

# Shortest Paths

Data Structures and Algorithms

# Announcements

Hey, you're not Kasey.

I'm Robbie

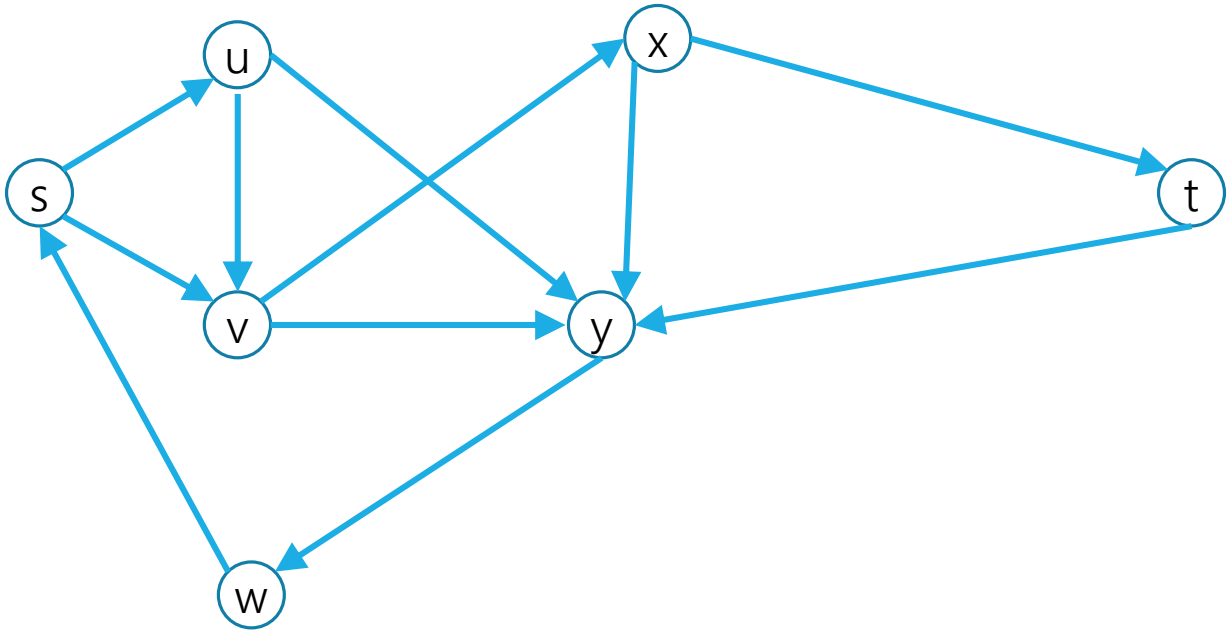
- Kasey's in Europe. She'll be back Monday.
- Kasey will be triaging email (but don't expect immediate responses)
- Office hours will shift a little (check the calendar on the webpage later in the week.)

# Warm Up

Run Breadth First Search on this graph starting from s.

What order are vertices placed on the queue?

When processing a vertex insert neighbors in alphabetical order.




```
search(graph)
    mark all vertices as unknown
    toVisit.enqueue(first vertex)
    while(toVisit is not empty)
        current = toVisit.dequeue()
        for (v : current.outNeighbors())
            if (v is unknown){
                toVisit.enqueue(v)
                mark v as known
        visited.add(current)
```

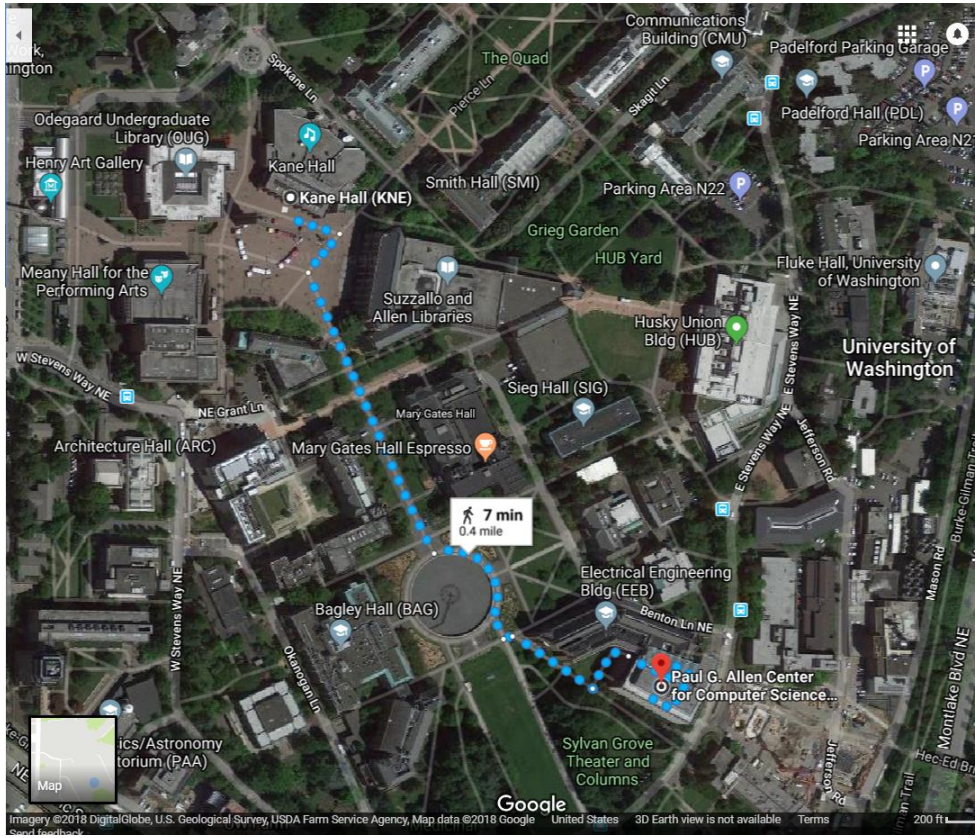
# This Week

Last week:
Kasey showed you a new object – graphs

This week:
How do we actually use graphs to solve problems?

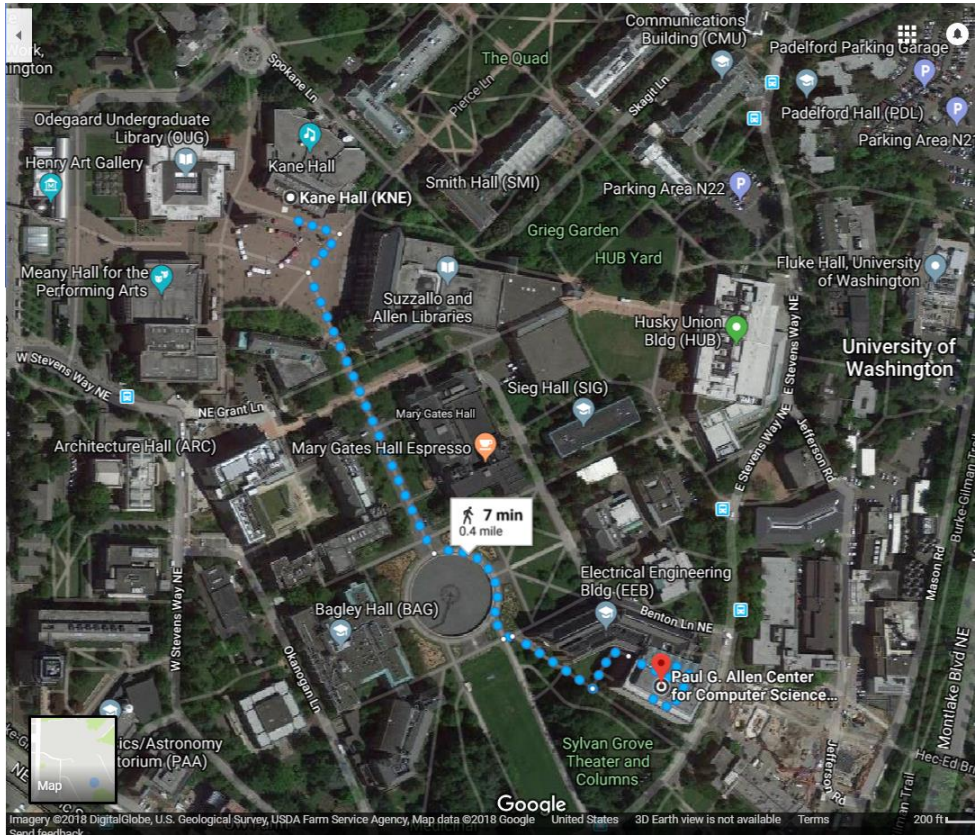
# Shortest Paths

How does Google Maps figure out this is the fastest way to get to office hours?

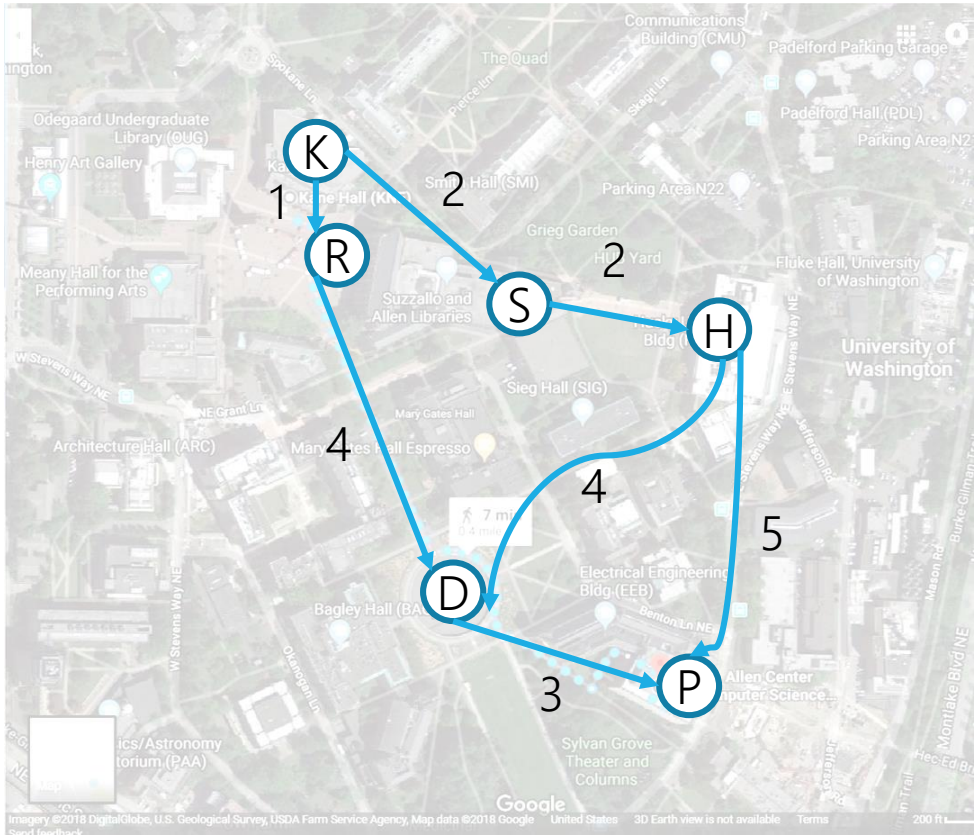


# Representing Maps as Graphs

How do we represent a map as a graph? What are the vertices and edges?



# Representing Maps as Graphs



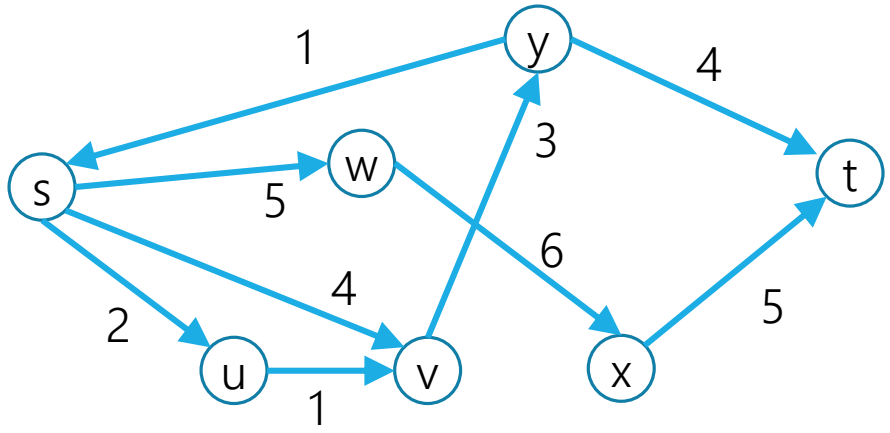
# Shortest Paths

The **length** of a path is the sum of the edge weights on that path.

**Shortest Path Problem**

**Given:** a directed graph G and vertices s and t

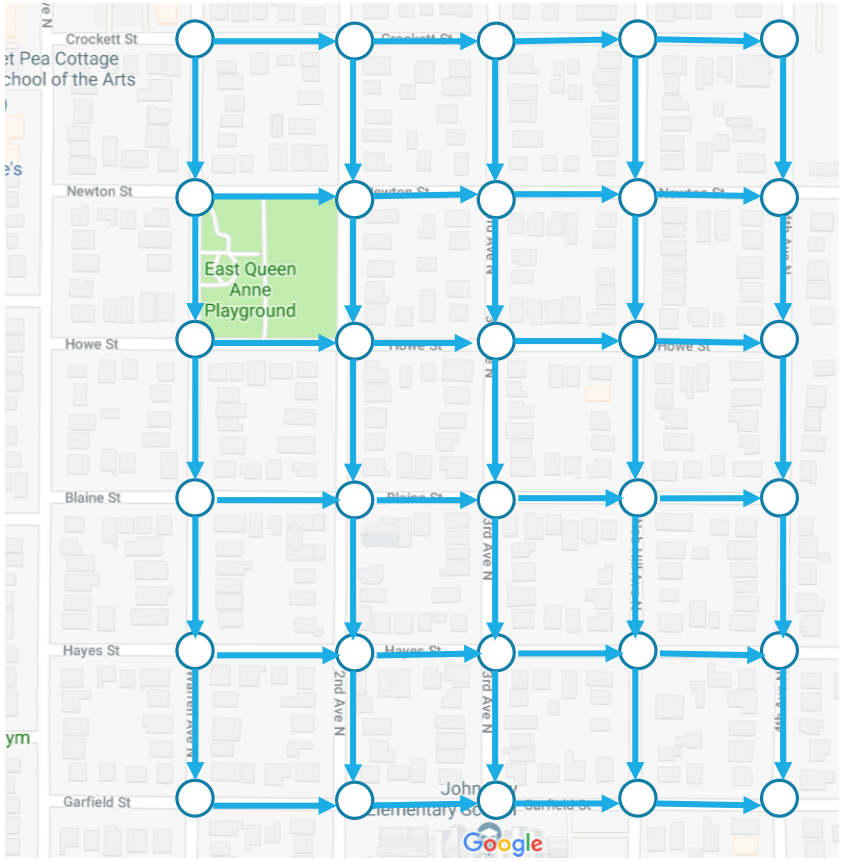
**Find:** the shortest path from s to t



# Unweighted graphs

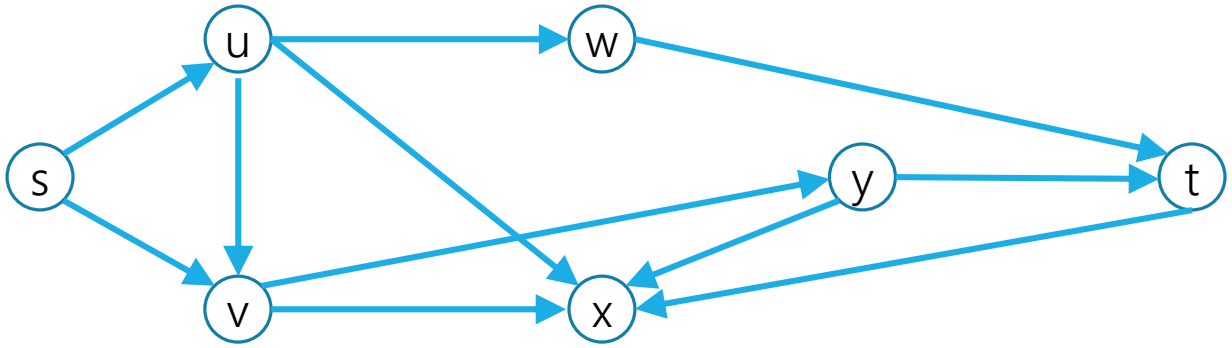
Let's start with a simpler version: the edges are all the same weight (**unweighted**)

If the graph is unweighted, how do we find a shortest paths?



# Unweighted Graphs

If the graph is unweighted, how do we find a shortest paths?




What's the shortest path from s to s?

- Well….we're already there.

What's the shortest path from s to u or v?

- Just go on the edge from s

From s to w,x, or y?

- Can't get there directly from s, if we want a length 2 path, have to go through u or v.

# Unweighted Graphs: Key Idea

To find the set of vertices at distance k, just find the set of vertices at distance k-1, and see if any of them have an outgoing edge to an undiscovered vertex.

Do we already know an algorithm that does something like that?

Yes! BFS!

```
bfsShortestPaths(graph G, vertex source)
   toVisit.enqueue(source)
    source.dist = 0
   while(toVisit is not empty){
      current = toVisit.dequeue()
      for (v : current.outNeighbors())
       {
         if (v is unknown){
              v.distance = current.distance + 1
              v.predecessor = current
             toVisit.enqueue(v)
              mark v as known
          }
       }
}
```

# Unweighted Graphs

If the graph is unweighted, how do we find a shortest paths?

```
bfsShortestPaths(graph G, vertex source)
    toVisit.enqueue(source)
    source.dist = 0
    while(toVisit is not empty){
        current = toVisit.dequeue()
        for (v : current.outNeighbors())
         {
            if (v is unknown){
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                v.predecessor = current
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                mark v as known
             }
         }
    }
```
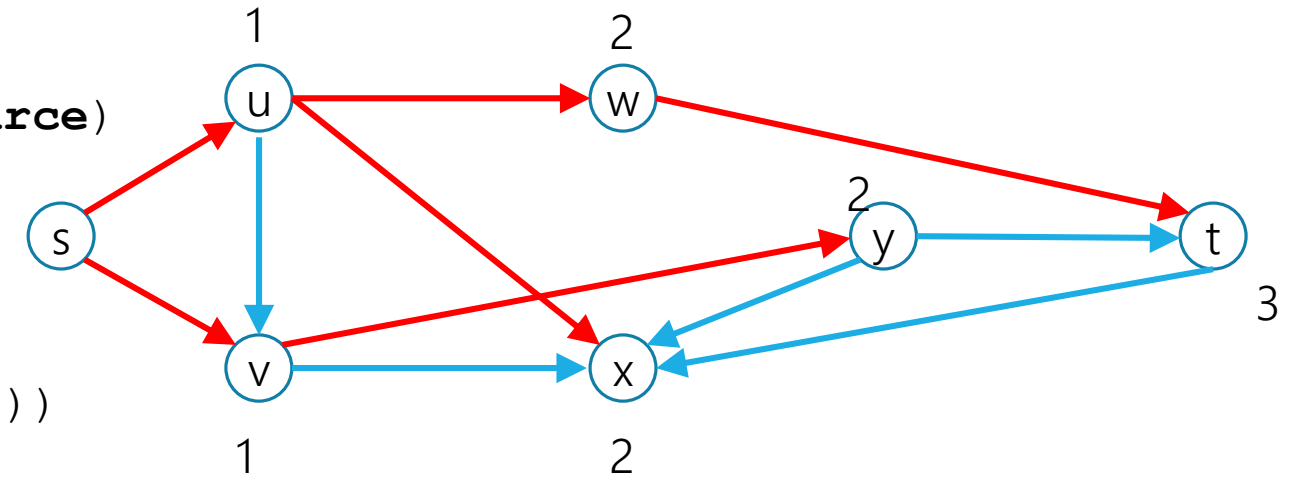



# What about the target vertex?

**Shortest Path Problem**

**Given:** a directed graph G and vertices s,t
**Find:** the shortest path from s to t.

BFS didn't mention a target vertex…
It actually finds the shortest path from s to every other vertex.

# Weighted Graphs

Each edge should represent the “time” or “distance” from one vertex to another.

Sometimes those aren’t uniform, so we put a weight on each edge to record that number.

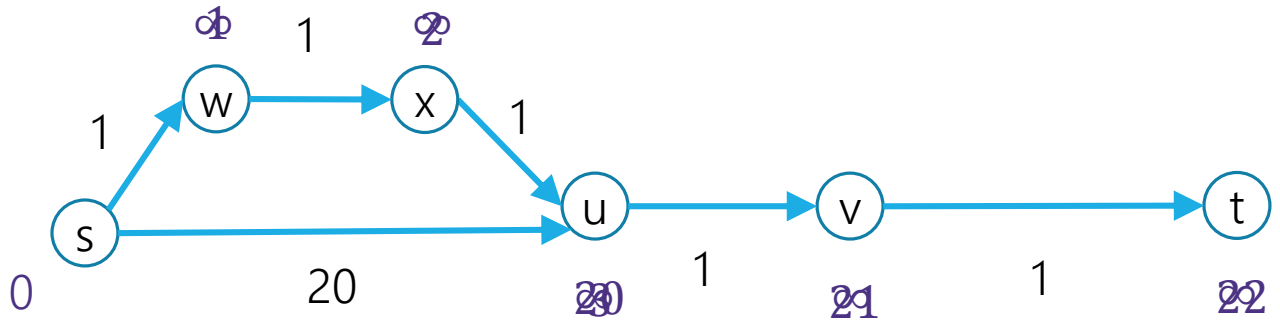
The length of a path in a weighted graph is the sum of the weights along that path.

We’ll assume all of the weights are positive

- For GoogleMaps that definitely makes sense.
- Sometimes negative weights make sense. **Today’s algorithm doesn’t work for those graphs**
- There are other algorithms that do work.

# Weighted Graphs: Take 1

BFS works if the graph is unweighted. Maybe it just works for weighted graphs too?




What went wrong? When we found a shorter path from s to u, we needed to update the distance to v (and anything whose shortest path went through u) but BFS doesn't do that.

# Weighted Graphs: Take 2

**Reduction (informally)**

Using an algorithm for Problem B to solve Problem A.

You already do this all the time.

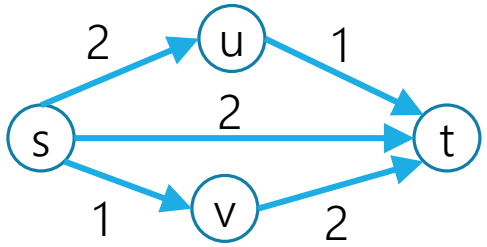
In project 2, you reduced implementing a hashset to implementing a hashmap.

Any time you use a library, you're reducing your problem to the one the library solves.

Can we reduce finding shortest paths on weighted graphs to finding them on unweighted graphs?

# Weighted Graphs: A Reduction

Given a weighted graph, how do we turn it into an unweighted one without messing up the edge lengths?



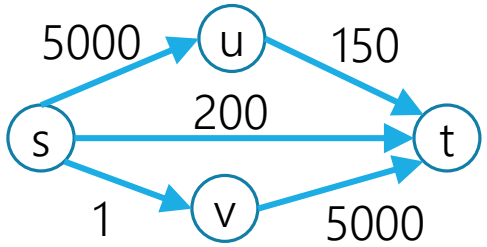

Transform Input

Unweighted Shortest Paths

Transform Output

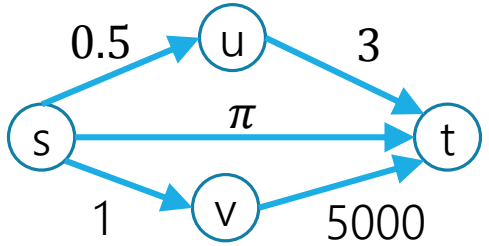
# Weighted Graphs: A Reduction

What is the running time of our reduction on this graph?




O(|V|+|E|) of the modified graph, which is…slow.

Does our reduction even work on this graph?




Ummm….

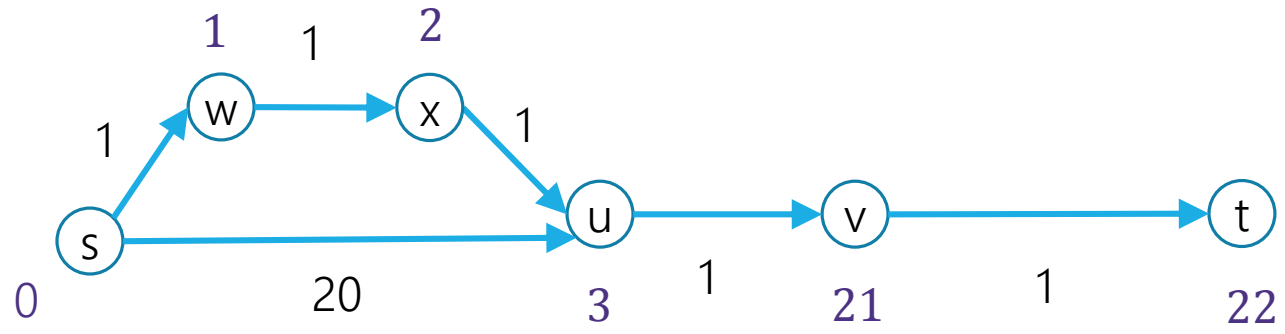
Tl;dr: If your graph's weights are all small positive integers, this reduction might work great. Otherwise we probably need a new idea.

# Weighted Graphs: Take 3

So we can't just do a reduction.

Instead let's try to figure out why BFS worked in the unweighted case, and try to make the same thing happen in the weighted case.

Why did BFS work on unweighted graphs? How did we avoid this problem:




When we used a vertex u to update shortest paths we already knew the exact shortest path to u. So we never ran into the update problem

So if we process the vertices in order of distance from s, we have a chance.

# Weighted Graphs: Take 3

Goal: Process the vertices in order of distance from s

Idea:

Have a set of vertices that are "known"

- (we know at least one path from s to them).

Record an estimated distance

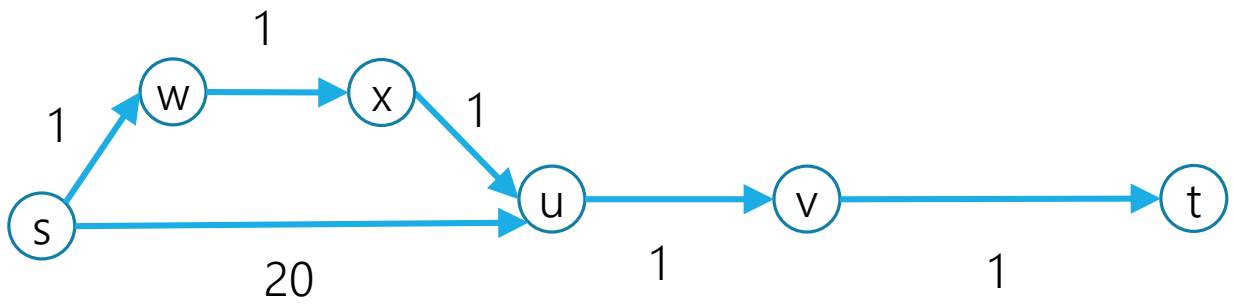
- (the best way we know to get to each vertex).

If we process only the vertex closest in estimated distance, we won't ever find a shorter path to a processed vertex.

# Dijkstra's Algorithm

```
Dijkstra(Graph G, Vertex source)
    initialize distances to ∞
    mark all vertices unprocesed
    while(there are unprocessed vertices){
        let u be the closest unprocessed vertex
        foreach(edge (u,v) leaving u){
            if(u.dist+w(u,v) < v.dist){
                v.dist = u.dist+w(u,v)
                v.predecessor = u
            }
        }
       mark u as processed
    }
```

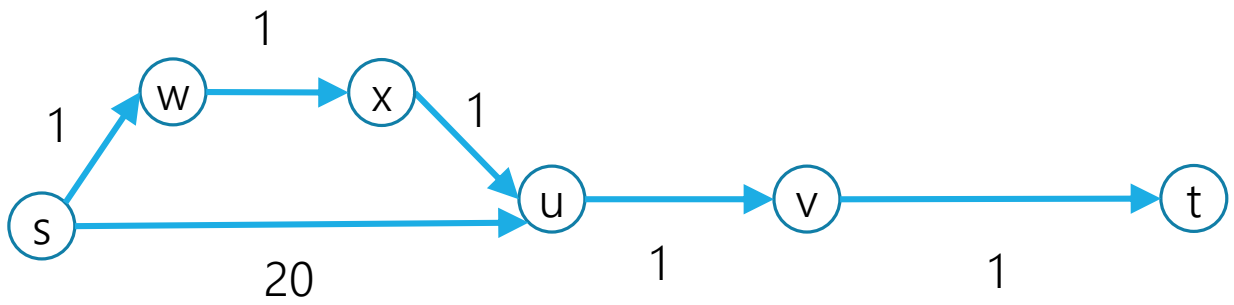
| Vertex | Distance | Predecessor | Processed |
|---|---|---|---|
| s | | | |
| w | | | |
| x | | | |
| u | | | |
| v | | | |
| t | | | |



# Dijkstra's Algorithm

```
Dijkstra(Graph G, Vertex source)
    initialize distances to ∞
    mark source as distance 0
    mark all vertices unprocessed
    while(there are unprocessed vertices){
        let u be the closest unprocessed vertex
        foreach(edge (u,v) leaving u){
            if(u.dist+w(u,v) < v.dist){
                v.dist = u.dist+w(u,v)
                v.predecessor = u
            }
        }
       mark u as processed
    }
```

| Vertex | Distance | Predecessor | Processed |
|---|---|---|---|
| s | 0 | -- | Yes |
| w | 1 | s | Yes |
| x | 2 | w | Yes |
| u | 3 | x | Yes |
| v | 4 | u | Yes |
| t | 5 | v | Yes |



# Implementation Details

One of those lines of pseudocode was a little sketchy

```
> let u be the closest unprocessed vertex
```

What ADT have we talked about that might work here?

Minimum Priority Queues!

**Min Priority Queue ADT**

**state**

Set of comparable values
- Ordered based on "priority"

**behavior**

**removeMin()** – returns the element with the smallest priority, removes it from the collection

**peekMin()** – find, but do not remove the element with the smallest priority

**insert(value)** – add a new element to the collection

# Making Minimum Priority Queues Work

They won't quite work "out of the box".

We don't have an update priority method. Can we add one?

- Percolate up!

To percolate u's entry in the heap up we'll have to get to it.

- Each vertex need pointer to where it appears in the priority queue
- I'm going to ignore this point for the rest of the lecture.

**Min Priority Queue ADT**

**state**

Set of comparable values
- Ordered based on "priority"

**behavior**

**removeMin()** – returns the element with the smallest priority, removes it from the collection

**peekMin()** – find, but do not remove the element with the smallest priority

**insert(value)** – add a new element to the collection

**DecreasePriority(e, p)** – decreases the priority of element e down to p.

# Running Time Analysis

```
Dijkstra(Graph G, Vertex source)
   initialize distances to ∞, source.dist to 0
   mark all vertices unprocessed
   initialize MPQ as a Min Priority Queue
   add source at priority 0
   while(MPQ is not empty){
      u = MPQ.getMin()
      foreach(edge (u,v) leaving u){
          if(u.dist+w(u,v) < v.dist){
             if(v.dist == ∞ )
                MPQ.insert(v, u.dist+w(u,v))
             else
                MPQ.decreasePriority(v, u.dist+w(u,v))
             v.dist = u.dist+w(u,v)
             v.predecessor = u
         }
      }
     mark u as processed
   }
```

# Another Application of Shortest Paths

Shortest path algorithms are obviously useful for GoogleMaps.

The wonderful thing about graphs is they can encode **arbitrary** relationships among objects.

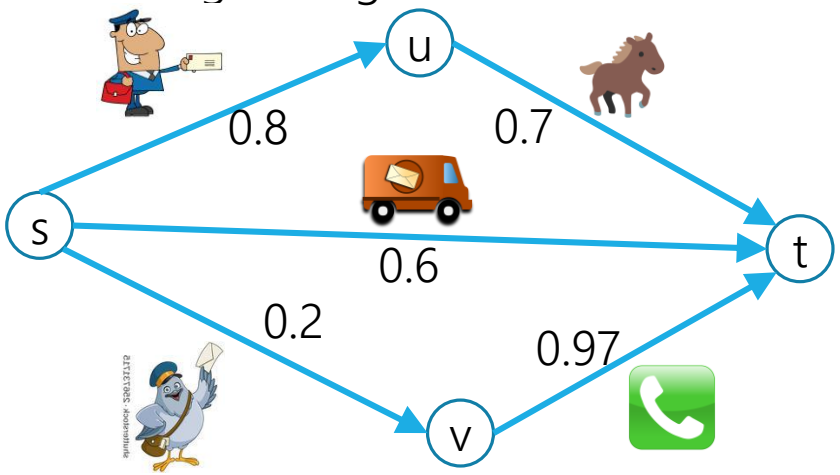
I don't care if you remember this problem

I don't care if you remember how we apply shortest paths.

I just want you to see that these algorithms have non-obvious applications.

# Another Application of Shortest Paths

I have a message I need to get from point s to point t.
But the connections are unreliable.
What path should I send the message along so it has the best chance of arriving?




**Maximum Probability Path**

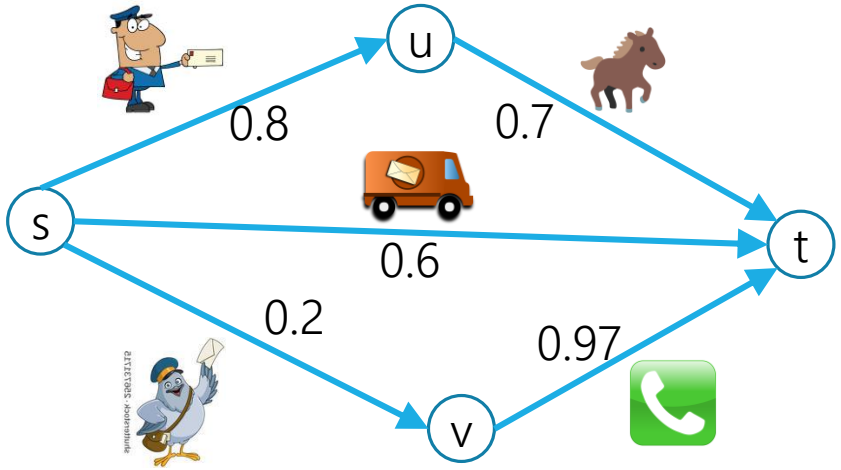
**Given:** a directed graph G, where each edge weight is the probability of successfully transmitting a message across that edge
**Find:** the path from s to t with maximum probability of message transmission

# Another Application of Shortest Paths

Let each edge's weight be the probability a message is sent successfully across the edge.

What's the probability we get our message all the way across a path?

- It's the product of the edge weights.

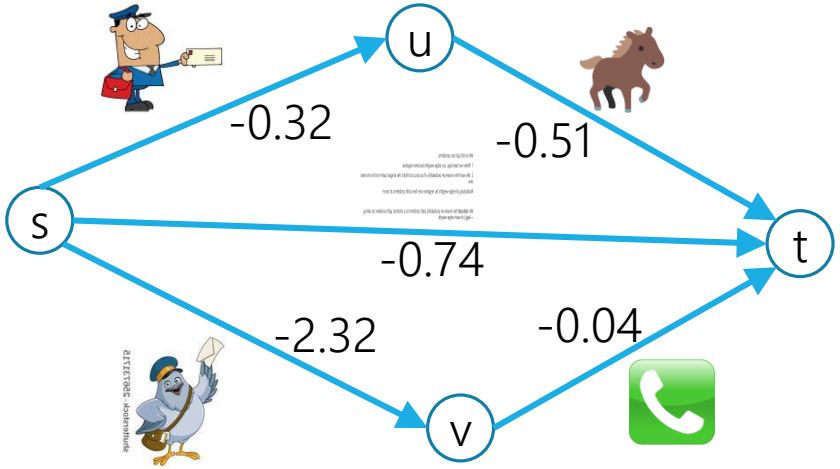



We only know how to handle sums of edge weights.

Is there a way to turn products into sums?

$\log(ab) = \log a + \log b$

# Another Application of Shortest Paths




We've still got two problems.

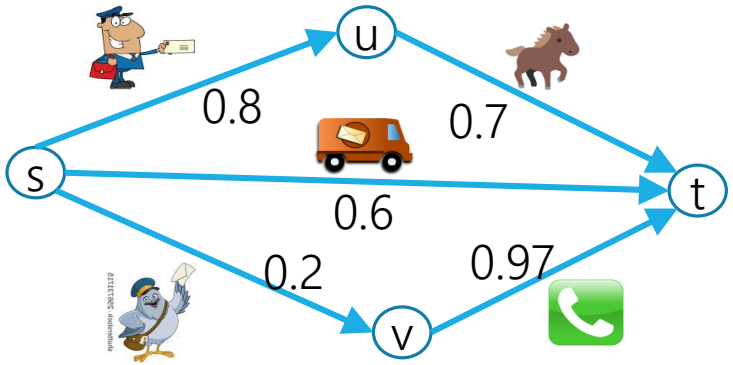
1. When we take logs, our edge weights become negative.

2. We want the *maximum* probability of success, but that's the longest path not the shortest one.

Multiplying all edge weights by negative one fixes both problems at once!

We **reduced** the maximum probability path problem to a shortest path problem by taking $-\log()$ of each edge weight.

# Maximum Probability Path Reduction




Transform Input

Weighted Shortest Paths

Transform Output