CSE 373: Data Structures and Algorithms

Disjoint Sets

Autumn 2018

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Announcements

Final exam info (logistics, topics covered, and practice material) is on course website.

Homework 6 is due this Friday at **noon.**

Homework 7 will be posted this Friday evening.

- Fill out the team sign up form by tomorrow 5pm to get the repo in time.
- Fill out the partner pool form by tomorrow 5pm to get assigned to a partner.

Today

Trees and Forests

Kruskal's algorithm

Disjoint Set ADT

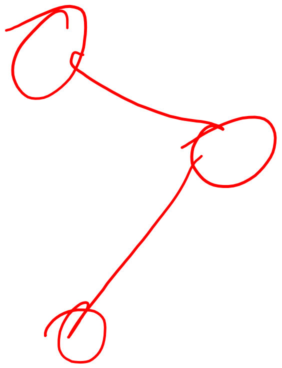
Trees and Forests

A tree is an undirected, connected, and acyclic graph.

A graph, however, can have unconnected vertices, or multiple trees.

Collection of trees is a called a forest.

A forest is any undirected and acyclic graph.
- By definition a tree is a forest



Worksheet question 1

Review

n Edges $= n-1$

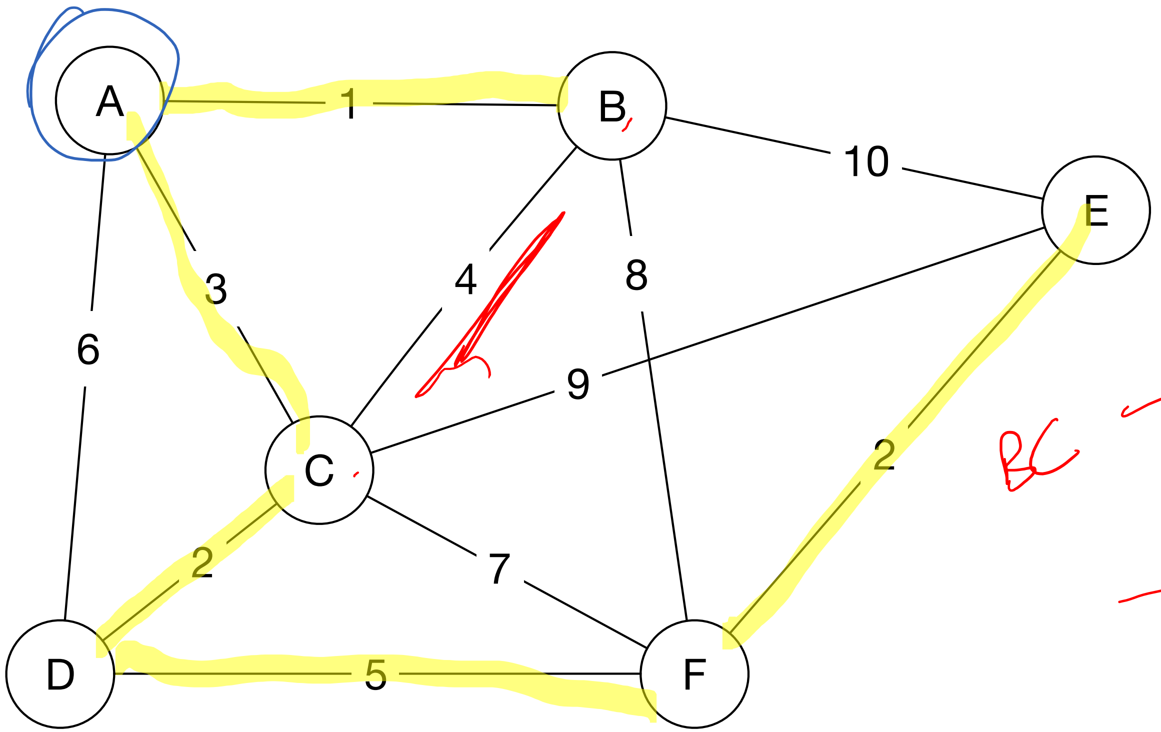
$n-1$

$|E| \propto |V|$

$|E| \quad |V|^2$

Review

Worksheet question 2a



BC

Step	Components	Edge	Include?
1	{A} {B} {C} {D} {E} {F}	A B	Yes
2	{A,B} {C} {D} — " —	D C	Yes
3	{A,B} {C,D} {E} {F}	E F	Yes
4	{E,F}	A,C	Yes No
	{A,B,C,D} {E,F}	D,F	Yes
	{A,B,C,D,E,F}	A,D	No
			No
			No
			No
			No
			No

Worksheet question 2b

Review

$|V| - 1$

function Kruskal(Graph G)
    initialize each vertex to be a component
    sort all edges by weight
    ~~**for** each edge (u, v) in sorted order~~ **do**
        if u and v are in different components **then**
            add edge (u,v) to the MST
            update u and v to be in the same component
        end if
    end for
end function

while (edgesAccepted < $|V| - 1$)
u, v = nextMinEdge()
edgesAccepted++;

Kruskal's Algorithm

```
function Kruskal(Graph G)
    initialize each vertex to be a component
    sort all edges by weight
    for each edge (u, v) in sorted order do
        if u and v are in different components then
            add edge (u,v) to the MST
            update u and v to be in the same component
        end if
    end for
end function
```

Minimum spanning forest

Given a forest, find a set of minimum spanning trees for each tree in the given forest.

Question: Which algorithm (Prims or Kruskals) would you use to find a minimum spanning forest?

Disjoint Set ADT

New ADT

Set ADT

state

Set of elements

- Elements must be unique!
- No required order

Count of Elements

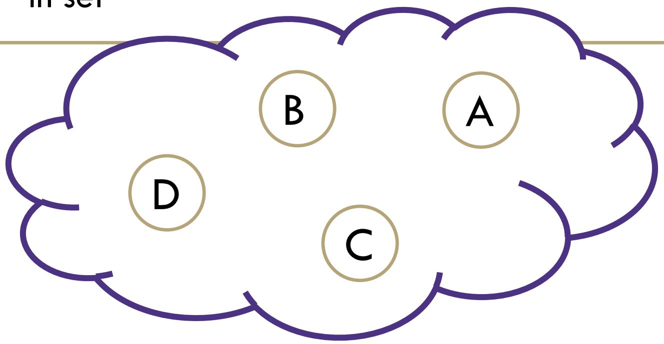
behavior

create(x) - creates a new set with a single member, x

add(x) - adds x into set if it is unique, otherwise add is ignored

remove(x) – removes x from set

size() – returns current number of elements in set



Disjoint-Set ADT

state

Set of Sets

- **Disjoint:** Elements must be unique across sets
- No required order
- Each set has representative

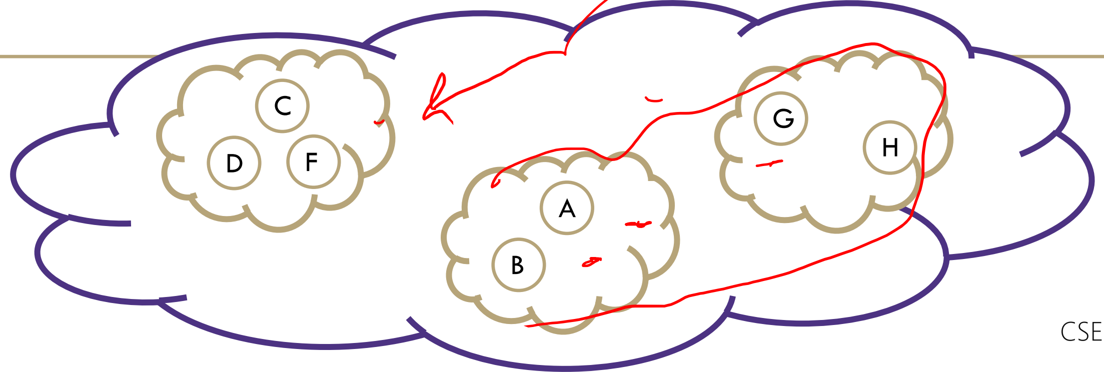
Count of Sets

behavior

makeSet(x) – creates a new set within the disjoint set where the only member is x. Picks representative for set

findSet(x) – looks up the set containing element x, returns representative of that set

union(x, y) – looks up set containing x and set containing y, combines two sets into one. Picks new representative for resulting set



Find(C)

Example

new()

makeSet(a)

makeSet(b)

makeSet(c)

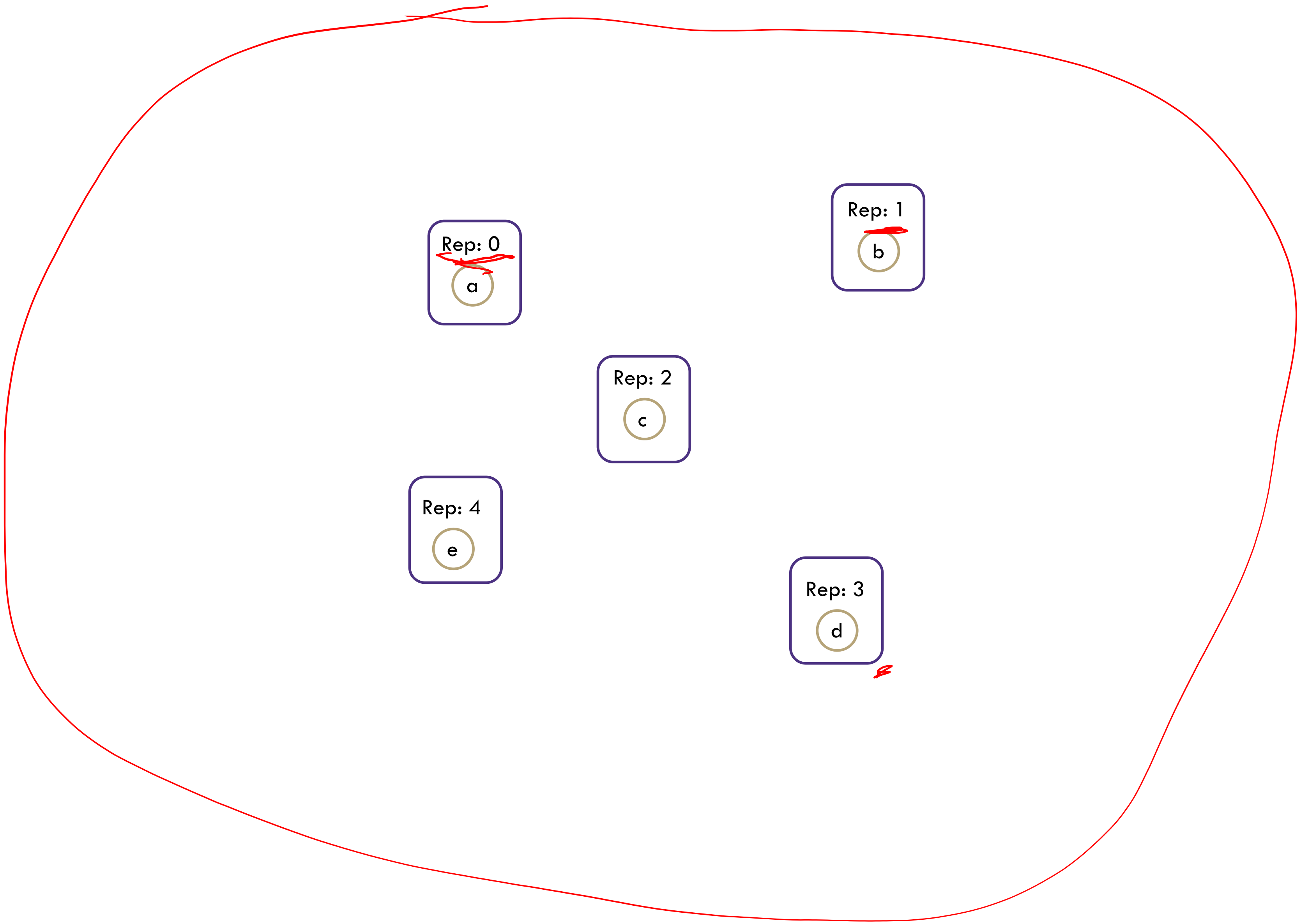
makeSet(d)

makeSet(e)

findSet(a) 0

findSet(d) 3

union(a, c)



Example

new()

makeSet(a)

makeSet(b)

makeSet(c)

makeSet(d)

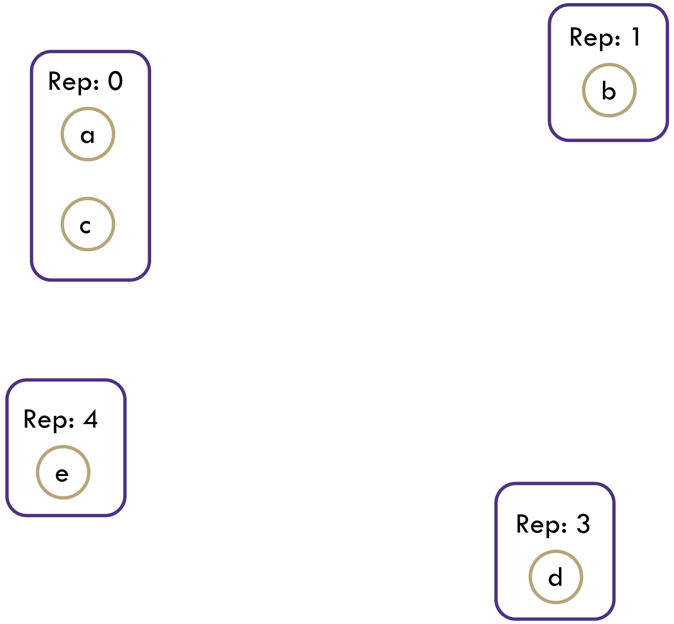
makeSet(e)

findSet(a)

findSet(d)

union(a, c)

union(b, d)



Example

new()

makeSet(a)

makeSet(b)

makeSet(c)

makeSet(d)

makeSet(e)

findSet(a)

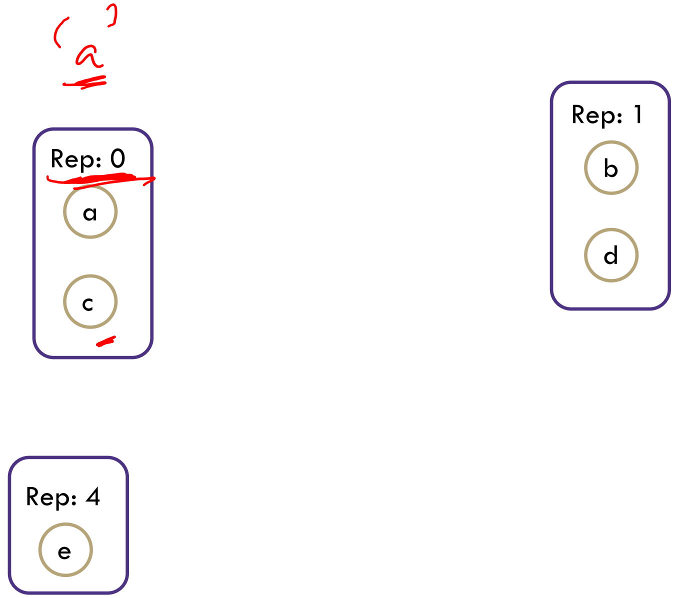
findSet(d)

union(a, c)

union(b, d) find set(c) 0

findSet(a) == findSet(c)

findSet(a) ~~==~~ findSet(d)



Implementation

Disjoint-Set ADT

state

Set of Sets
- **Disjoint:** Elements must be unique across sets
- No required order
- Each set has representative

Count of Sets

behavior

makeSet(x) – creates a new set within the disjoint set where the only member is x. Picks representative for set

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TreeDisjointSet<E>

state

`Collection<TreeSet> forest`

`Dictionary<NodeValues, NodeLocations> nodeInventory`

behavior

`makeSet(x)`-create a new tree of size 1 and add to our forest

`findSet(x)`-locates node with x and moves up tree to find root

`union(x, y)`-append tree with y as a child of tree with x

TreeSet<E>

state

`SetNode overallRoot`

behavior

`TreeSet(x)`

`add(x)`

`remove(x, y)`

`getRep()`-returns data of overallRoot

SetNode<E>

state

`E data`

`Collection<SetNode> children`

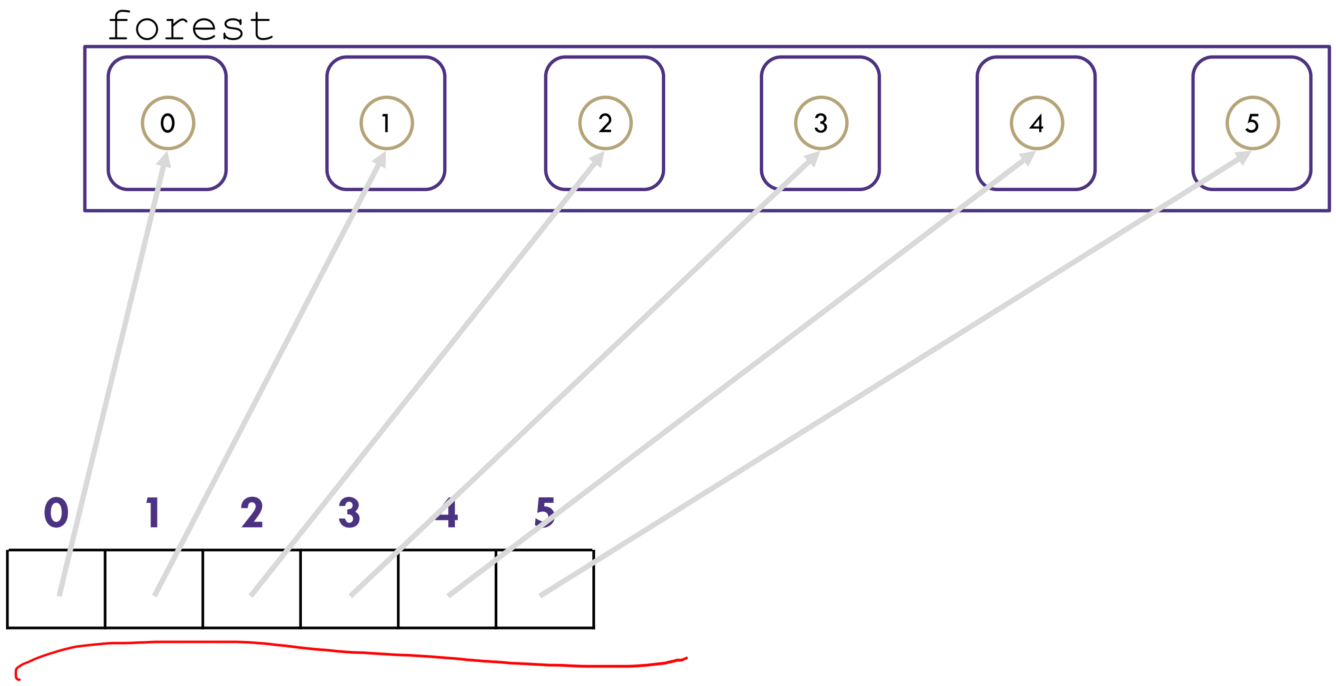
behavior

`SetNode(x)`

`addChild(x)`

`removeChild(x, y)`

Implement makeSet(x)




```
makeSet(0)
makeSet(1)
makeSet(2)
makeSet(3)
makeSet(4)
makeSet(5)
```

TreeDisjointSet<E>

state

```
Collection<TreeSet> forest
Dictionary<NodeValues,
NodeLocations> nodeInventory
```

behavior

```
makeSet(x)-create a new tree
of size 1 and add to our
forest
findSet(x)-locates node with x
and moves up tree to find root
union(x, y)-append tree with y
as a child of tree with x
```

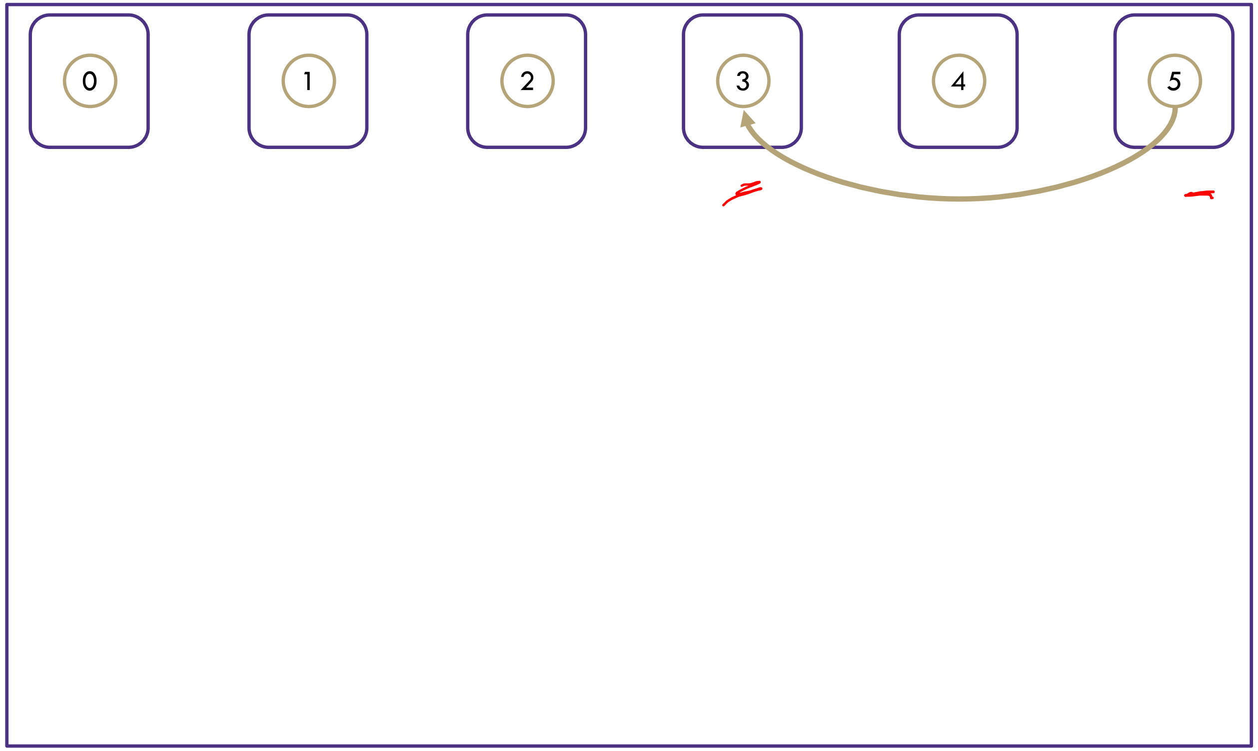
Worst case runtime?

O(1)

Implement union(x, y)

```
union(3, 5)
```

forest



0	1	2	3	4	5
->	->	->	->	->	->

TreeDisjointSet<E>

state

```
Collection<TreeSet> forest
Dictionary<NodeValues,
NodeLocations> nodeInventory
```

behavior

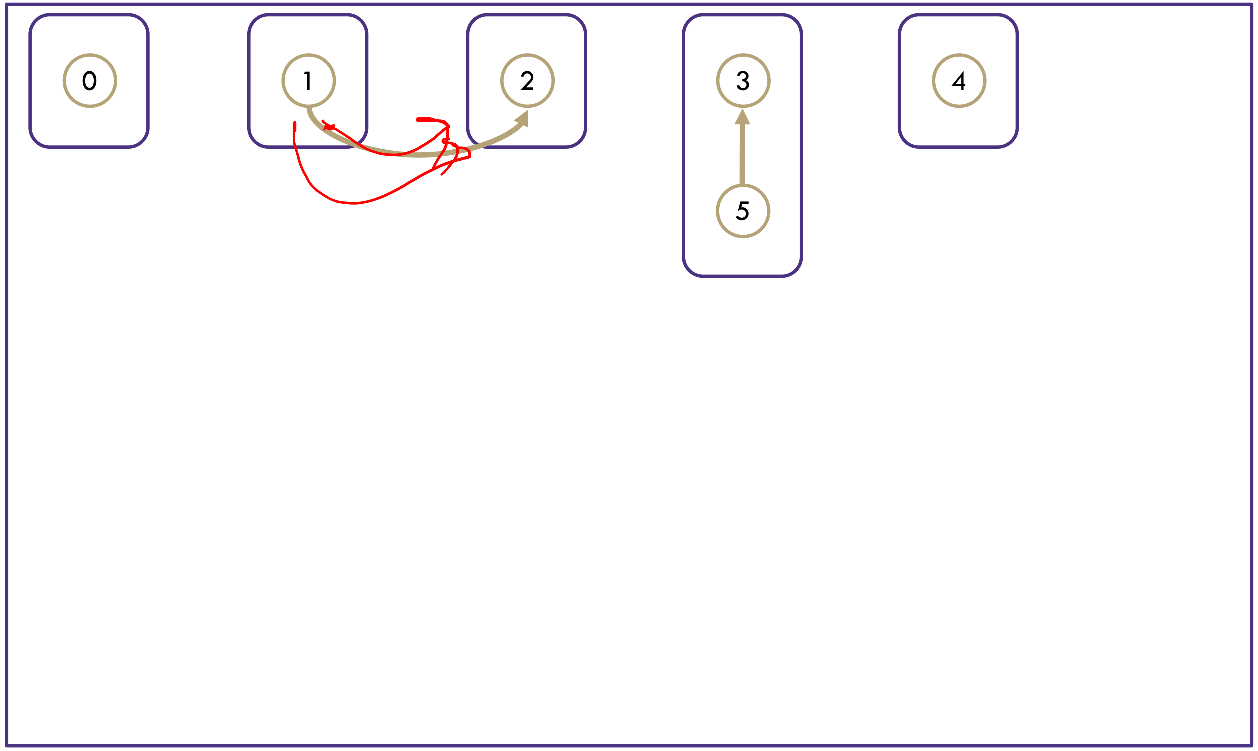
```
makeSet(x)-create a new tree
of size 1 and add to our
forest
findSet(x)-locates node with x
and moves up tree to find root
union(x, y)-append tree with y
as a child of tree with x
```

Implement union(x, y)

```
union(3, 5)

union(2, 1)
```

forest



0	1	2	3	4	5
->	->	->	->	->	->

TreeDisjointSet<E>

state

```
Collection<TreeSet> forest
Dictionary<NodeValues,
NodeLocations> nodeInventory
```

behavior

```
makeSet(x)-create a new tree
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```
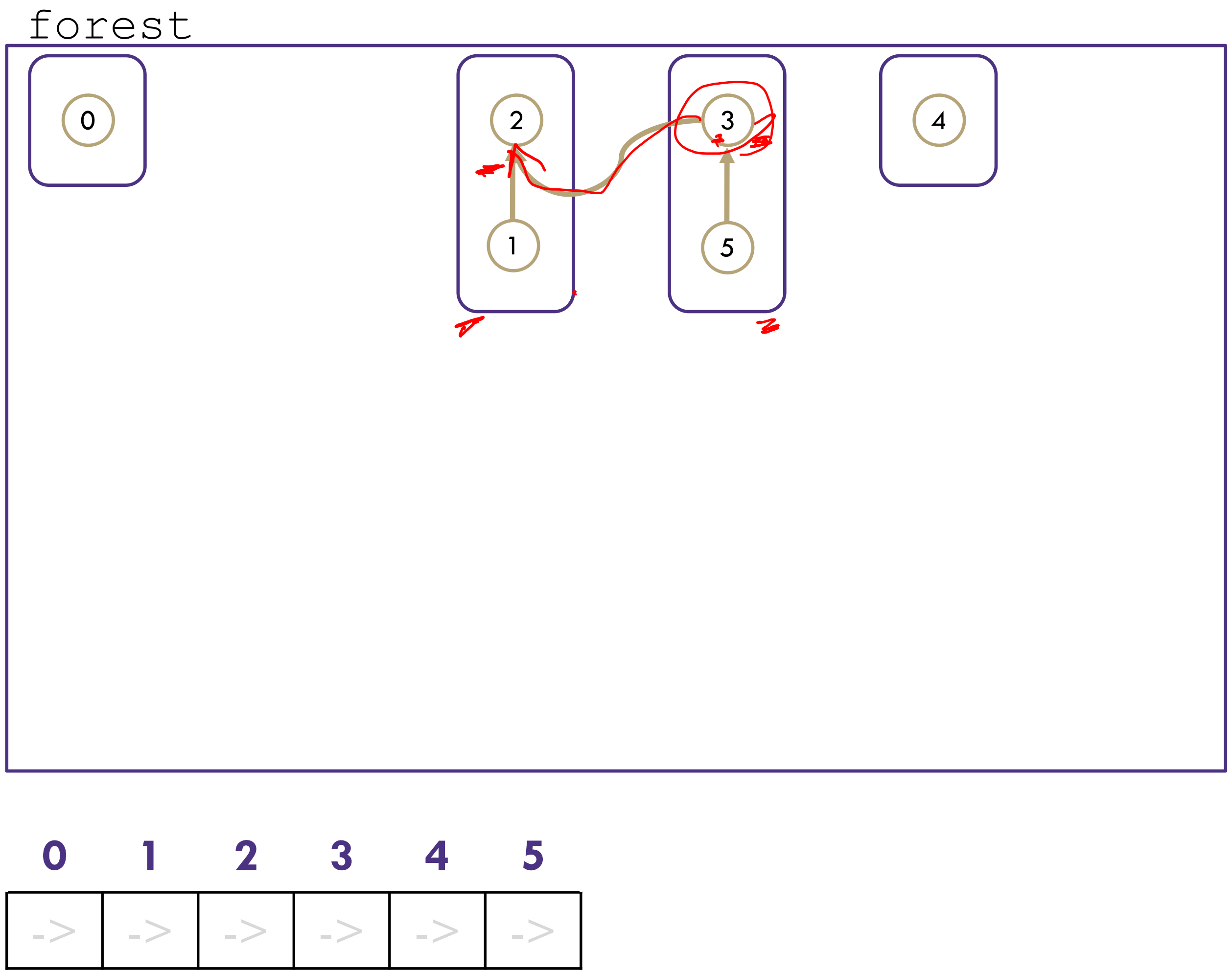

Implement union(x, y)

```
union(3, 5)

union(2, 1)

union(2, 5)
```



TreeDisjointSet<E>

state

Collection<TreeSet> forest

Dictionary<NodeValues, NodeLocations> nodeInventory

behavior

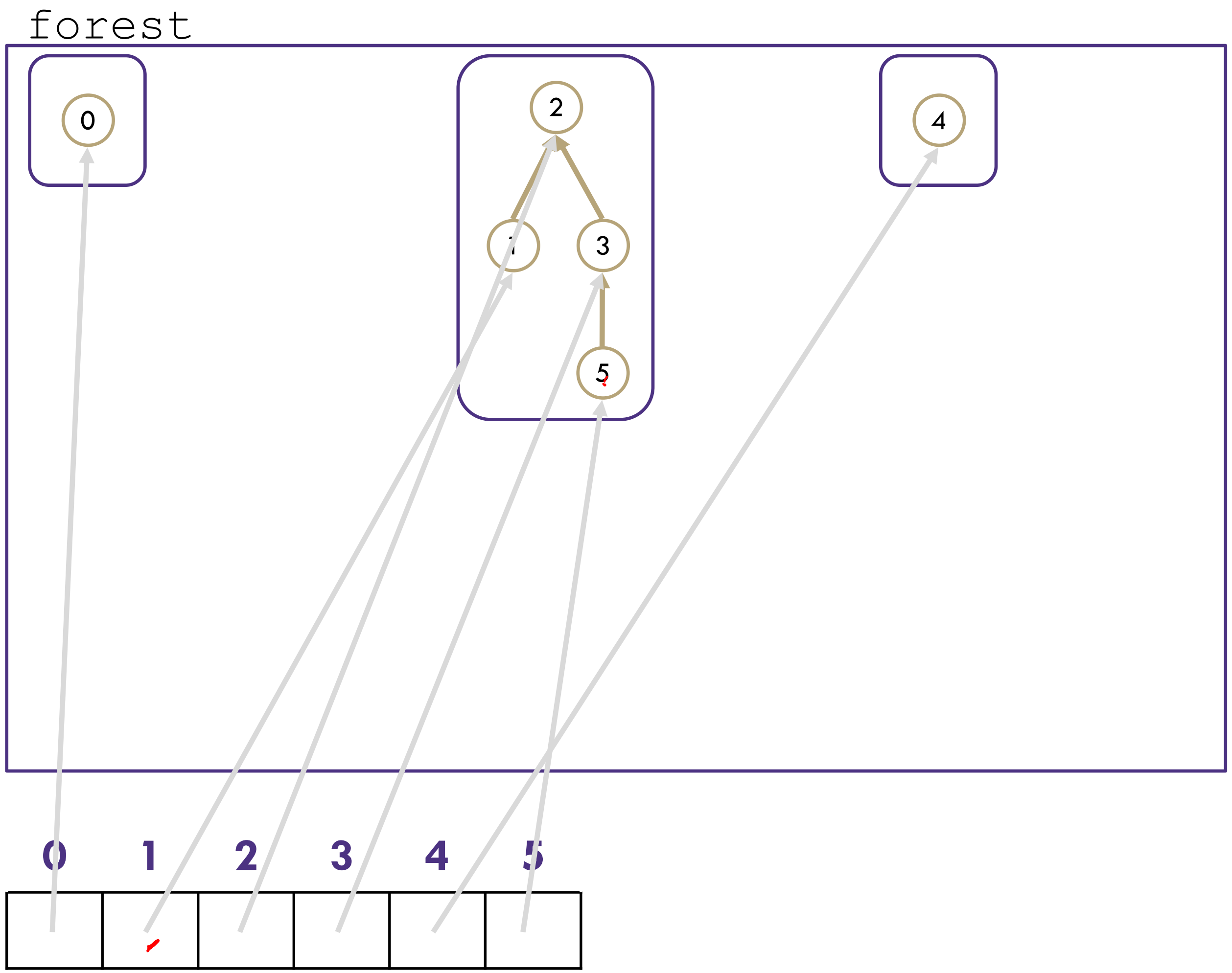
makeSet(x)-create a new tree of size 1 and add to our forest

findSet(x)-locates node with x and moves up tree to find root

union(x, y)-append tree with y as a child of tree with x

Implement union(x, y)

```
union(3, 5)

union(2, 1)

union(2, 5)
```



TreeDisjointSet<E>

state

```
Collection<TreeSet> forest
Dictionary<NodeValues, NodeLocations> nodeInventory
```

behavior

`makeSet(x)`-create a new tree of size 1 and add to our forest

`findSet(x)`-locates node with x and moves up tree to find root

`union(x, y)`-append tree with y as a child of tree with x

Implement findSet(x)

findSet(0) 0

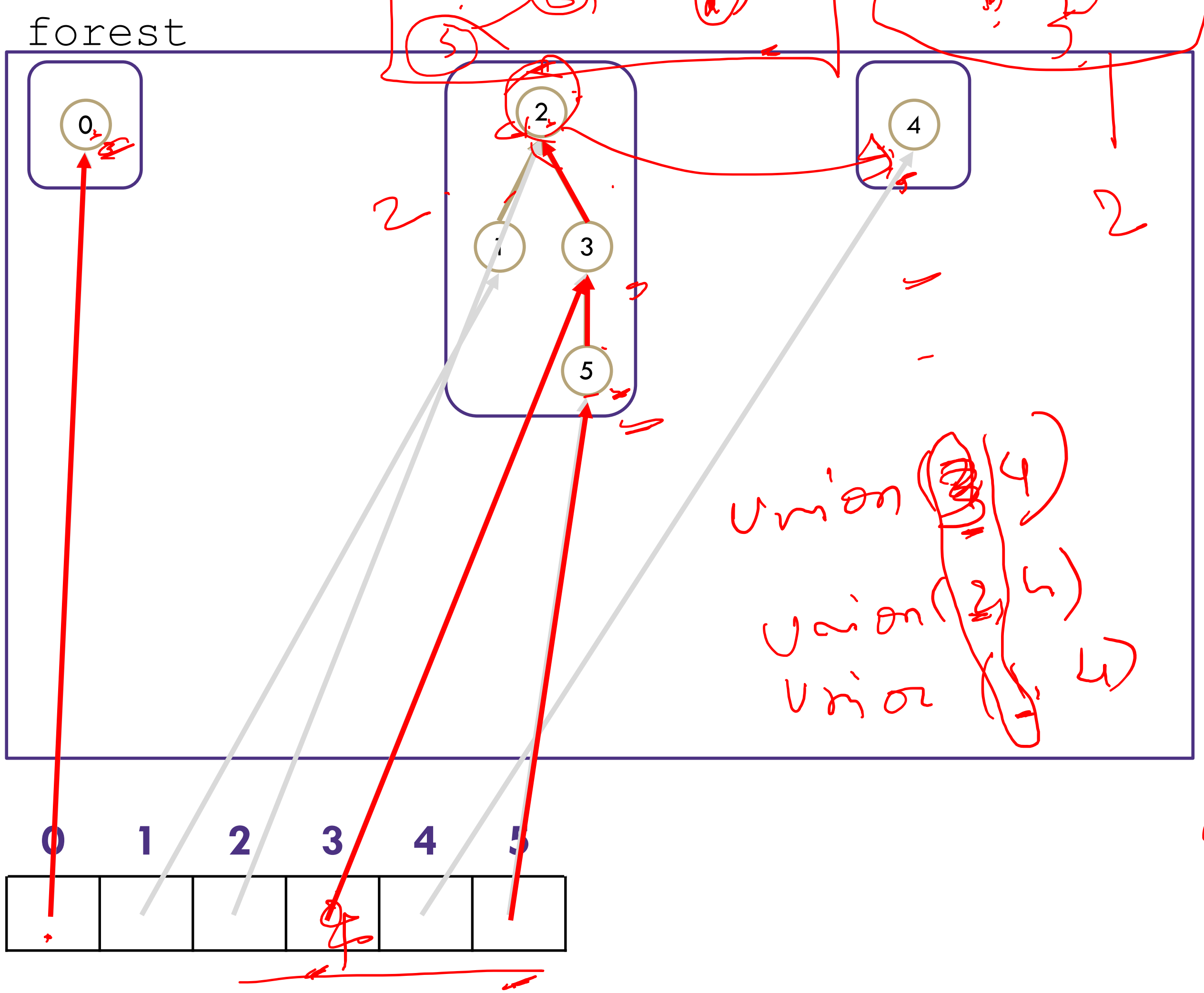
findSet(3) 2

findSet(5) 2

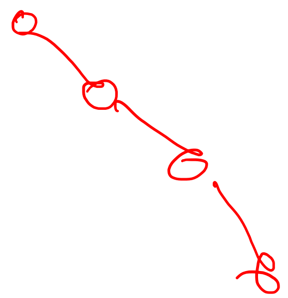
union(0,5)

- findSet(0)

- findSet(5)



Log(n)



Worst case runtime?

O(n)

Worst case runtime of union?

O(n)

TreeDisjointSet<E>

state

Collection<TreeSet> forest

Dictionary<NodeValues, NodeLocations> nodeInventory

behavior

makeSet(x)-create a new tree of size 1 and add to our forest

findSet(x)-locates node with x and moves up tree to find root

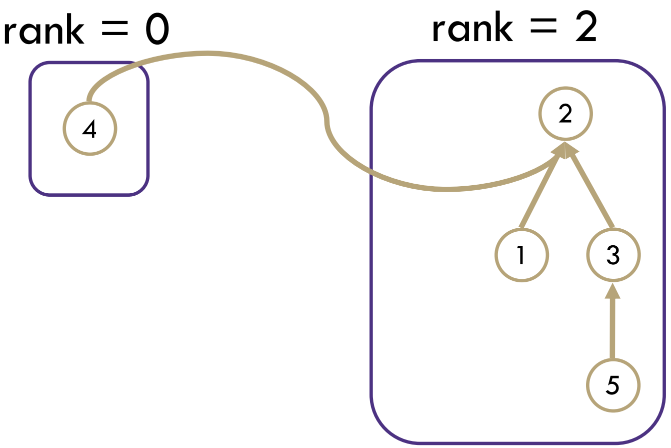
union(x, y)-append tree with y as a child of tree with x

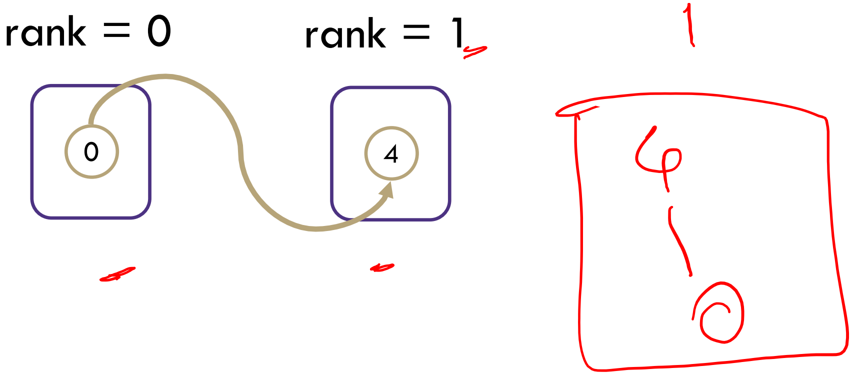
Improving union

Problem: Trees can be unbalanced

Solution: Union-by-rank!

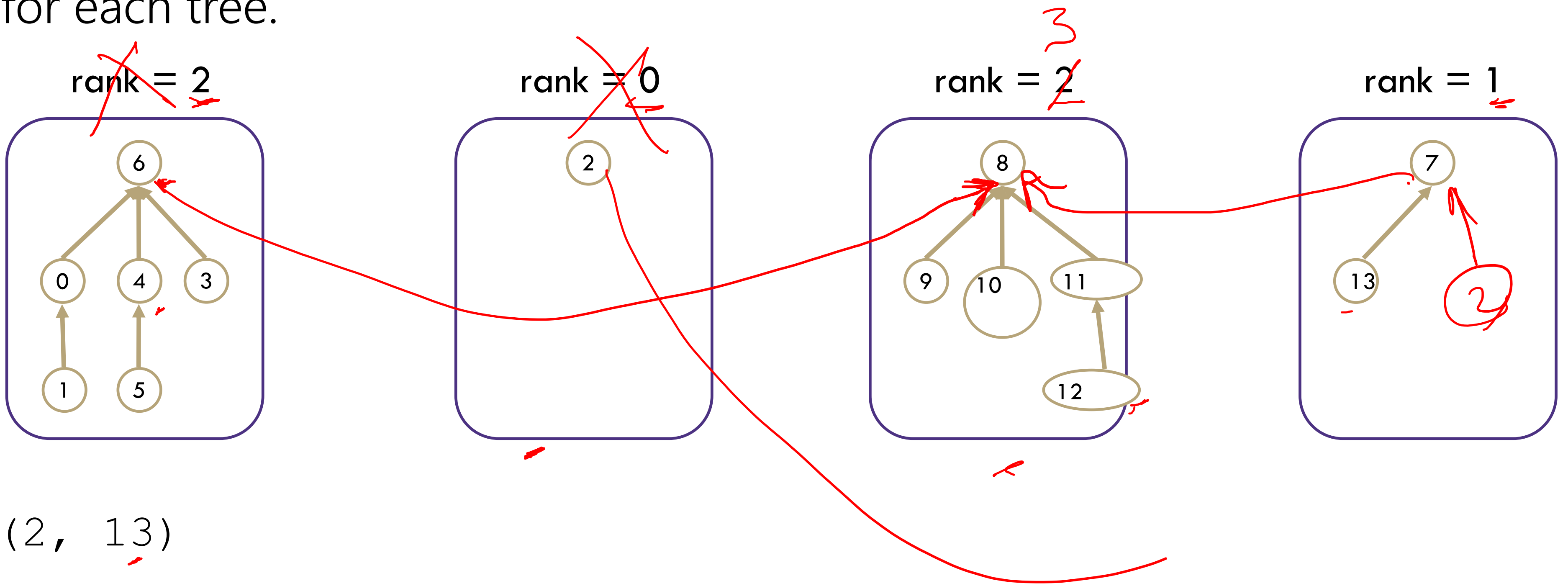
- let rank(x) be a number representing the upper bound of the height of x so rank(x) >= height(x)
- Keep track of rank of all trees
- When unioning make the tree with larger rank the root
- If it's a tie, pick one randomly and increase rank by one





Worksheet question 3

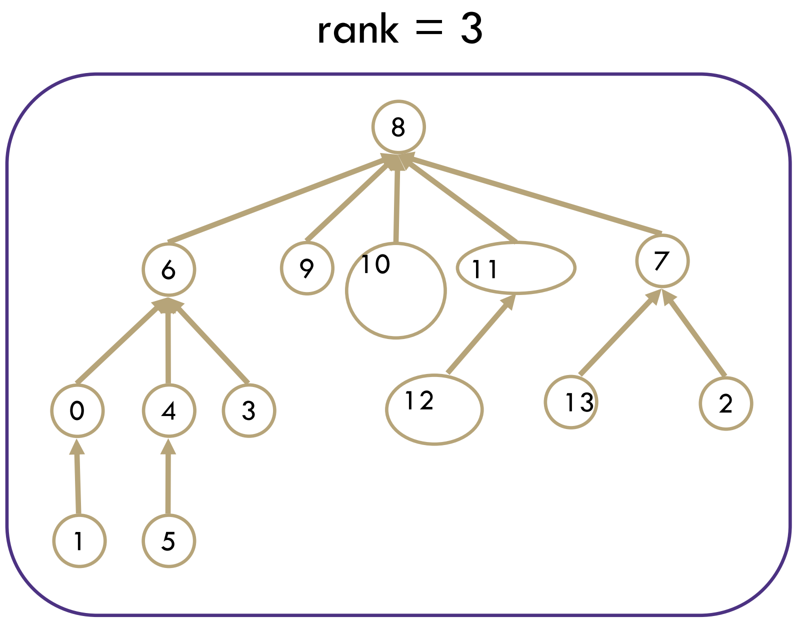
Given the following disjoint-set what would be the result of the following calls on union if we add the "union-by-rank" optimization. Draw the forest at each stage with corresponding ranks for each tree.




```
union(2, 13)
union(4, 12)
union(2, 8)
```

Worksheet question 3

Given the following disjoint-set what would be the result of the following calls on union if we add the “union-by-rank” optimization. Draw the forest at each stage with corresponding ranks for each tree.




```
union(2, 13)
union(4, 12)
union(2, 8)
```

Does this improve the worst case runtimes?

findSet is more likely to be O(log(n)) than O(n)

Improving findSet()

Problem: Every time we call findSet() you must traverse all the levels of the tree to find representative

Solution: Path Compression

- Collapse tree into fewer levels by updating parent pointer of each node you visit
- Whenever you call findSet() update each node you touch's parent pointer to point directly to overallRoot

findSet(5)

findSet(4)

Does this improve the worst case runtimes?

findSet is more likely to be O(1) than O(log(n))