Section 03: Asymptotic Analysis

Review Problems

1. Code Analysis

For each of the following code blocks, what is the worst-case runtime? Give a big-Θ bound.

(a)
```
public IList<String> repeat(DoubleLinkedList<String> list, int n) {
    IList<String> result = new DoubleLinkedList<String>();
    for(String str : list) {
        for(int i = 0; i < n; i++) {
            result.add(str);
        }
    }
    return result;
}
```

(b)
```
public void foo(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 5; j < i; j++) {
            System.out.println("Hello!");
        }

        for (int j = i; j >= 0; j -= 2) {
            System.out.println("Hello!");
        }
    }
}
```

(c)
```
public int num(int n){
    if (n < 10) {
        return n;
    } else if (n < 1000) {
        return num(n - 2);
    } else {
        return num(n / 2);
    }
}
```

(d)
```
public int foo(int n) {
    if (n <= 0) {
        return 3;
    }
    int x = foo(n - 1);
    System.out.println("hello");
    x += foo(n - 1);
    return x;
}
```

2. Binary Search Trees

(a) Write a method `validate` to validate a BST. Although the basic algorithm can be converted to any data structure and work in any format, if it helps, you may write this method for the `IntTree` class:

```
public class IntTree {
    private IntTreeNode overallRoot;

    // constructors and other methods omitted for clarity

    private class IntTreeNode {
        public int data;
        public IntTreeNode left;
        public IntTreeNode right;

        // constructors omitted for clarity
    }
}
```

Section Problems

3. Recurrences

For each of the following recurrences, use the unfolding method to first convert the recurrence into a summation. Then, find a big-Θ bound on the function in terms of n. Assume all division operations are integer division.

(a) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 3 & \text{otherwise} \end{cases}$

(b) $T(n) = \begin{cases} 1 & \text{if } n = 0 \\ T(n-1) + 2 & \text{otherwise} \end{cases}$

(c) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + n & \text{otherwise} \end{cases}$

(d) $T(n) = \begin{cases} 1 & \text{if } n = 0 \\ T(n/3) + 4 & \text{otherwise} \end{cases}$ Use integer division in n is not a multiple of 3.

(e) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n-1) + 1 & \text{otherwise} \end{cases}$

(f) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + 100 & \text{otherwise} \end{cases}$

4. Modeling recursive functions

(a) Consider the following method.

```
public static int f(int n) {
    if (n == 0) {
        return 0;
    }

    int result = 0;
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < i; j++) {
            result++;
        }
    }
    return 5 * f(n / 2) + 3 * result + 2 * f(n / 2);
}
```

(i) Find a recurrence $T(n)$ modeling the worst-case runtime of `f(n)`.

(ii) Find a recurrence $W(n)$ modeling the *returned integer output* of `f(n)`.

(b) Consider the following method.

```
public static int g(n) {
    if (n <= 1) {
        return 1000;
    }
    if (g(n / 3) > 5) {
        for (int i = 0; i < n; i++) {
            System.out.println("Hello");
        }
        return 5 * g(n / 3);
    } else {
        for (int i = 0; i < n * n; i++) {
            System.out.println("World");
        }
        return 4 * g(n / 3);
    }
}
```

(i) Find a recurrence $S(n)$ modeling the worst-case runtime of `g(n)`.

(ii) Find a recurrence $X(n)$ modeling the *returned integer output* of `g(n)`.

(iii) Find a recurrence $P(n)$ modeling the *printed output* of `g(n)`.

(c) Consider the following set of recursive methods.

```
public int test(int n) {
    IDictionary<Integer, Integer> dict = new AvlDictionary<>();
    populate(n, dict);
    int counter = 0;
    for (int i = 0; i < n; i++) {
        counter += dict.get(i);
    }
    return counter;
}

private void populate(int k, IDictionary<Integer, Integer> dict) {
    if (k == 0) {
        dict.put(0, k);
    } else {
        for (int i = 0; i < k; i++) {
            dict.put(i, i);
        }
        populate(k / 2, dict);
    }
}
```

(i) Write a mathematical function representing the *worst-case runtime* of `test`.

You should write two functions, one for the runtime of `test` and one for the runtime of `populate`.

(ii) Write a mathematical function $Y(n)$ representing the *returned integer output* of `test`.

The reason is that all keys in this dictionary are integers in the range [0, n-1], and each value is equal to its key.

5. AVL Trees

(a) Draw an AVL Tree as each of the following keys are added in the order given. Show intermediate steps.

(i)

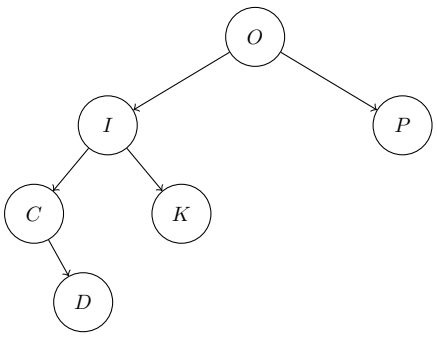
$$\{13, 17, 14, 19, 22, 18, 11, 10, 21\}$$

(ii)

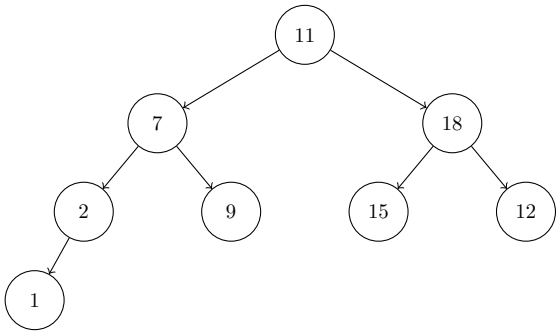
$$\{1, 2, 3, 4, 5, 6\}$$

(b) Identify if the following trees are AVL trees. Explain your answer.

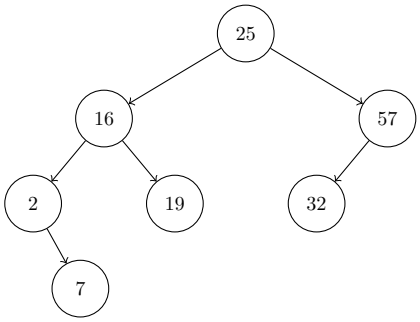
(i) Tree 1



(ii) Tree 2



(iii) Tree 3



Food for thought

6. AVL trees

What is the minimum number of nodes in an AVL tree, given the following heights? Draw a picture of such a tree. (**Reminder**: an AVL tree's height is 0 for a tree with only 1 node in it!)

(a) 1

(b) 2

(c) 4

7. Algorithm Design

(a) Given a binary search tree, describe how you could convert it into an AVL tree with worst-case time $\mathcal{O}(n \log(n))$. What is the best case runtime of your algorithm?

(b) Given an AVL tree, describe how would you do a level order tree traversal. What is the worst-case runtime of your algorithm?

Challenge Problems

8. Recurrences

(a) For the following recurrence, use the unfolding method to first convert the recurrence into a summation. Then, find a big-Θ bound on the function in terms of n. Assume all division operations are integer division.

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2T(n/3) + n & \text{otherwise} \end{cases}$$

9. Modeling recursive functions

Consider the following recursive function. You may assume that the input will be a multiple of 3.

```
public int test(int n) {
    if (n <= 6) {
        return 2;
    } else {
        int curr = 0;
        for (int i = 0; i < n * n; i++) {
            curr += 1;
        }
        return curr + test(n - 3);
    }
}
```

(a) Write a recurrence modeling the *worst-case runtime* of `test`.

(b) Unfold the recurrence into a summation (for $n \geq 6$).

(c) Simplify the summation into a closed form (for $n \geq 6$).

10. AVL Trees

(a) Is there a relationship between an AVL tree's height, and its minimum or maximum number of nodes? If so, what is it?

(b) Write a method `isAVLTree` to check if a given tree (which is guaranteed to be a valid BST) is a valid AVL tree. If it helps, you may write this method for this tree class, `HeightTree`, which keeps track of the height of a tree at each node:

```
public class HeightTree {
    private IntHeightNode overallRoot;

    // constructors and other methods omitted for clarity

    private class IntHeightNode {
        public int data;
        public int height;
        public IntHeightNode left;
        public IntHeightNode right;

        // constructors omitted for clarity
    }
}
```

(c) Now write `isAVLTree` *without* assuming that the tree is a valid BST.

(d) Now write `isAVLTree` for the `IntTree` class (you may assume again that the tree is guaranteed to be a valid BST).