

CSE 373: Data Structures and Algorithms

# Minimum Spanning Tree

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# Four classes of graph problem

.. that can be solved efficiently (in polynomial time)

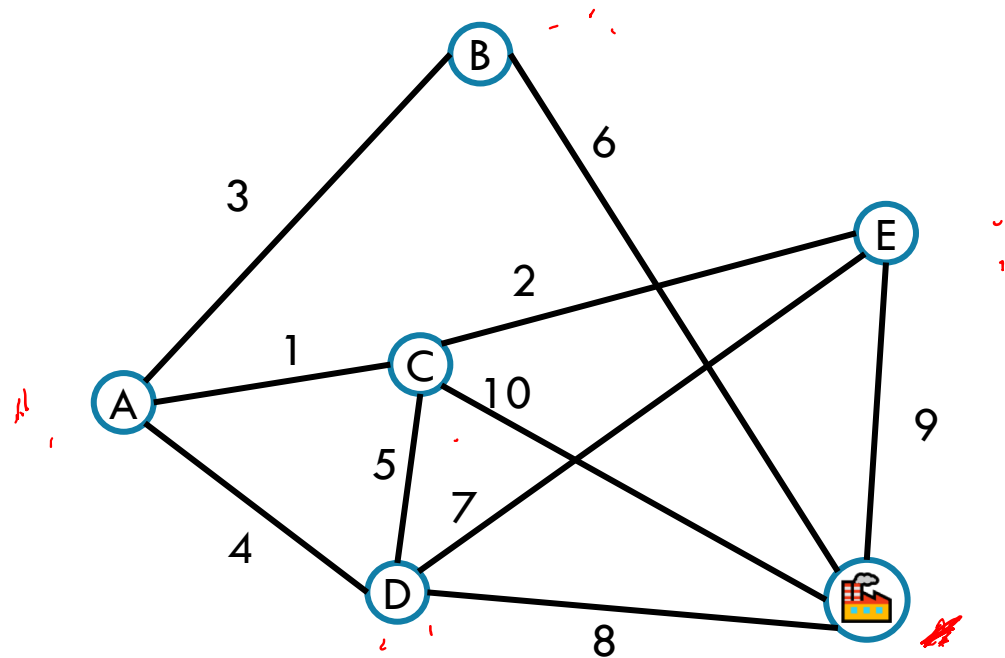
1. **Shortest path** – find a shortest path between two vertices in a graph
2. **Minimum spanning tree** – find subset of edges with minimum total weights
3. Matching – find set of edges without common vertices
4. Maximum flow – find the maximum flow from a source vertex to a sink vertex

A wide array of graph problems that can be solved in polynomial time are variants of these above problems.

In this class, we'll cover the first two problems – shortest path and minimum spanning tree

# Minimum Spanning Trees

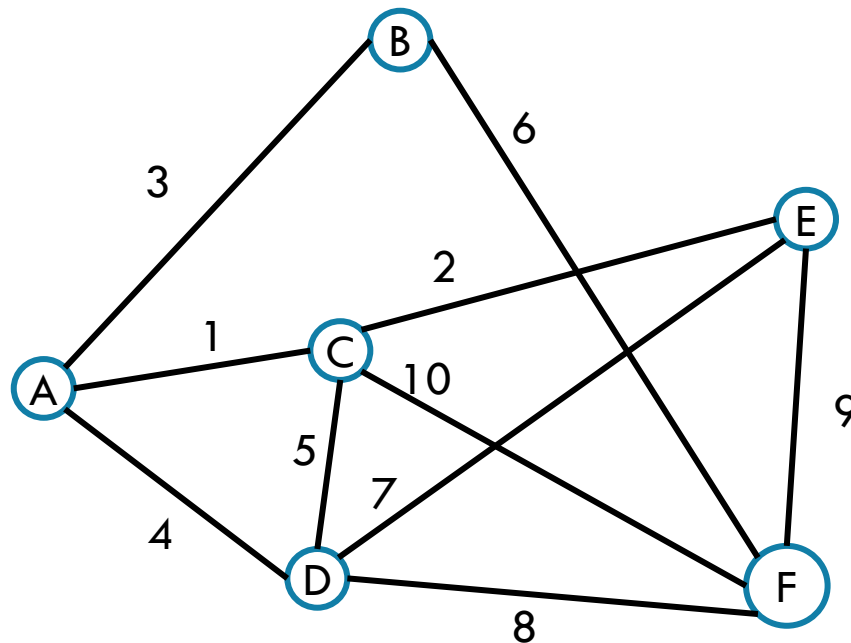
It's the 1920's. Your friend at the electric company needs to choose where to build wires to connect all these cities to the plant.



She knows how much it would cost to lay electric wires between any pair of locations, and wants the cheapest way to make sure electricity from the plant to every city.

# Minimum Spanning Trees

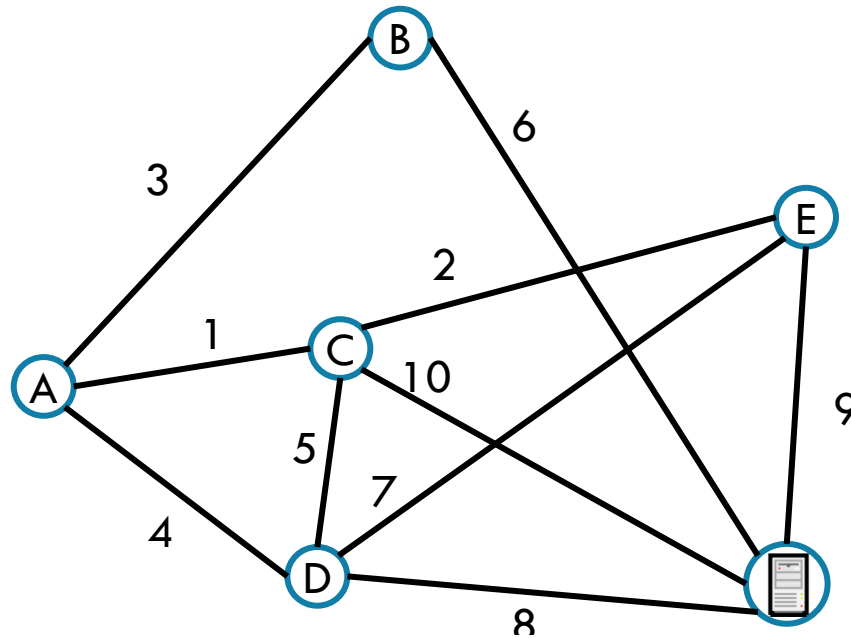
It's the 1950's Your boss at the phone company needs to choose where to build wires to connect all these cities to each other.



She knows how much it would cost to lay elec phone between any pair of locations, and wants the cheapest way to make sure electricit Everyone can call everyone else.

# Minimum Spanning Trees

It's **today**. Your friend at the **ISP** needs to choose where to build wires to connect all these cities to the **Internet with fiber optic cable**



She knows how much it would cost to lay electric **cable** between any pair of locations, and wants the cheapest way to make sure electricity **Everyone can reach the server**

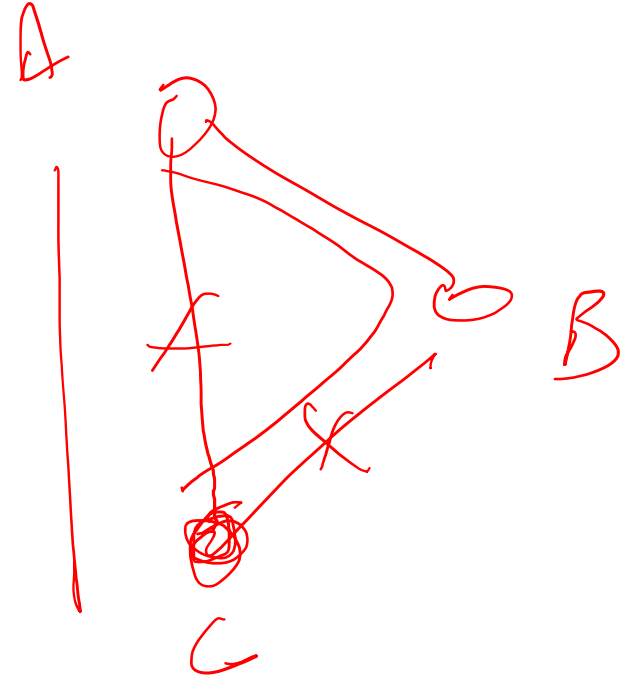
# Minimum Spanning Trees

What do we need? A set of edges such that:

- Every vertex touches at least one of the edges. (the edges **span** the graph)
- The graph on just those edges is **connected**.
- The minimum weight set of edges that meet those conditions.

Assume all edge weights are positive.

Claim: The set of edges we pick never has a cycle. Why?



# Aside: Trees

Our BSTs had:

- A root
- Left and/or right children
- Connected and no cycles

Our heaps had:

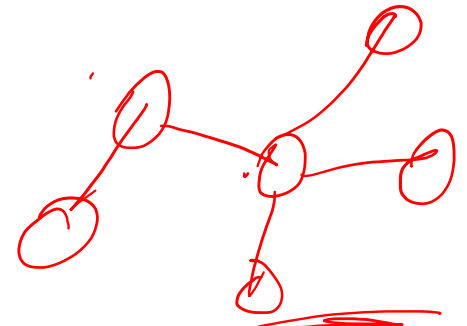
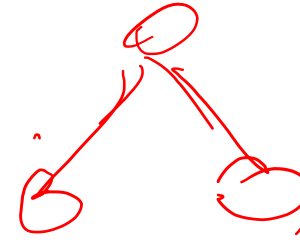
- A root
- Varying numbers of children (but same at each level of the tree)
- Connected and no cycles

On graphs our trees:

- Don't need a root (the vertices aren't ordered, and we can start BFS from anywhere)
- Varying numbers of children (can also vary between levels)
- Connected and no cycles

**Tree (when talking about graphs)**

An undirected, connected acyclic graph.



# MST Problem

What do we need? A set of edges such that:

- Every vertex touches at least one of the edges. (the edges **span** the graph)
- The graph on just those edges is **connected**.
- The minimum weight set of edges that meet those conditions.

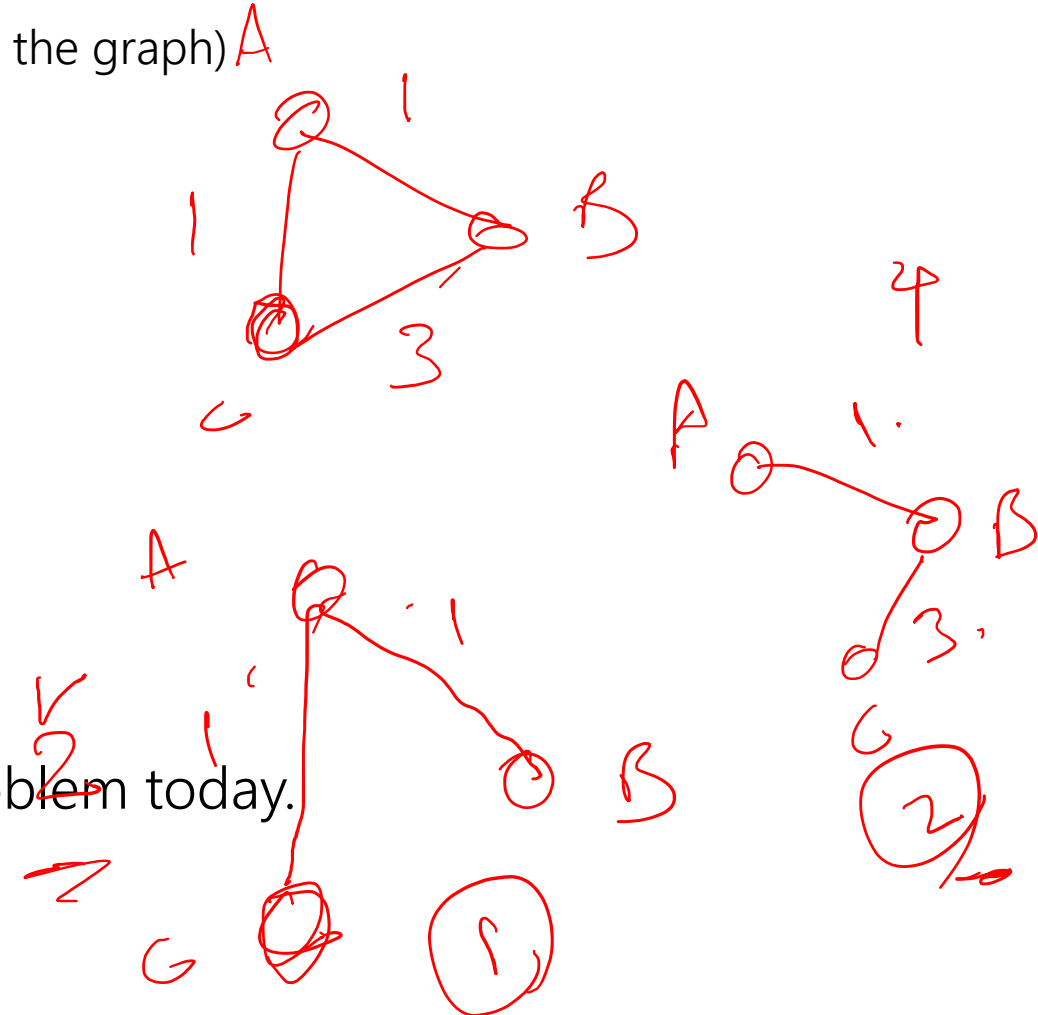
Our goal is a tree!

## Minimum Spanning Tree Problem

**Given:** an undirected, weighted graph  $G$

**Find:** A minimum-weight set of edges such that you can get from any vertex of  $G$  to any other on only those edges.

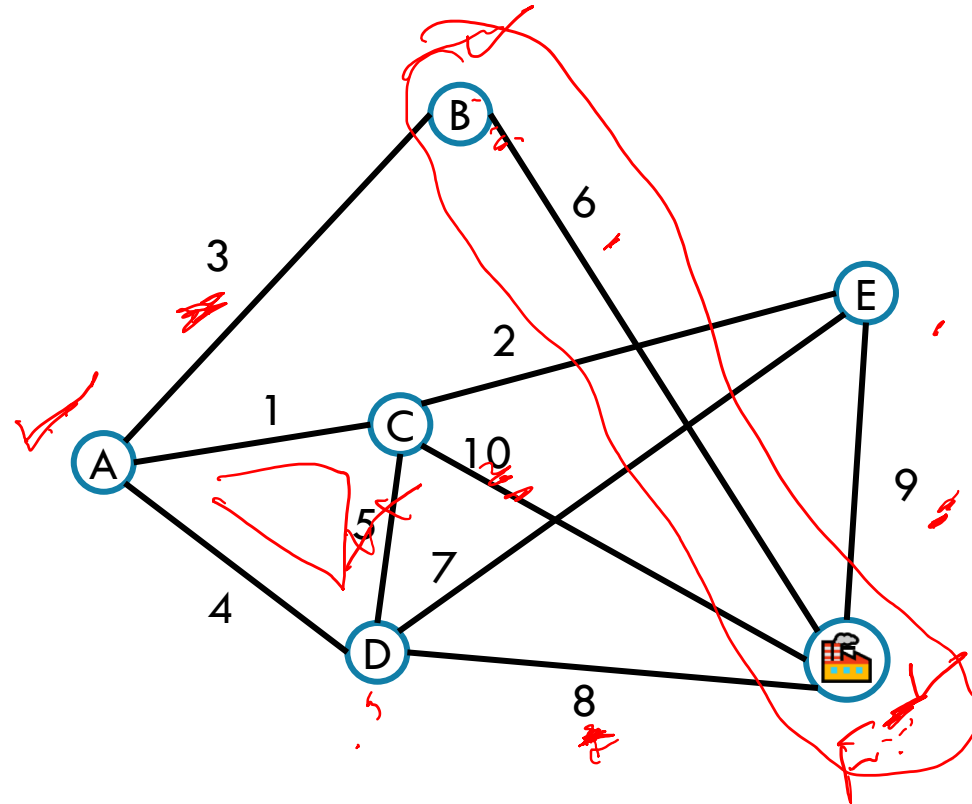
We'll go through two different algorithms for this problem today.





# Example

Try to find a MST of this graph:



# Prim's Algorithm

Algorithm idea: choose an arbitrary starting point. Add a new edge that:

- Will let you reach more vertices.
- Is as light as possible

We'd like each not-yet-connected vertex to be able to tell us the lightest edge we could add to connect it.

# Code

PrimMST(Graph G)

```
    initialize distances to  $\infty$ 
    mark source as distance 0
    mark all vertices unprocessed
    foreach(edge (source, v) )
        v.dist = w(source, v)
    while(there are unprocessed vertices){
        let u be the closest unprocessed vertex
        add u.bestEdge to spanning tree
        foreach(edge (u, v) leaving u){
            if(w(u, v) < v.dist){
                v.dist = w(u, v)
                v.bestEdge = (u, v)
            }
        }
        mark u as processed
    }
```

# Try it Out

PrimMST(Graph G)

initialize distances to  $\infty$

mark source as distance 0

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foreach(edge (source, v) )

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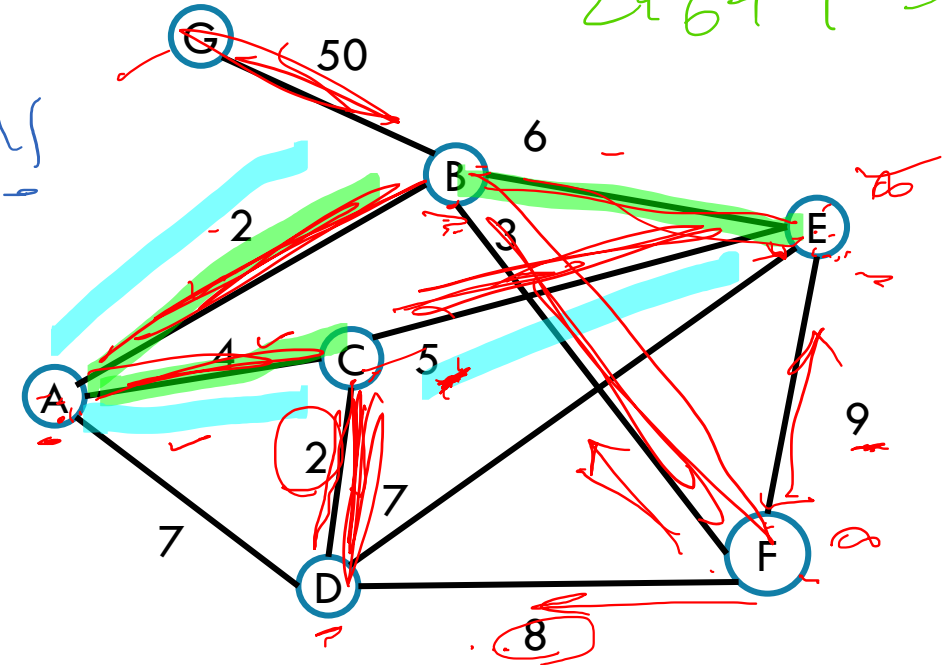
}

mark u as processed

}

$$2 + 4 + 5 = 11$$

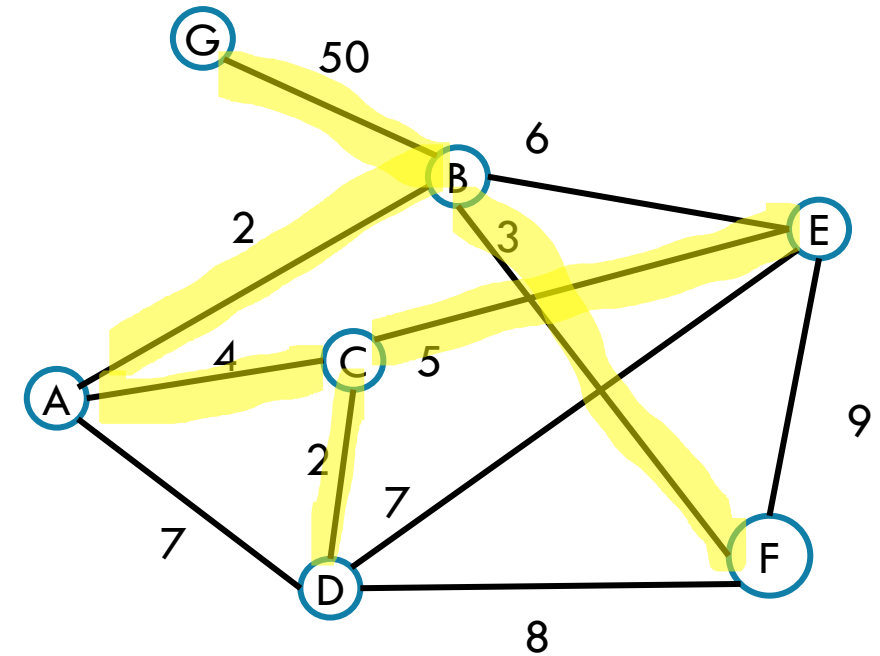
$$2 + 6 + 4 = 12$$



Vertex	Distance	Best Edge	Processed
A	0		✓
B	2	A, B	✓
C	4	A, C	✓
D	7	C, D	✓
E	5	C, E	✓
F	8	D, F	✓
G	50		✓

# Try it Out

```
PrimMST(Graph G)
  initialize distances to  $\infty$ 
  mark source as distance 0
  mark all vertices unprocessed
  foreach(edge (source, v) )
    v.dist = w(source,v)
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      }
    }
    mark u as processed
  }
```



Vertex	Distance	Best Edge	Processed
A			
B			
C			
D			
E			
F			
G			

# Does This Algorithm Always Work?

Prim's Algorithm is a **greedy** algorithm. Once it decides to include an edge in the MST it never reconsiders its decision.

Greedy algorithms rarely work.

There are special properties of MSTs that allow greedy algorithms to find them.

In fact MSTs are so *magical* that there's more than one greedy algorithm that works.

# A different Approach

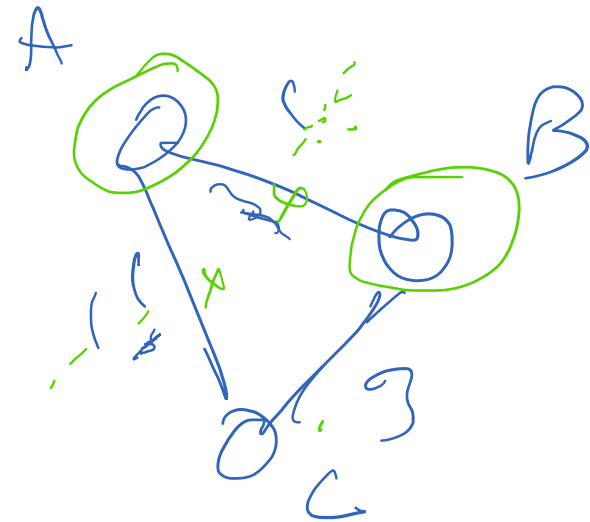
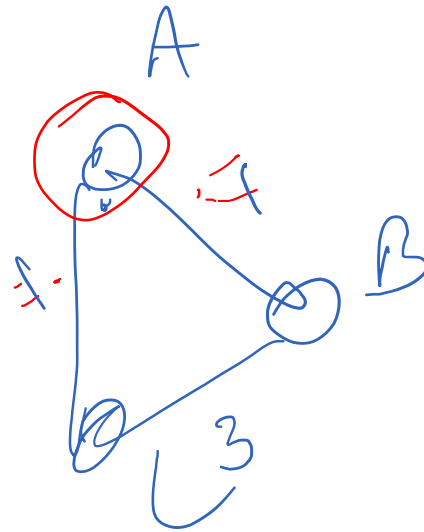
Prim's Algorithm started from a single vertex and reached more and more other vertices.

Prim's thinks vertex by vertex (add the closest vertex to the currently reachable set).

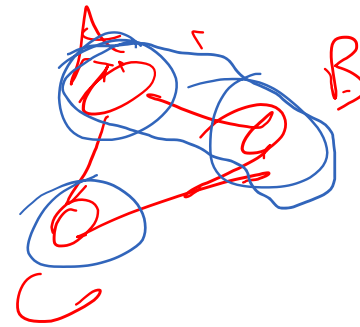
What if you think edge by edge instead?

Start from the lightest edge; add it if it connects new things to each other (don't add it if it would create a cycle)

This is Kruskal's Algorithm.



# Kruskal's Algorithm



$|E|$

KruskalMST(Graph G)

initialize each vertex to be a connected component

sort the edges by weight

$|E|$   $|V|$   $\rightarrow$  foreach(edge (u, v) in sorted order){

if(~~u and v~~ are in different components){

add (u,v) to the MST

Update u and v to be in the same component

}

}

$O(|E| \log |E|)$

$O(|E| |V|)$



# Try It Out

KruskalMST(Graph G)

initialize each vertex to be a connected component

sort the edges by weight

foreach(edge (u, v) in sorted order){

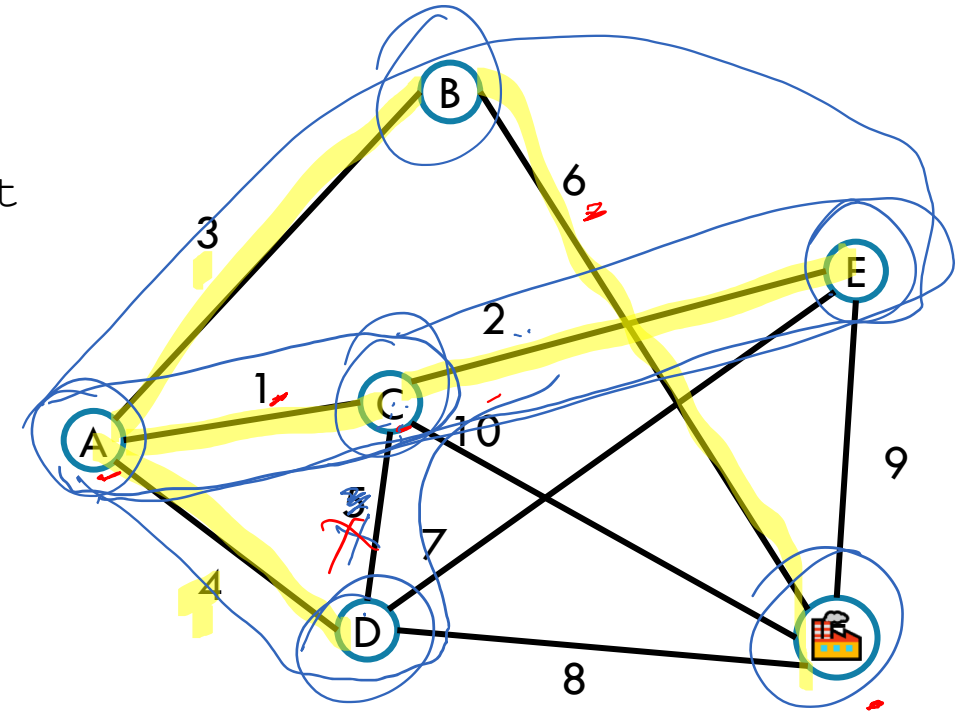
if(u and v are in different components){

add (u,v) to the MST

Update u and v to be in the same component

}

}



# Kruskal's Algorithm: Running Time

```
KruskalMST(Graph G)
  initialize each vertex to be a connected component
  sort the edges by weight
  foreach(edge (u, v) in sorted order){
    if(u and v are in different components){
      add (u,v) to the MST
      Update u and v to be in the same component
    }
  }
```

# Kruskal's Algorithm: Running Time

Running a new [B/D]FS in the partial MST, at every step seems inefficient.

Do we have an ADT that will work here?

Not yet...

We will cover “Union-Find” next lecture.

# Try it Out

KruskalMST(Graph G)

  initialize each vertex to be a connected component

  sort the edges by weight

  foreach(edge (u, v) in sorted order){

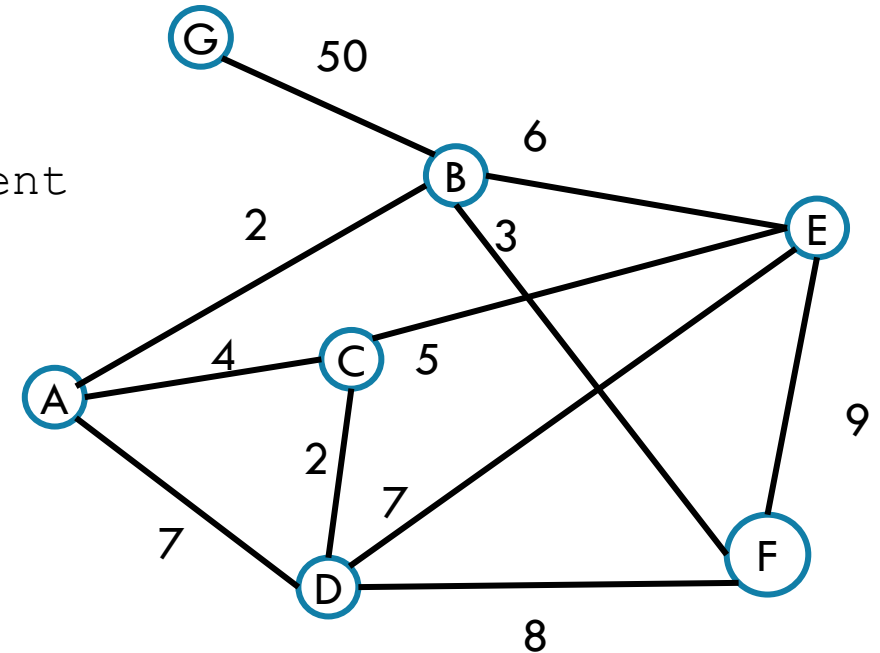
    if(u and v are in different components){

      add (u,v) to the MST

      Update u and v to be in the same component

  }

}



# Aside: A Graph of Trees

A tree is an undirected, connected, and acyclic graph.

How would we describe the graph Kruskal's builds.

It's not a tree until the end.

It's a forest!

A forest is any undirected and acyclic graph



## Appendix: MST Properties, Another MST Application

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# Some Extra Comments

Prim was the employee at Bell Labs in the 1950's

The mathematician in the 1920's was Boruvka

- He had a different *also greedy* algorithm for MSTs.
- Boruvka's algorithm is trickier to implement, but is useful in some cases.

There's at least a fourth greedy algorithm for MSTs...

If all the edge weights are distinct, then the MST is unique.

If some edge weights are equal, there may be multiple spanning trees. Prim's/Dijkstra's are only guaranteed to find you one of them.

# Why do all of these MST Algorithms Work?

MSTs satisfy two very useful properties:

**Cycle Property:** The heaviest edge along a cycle is NEVER part of an MST.

**Cut Property:** Split the vertices of the graph any way you want into two sets A and B. The lightest edge with one endpoint in A and the other in B is ALWAYS part of an MST.

Whenever you add an edge to a tree you create exactly one cycle, you can then remove any edge from that cycle and get another tree out.

This observation, combined with the cycle and cut properties form the basis of all of the greedy algorithms for MSTs.



# One More MST application

Let's say you're building a new building.

There are very important building donors coming to visit TOMORROW,  
- and the hallways are not finished.

You have  $n$  rooms you need to show them, connected by the unfinished hallways.

Thanks to your generous donors you have  $n-1$  construction crews, so you can assign one to each of that many hallways.

- Sadly the hallways are narrow and you can't have multiple crews working on the same hallway.

Can you finish enough hallways in time to give them a tour?

## Minimum **Bottleneck** Spanning Tree Problem

**Given:** an undirected, weighted graph  $G$

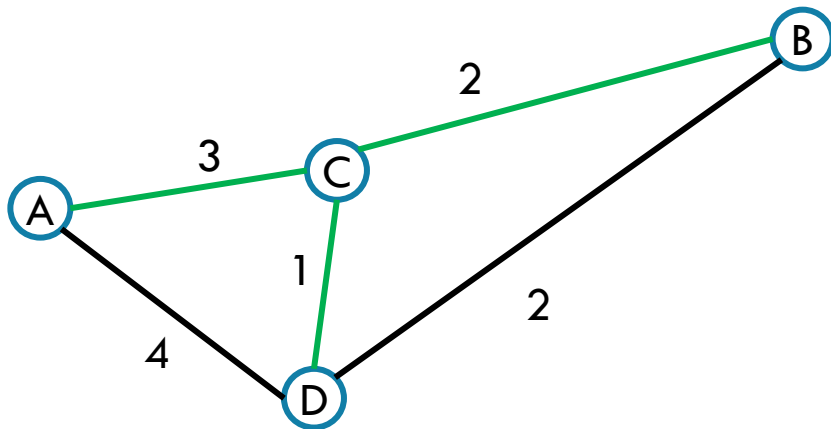
**Find:** A spanning tree such that the weight of the maximum edge is minimized.

# MSTs and MBSTs

## Minimum Spanning Tree Problem

**Given:** an undirected, weighted graph  $G$

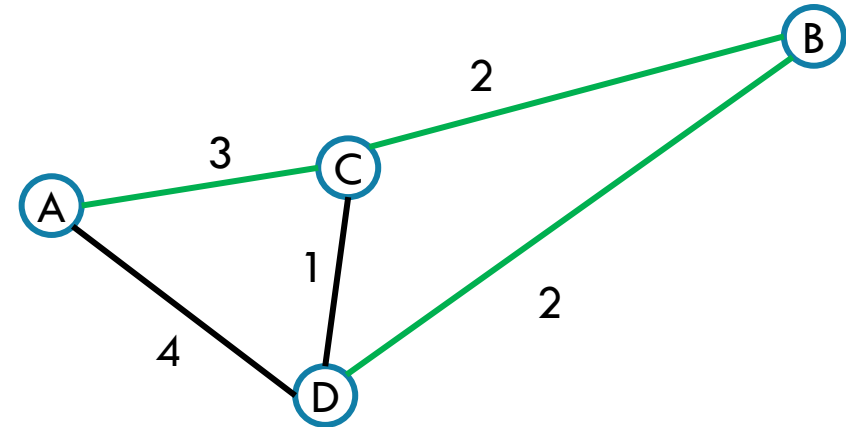
**Find:** A minimum-weight set of edges such that you can get from any vertex of  $G$  to any other on only those edges.



## Minimum **Bottleneck** Spanning Tree Problem

**Given:** an undirected, weighted graph  $G$

**Find:** A spanning tree such that the weight of the maximum edge is minimized.



Graph on the right is a minimum bottleneck spanning tree, but not a minimum spanning tree.

# Finding MBSTs

Algorithm Idea: want to use smallest edges. Just start with the smallest edge and add it if it connects previously unrelated things (and don't if it makes a cycle).

Hey wait...that's Kruskal's Algorithm!

Every MST is an MBST (because Kruskal's can find any MST when looking for MBSTs) but not vice versa (see the example on the last slide).

If you need an MBST, any MST algorithm will work.

There are also some specially designed MBST algorithms that are *faster* (see Wikipedia)

Takeaway: When you're modeling a problem, be careful to really understand what you're looking for. There may be a better algorithm out there.