FS from every vertex is you recompute a lot of information.

The time you are popped off the stack in DFS contains a "smart" ordering to do a second DFS where you don't need to recompute that information.

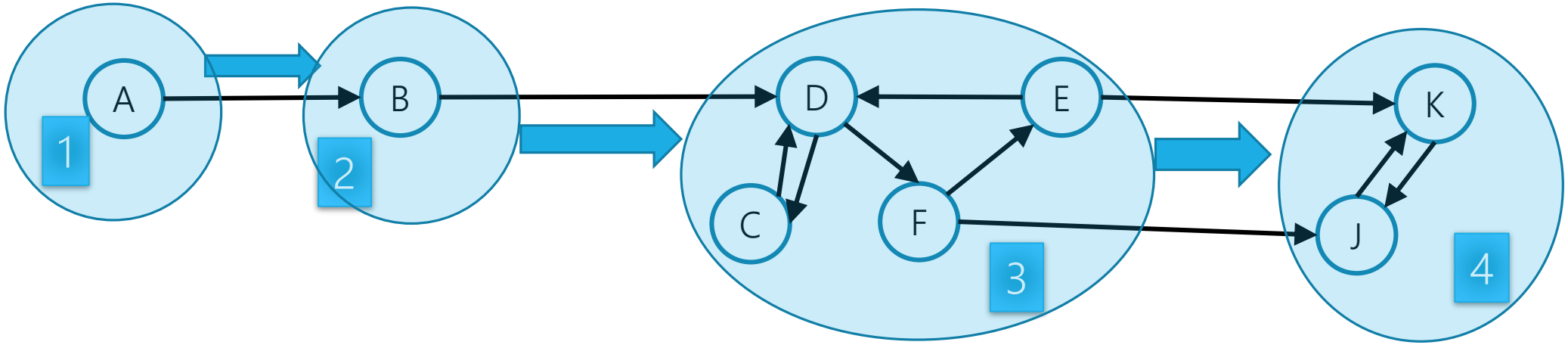
# Why Find SCCs?

Graphs are useful because they encode relationships between arbitrary objects.

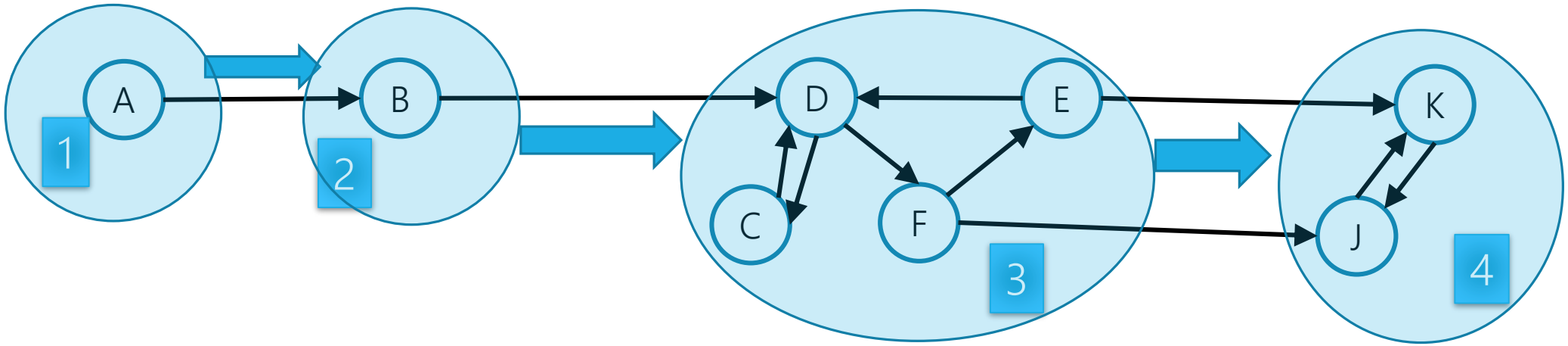
We've found the strongly connected components of G.

Let's build a new graph out of them! Call it H

- Have a vertex for each of the strongly connected components
- Add an edge from component 1 to component 2 if there is an edge from a vertex inside 1 to one inside 2.



# Why Find SCCs?




That's awful meta. Why?

This new graph summarizes reachability information of the original graph.

- I can get from A (of G) in 1 to F (of G) in 3 if and only if I can get from 1 to 3 in H.

# Why Must H Be a DAG?

H is always a DAG (do you see why?).

# Takeaways

Finding SCCs lets you **collapse** your graph to the meta-structure.
If (and only if) your graph is a DAG, you can find a topological sort of your graph.

Both of these algorithms run in linear time.

Just about everything you could want to do with your graph will take at least as long.

You should think of these as **"almost free" preprocessing** of your graph.

- Your other graph algorithms only need to work on
  - topologically sorted graphs and
  - strongly connected graphs.

# A Longer Example

The best way to really see why this is useful is to do a bunch of examples.

Take CSE 417 for that. The second best way is to see one example right now…

This problem doesn't *look like* it has anything to do with graphs

- no maps
- no roads
- no social media friendships

Nonetheless, a graph representation is the best one.

I don't expect you to remember this problem.

I just want you to see

- graphs can show up anywhere.
- SCCs and Topological Sort are useful algorithms.

# Example Problem: Final Creation

We have a long list of types of problems we might want to put on the final.

- Heap insertion problem, big-O problems, finding closed forms of recurrences, testing…

To try to make you all happy, we might ask for your preferences. Each of you gives us two preferences of the form "I [do/don't] want a [] problem on the final" *

We'll assume you'll be happy if you get at least one of your two preferences.

**Final Creation Problem**

**Given**: A list of 2 preferences per student.
**Find**: A set of questions so every student gets at least one of their preferences (or accurately report no such question set exists).

*This is NOT how Kasey is making the final.

# Final Creation: Take 1

We have Q kinds of questions and S students.

What if we try every possible combination of questions.

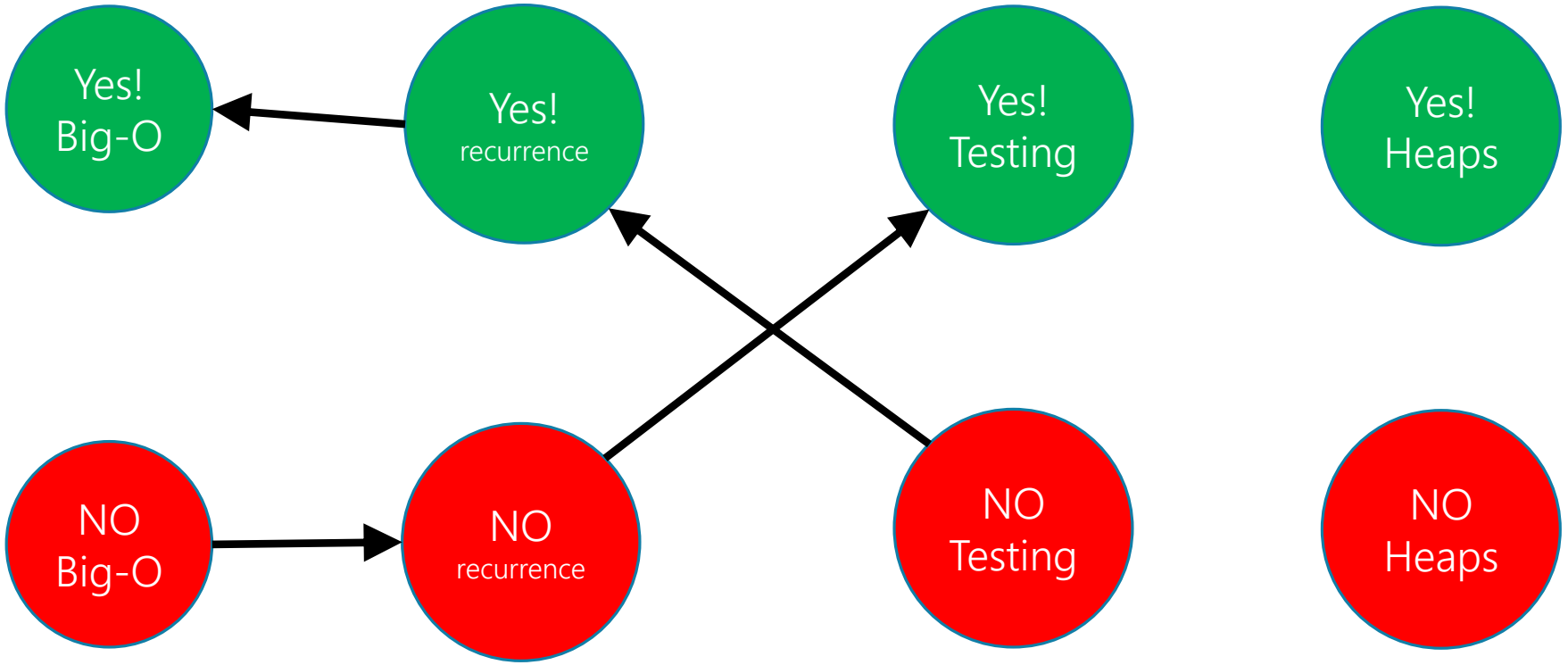
How long does this take? O($2^Q S$)

If we have a lot of questions, that's **really** slow.

# Final Creation: Take 2

Each student introduces new relationships for data:

Let's say your preferences are represented by this table:




| Problem | YES | NO |
|---|---|---|
| Big-O | X | |
| Recurrence | | X |
| Testing | | |
| Heaps | | |

| Problem | YES | NO |
|---|---|---|
| Big-O | | |
| Recurrence | X | |
| Testing | X | |
| Heaps | | |

If we don't include a big-O proof, can you still be happy?
If we do include a recurrence can you still be happy?

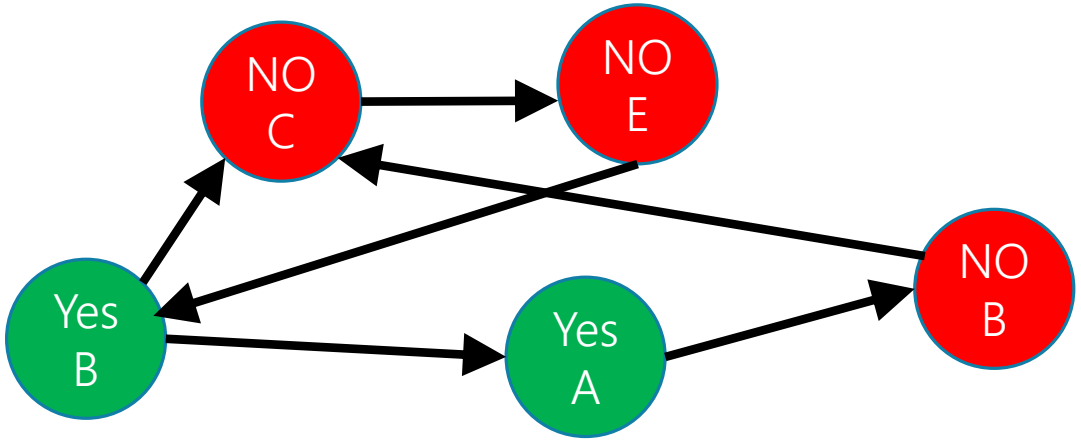
# Final Creation: Take 2

Hey we made a graph!

What do the edges mean?

- We need to avoid an edge that goes TRUE THING → FALSE THING

Let's think about a single SCC of the graph.




Can we have a true and false statement in the same SCC?

What happens now that Yes B and NO B are in the same SCC?

# Final Creation: SCCs

The vertices of a SCC must either be all true or all false.

**Algorithm Step 1:** Run SCC on the graph. Check that each question-type-pair are in different SCC.

Now what? Every SCC gets the same value.

- Treat it as a single object!

We want to avoid edges from true things to false things.

- "Trues" seem more useful for us at the end.

Is there some way to start from the end?

YES! Topological Sort

# Making the Final

Algorithm:
Make the requirements graph.

Find the SCCs.

If any SCC has including and not including a problem, we can't make the final.

Run topological sort on the graph of SCC.

Starting from the end:

- if everything in a component is unassigned, set them to true, and set their opposites to false.
- Else If one thing in a component is assigned, assign the same value to the rest of the nodes in the component and the opposite value to their opposites.

This works!!

How fast is it?

O(Q + S). That's a HUGE improvement.

# Some More Context

The Final Making Problem was a type of "Satisfiability" problem.

We had a bunch of variables (include/exclude this question), and needed to satisfy everything in a list of requirements.

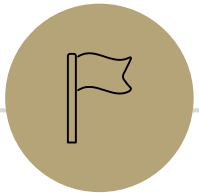
SAT is a general way to encode lots of hard problems.

Because every requirement was "do at least one of these 2" this was a 2-SAT instance.

If we change the 2 into a 3, no one knows an algorithm that runs efficiently.

And finding one (or proving one doesn't exist) has a $1,000,000 prize.

If we get to P vs. NP at the end of the quarter Kasey will tell you more.



# Appendix: Strongly Connected Components Algorithm

# Efficient SCC

We'd like to find all the vertices in our strongly connected component in time corresponding to the size of the component, not for the whole graph.

We can do that with a DFS (or BFS) as long as we don't leave our connected component.

If we're a "sink" component, that's guaranteed. I.e. a component whose vertex in the meta-graph has no outgoing edges.

How do we find a sink component? We don't have a meta-graph yet (we need to find the components first)

DFS can find a vertex in a source component, i.e. a component whose vertex in the meta-graph has no incoming edges.

- That vertex is the last one to be popped off the stack.

So if we run DFS in the *reversed* graph (where each edge points the opposite direction) we can find a sink component.

# Efficient SCC

So from a DFS in the reversed graph, we can use the order vertices are popped off the stack to find a sink component (in the original graph).

Run a DFS from that vertex to find the vertices in that component *in size of that component time.*

Now we can delete the edges coming into that component.

The last remaining vertex popped off the stack is a sink of the remaining graph, and now a DFS from them won't leave the component.

Iterate this process (grab a sink, start DFS, delete edges entering the component).

In total we've run two DFSs. (since we never leave our component in the second DFS).

More information, and pseudocode:

https://en.wikipedia.org/wiki/Kosaraju%27s_algorithm

http://jeffe.cs.illinois.edu/teaching/algorithms/notes/19-dfs.pdf (mathier)