Section 03: Solutions

Review Problems

1. Code Analysis

For each of the following code blocks, what is the worst-case runtime? Give a big-Θ bound.

(a)
```
public IList<String> repeat(DoubleLinkedList<String> list, int n) {
    IList<String> result = new DoubleLinkedList<String>();
    for(String str : list) {
        for(int i = 0; i < n; i++) {
            result.add(str);
        }
    }
    return result;
}
```

Solution:

The runtime is $\Theta(nm)$, where m is the length of the input list and n is equal to the int `n` parameter.

One thing to note here is that unlike many of the methods we've analyzed before, we can't quite describe the runtime of this algorithm using just a single variable: we need two, one for each loop.

(b)
```
public void foo(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 5; j < i; j++) {
            System.out.println("Hello!");
        }

        for (int j = i; j >= 0; j -= 2) {
            System.out.println("Hello!");
        }
    }
}
```

Solution:

The inner loop executes about $i - 5 + i/2$ operations per loop. So we execute about

$$\sum_{i=0}^{n-1} i - 5 + i/2 = \frac{3}{2}\sum_{i=0}^{n-1} i - \sum_{i=0}^{n-1} 5 = \frac{3}{2} * \frac{(0 + n - 1) * n}{2} - 5n = \frac{3n(n-1)}{4} - 5n$$

which means the runtime is $\Theta\left(n^2\right)$.

(c)
```java
public int num(int n){
    if (n < 10) {
        return n;
    } else if (n < 1000) {
        return num(n - 2);
    } else {
        return num(n / 2);
    }
}
```

Solution:

The answer is $\Theta\left(\log(n)\right)$.

One thing to note is that the second case effectively has no impact on the runtime. That second case occurs only for $n < 1000$ – when discussing asymptotic analysis, we only care what happens with the runtime as n grows large.

(d)
```java
public int foo(int n) {
    if (n <= 0) {
        return 3;
    }
    int x = foo(n - 1);
    System.out.println("hello");
    x += foo(n - 1);
    return x;
}
```

Solution:

The answer is $\Theta\left(2^n\right)$.

In order to determine that this is exponential, let's start by considering the following recurrence:

$$T(n) = \begin{cases} 1 & \text{If } n = 0 \\ 2T(n-1) + 1 & \text{Otherwise} \end{cases}$$

While we could unfold this to get an exact closed form, we can approximate the final asymptotic behavior by taking a step back and thinking on a higher level what this is doing.

Basically, what happens is we take the work done by $T(n-1)$ and multiply it by 2. If we ignore the $+1$ constant work done in the recursive case, the net effect is that we multiply 2 approximately n times. This simplifies to 2^n.

2. Binary Search Trees

(a) Write a method `validate` to validate a BST. Although the basic algorithm can be converted to any data structure and work in any format, if it helps, you may write this method for the `IntTree` class:

```
public class IntTree {
    private IntTreeNode overallRoot;

    // constructors and other methods omitted for clarity

    private class IntTreeNode {
        public int data;
        public IntTreeNode left;
        public IntTreeNode right;

        // constructors omitted for clarity
    }
}
```

Solution:

```
public boolean validate() {
    return validate(overallRoot, Integer.MIN_VALUE, Integer.MAX_VALUE);
}

private boolean validate(IntTreeNode root, int min, int max) {
    if (root == null) {
        return true;
    } else if (root.data > max || root.data < min) {
        return false;
    } else {
        return validate(root.left, min, root.data - 1) &&
            validate (root.right, root.data + 1, max);
    }
}
```

Section Problems

3. Recurrences

For each of the following recurrences, use the unfolding method to first convert the recurrence into a summation. Then, find a big-Θ bound on the function in terms of n. Assume all division operations are integer division.

(a) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(n/2) + 3 & \text{otherwise} \end{cases}$

Solution:

Unfolding a few levels to find a pattern:

$$\begin{aligned} T(n) &= T(n/2) + 3 \\ &= T(n/4) + 3 + 3 \\ &= T(n/8) + 3 + 3 + 3 \\ &= T(n/2^i) + 3i \end{aligned}$$

Setting $i = \log n$ to force the input into a base case, we get: $1 + \sum_{j=1}^{\log(n)} 3$. The big-$\Theta$ bound is $\Theta(\log(n))$.

Something you may notice is that depending on what exactly n is, the expression $\log(n)$ may not evaluate to an integer. In that case, what does it mean to have $\log(n)$ as the upper limit of a summation?

What exactly this mean differs based on convention, but for the purposes of this class, we'll assume that j varies starting at 1 up to the largest possible integer that is $\leq \log(n)$. We could write this more explicitly using floors: $1 + \sum_{j=1}^{\lfloor \log(n) \rfloor} 3$.

(b) $T(n) = \begin{cases} 1 & \text{if } n = 0 \\ T(n-1) + 2 & \text{otherwise} \end{cases}$

Solution:

$$\begin{aligned} T(n) &= T(n-1) + 2 \\ &= T(n-2) + 2 + 2 \\ &= T(n-3) + 2 + 2 + 2 \\ &= T(n-i) + 2i \end{aligned}$$

The summation is $1 + \sum_{i=1}^{n} 2 = 2n + 1$. The big-$\Theta$ bound is $\Theta(n)$.

(c) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + n & \text{otherwise} \end{cases}$

Solution:

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= 2\left(2T\left(\frac{n}{2\cdot 2}\right) + \frac{n}{2}\right) + n \\ &= 2^2T\left(\frac{n}{2\cdot 2}\right) + n + n \\ &= 2^2\left(2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + n + n \\ &= 2^3T\left(\frac{n}{2^3}\right) + n + n + n \\ &= 2^iT\left(\frac{n}{2^i}\right) + i\cdot n \end{aligned}$$

Setting $i = \log(n)$ to force the input into a base case, we get: $2^{\log n} \cdot 1 + \log(n) \cdot n = n + n\log(n)$. The big-$\Theta$ bound is $\Theta\left(n\log(n)\right)$.

(d) $T(n) = \begin{cases} 1 & \text{if } n = 0 \\ T(n/3) + 4 & \text{otherwise} \end{cases}$ Use integer division in n is not a multiple of 3. **Solution:**

Unrolling for a few levels we get:

$$\begin{aligned} T(n) &= T(n/3) + 4 \\ &= T(n/3^2) + 4 + 4 \\ &= T(n/3^3) + 4 + 4 + 4 \\ &= T(n/3^i) + 4i \end{aligned}$$

The summation is $1 + \sum_{j=1}^{\log_3(n)+1} 4$. Note that the "+1" in the upper limit of the summation corresponds to our base case being at $n = 0$ instead of $n = 1$. The big-Θ bound is $\Theta\left(\log n\right)$.

(e) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n-1) + 1 & \text{otherwise} \end{cases}$

Solution:

Using a similar process, we get the following expression: $2^{n-1} + \sum_{k=0}^{n-2} 2^k = 2^{n-1} + 2^{n-1} - 1$. Both of these terms are $\Theta\left(2^n\right)$ (because $2^{n-1} = \frac{1}{2}2^n = \Theta\left(2^n\right)$) This ends up being in $\Theta\left(2^n\right)$.

(f) $T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + 100 & \text{otherwise} \end{cases}$

Solution:

$$\begin{aligned} T(n) &= 2T(n/2) + 100 \\ &= 2[2T(n/2^2) + 100] + 100 \\ &= 2^2T(n/2^2) + 2 \cdot 100 + 100 \\ &= 2^2[2T(n/2^3) + 100] + 2 \cdot 100 + 100 \\ &= 2^3T(n/2^3) + 2^2 \cdot 100 + 2 \cdot 100 + 100 \end{aligned}$$

We first get $T(n) = 2^iT(n/2^i) + \sum_{j=0}^{i-1} 2^j \cdot 100$. Setting $i = \log n$ will let us hit the base case, where we get

$$T(n) = 2^{\log n}T(1) + \sum_{j=0}^{\log n - 1} 2^j \cdot 100 = n \cdot 1 + 100 \sum_{j=0}^{\log n - 1} 2^j = n + 100\left(2^{\log n} - 1\right) = 101n - 100$$

Where we applied the "factoring out a constant" and "finite geometric series" identities.

Therefore, we have $\Theta(n)$.

4. Modeling recursive functions

(a) Consider the following method.

```
public static int f(int n) {
    if (n == 0) {
        return 0;
    }

    int result = 0;
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < i; j++) {
            result++;
        }
    }
    return 5 * f(n / 2) + 3 * result + 2 * f(n / 2);
}
```

(i) Find a recurrence $T(n)$ modeling the worst-case runtime of `f(n)`.

Solution:

$$T(n) = \begin{cases} 1 & \text{When } n = 0 \\ \frac{n(n-1)}{2} + 2T(n/2) & \text{Otherwise} \end{cases}$$

The runtime for the two loops is $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1 = \sum_{i=0}^{n-1} i = \dfrac{n(n-1)}{2}$.

(ii) Find a recurrence $W(n)$ modeling the *returned integer output* of `f(n)`.

Solution:

$$W(n) = \begin{cases} 0 & \text{When } n = 0 \\ \frac{3n(n-1)}{2} + 7W(n/2) & \text{Otherwise} \end{cases}$$

After the loop, `result` is $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} 1 = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$. And we use `3 * result`.

(b) Consider the following method.

```
public static int g(n) {
    if (n <= 1) {
        return 1000;
    }
    if (g(n / 3) > 5) {
        for (int i = 0; i < n; i++) {
            System.out.println("Hello");
        }
        return 5 * g(n / 3);
    } else {
        for (int i = 0; i < n * n; i++) {
            System.out.println("World");
        }
        return 4 * g(n / 3);
    }
}
```

(i) Find a recurrence $S(n)$ modeling the worst-case runtime of `g(n)`.

Solution:

$$S(n) = \begin{cases} 1 & \text{When } n \leq 1 \\ 2S(n/3) + n & \text{Otherwise} \end{cases}$$

Important: note that the if statement contains a recursive call that must be evaluated for $n > 1$.

(ii) Find a recurrence $X(n)$ modeling the *returned integer output* of `g(n)`.

Solution:

$$X(n) = \begin{cases} 1000 & \text{When } n \leq 1 \\ 5T(n/3) & \text{Otherwise} \end{cases}$$

(iii) Find a recurrence $P(n)$ modeling the *printed output* of `g(n)`.

Solution:

$$P(n) = 2P(n/3) + n$$

(c) Consider the following set of recursive methods.

```
public int test(int n) {
    IDictionary<Integer, Integer> dict = new AvlDictionary<>();
    populate(n, dict);
    int counter = 0;
    for (int i = 0; i < n; i++) {
        counter += dict.get(i);
    }
    return counter;
}

private void populate(int k, IDictionary<Integer, Integer> dict) {
    if (k == 0) {
        dict.put(0, k);
    } else {
        for (int i = 0; i < k; i++) {
            dict.put(i, i);
        }
        populate(k / 2, dict);
    }
}
```

(i) Write a mathematical function representing the *worst-case runtime* of `test`.

You should write two functions, one for the runtime of `test` and one for the runtime of `populate`.

Solution:

The runtime of the `populate` method is:

$$P(k) = \begin{cases} \log(N) & \text{When } k = 0 \\ k\log(N) + P(k/2) & \text{Otherwise} \end{cases}$$

Here, N is the maximum possible value of n.

The runtime of the `test` method is then $R(n) = P(n) + n\log(n)$.

(ii) Write a mathematical function $Y(n)$ representing the *returned integer output* of `test`.

Solution:

$$Y(n) = \frac{n(n-1)}{2}$$

The reason is that all keys in this dictionary are integers in the range [0, n-1], and each value is equal to its key.

5. AVL Trees

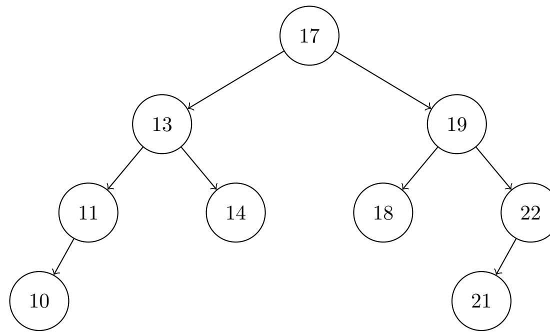
(a) Draw an AVL Tree as each of the following keys are added in the order given. Show intermediate steps.

(i)

$$\{13, 17, 14, 19, 22, 18, 11, 10, 21\}$$

Solution:

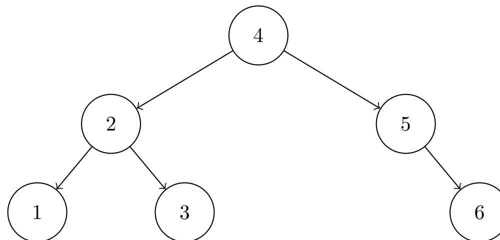
We've shown just the final tree here. There are various websites that will show what to do step-by-step. Here's one for BSTs/AVL trees: https://visualgo.net/en/bst



(ii)

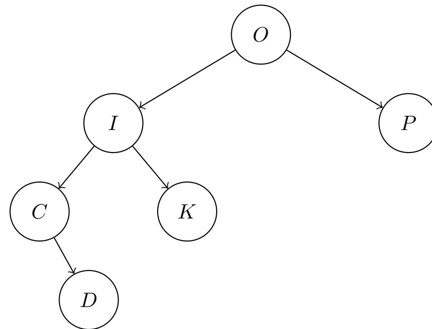
$$\{1, 2, 3, 4, 5, 6\}$$

Solution:



(b) Identify if the following trees are AVL trees. Explain your answer.

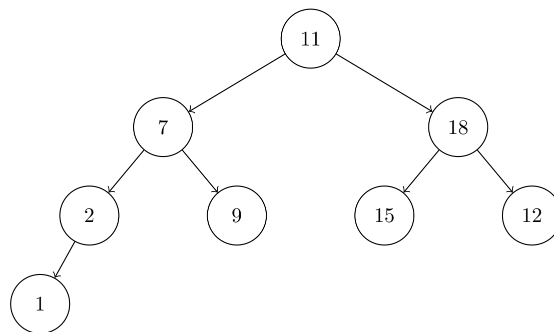
(i) Tree 1



Solution:

No, does not meet the balance property.

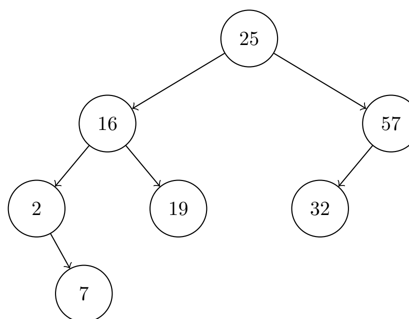
(ii) Tree 2



Solution:

No, does not meet the BST property. 12 is not greater than 18.

(iii) Tree 3



Solution:

Yes, it satisfies the balance and BST properties.

Food for thought

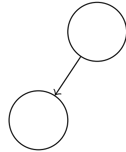
6. AVL trees

What is the minimum number of nodes in an AVL tree, given the following heights? Draw a picture of such a tree. (**Reminder**: an AVL tree's height is 0 for a tree with only 1 node in it!)

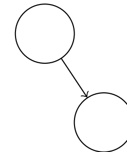
(a) 1

Solution:

Minimum number of nodes is 2, for a height of 1.



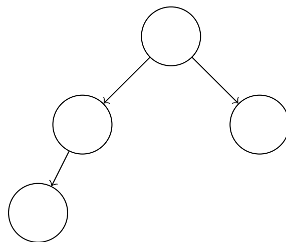
Solution:



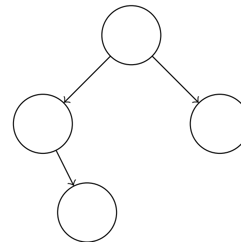
(b) 2

Solution:

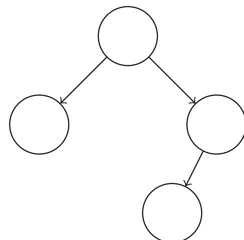
Minimum number of nodes is 4, for a height of 2.



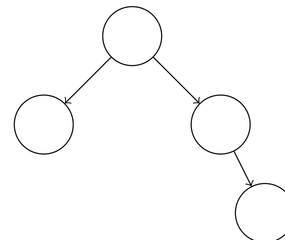
Solution:



Solution:



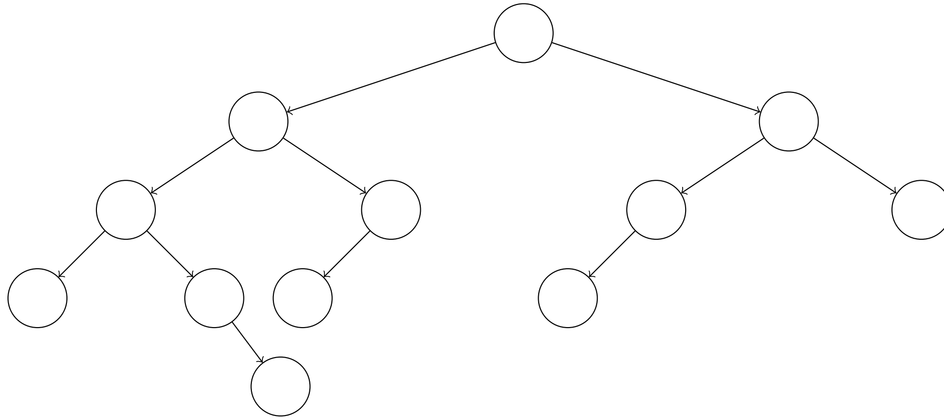
Solution:



(c) 4

Solution:

Minimum number of nodes is 12, for a height of 4. There are many possible solutions. Here is one!



7. Algorithm Design

(a) Given a binary search tree, describe how you could convert it into an AVL tree with worst-case time $\mathcal{O}(n \log(n))$. What is the best case runtime of your algorithm?

Solution:

Since we already have a BST, we can do an in-order traversal on the tree to get a sorted array of nodes. We could now simply insert all of these nodes back into an AVL tree using rotations which would give us an $\mathcal{O}(n \log(n))$ runtime.

(b) Given an AVL tree, describe how would you do a level order tree traversal. What is the worst-case runtime of your algorithm?

Solution:

Since an AVL tree is just a balanced BST, we can use a queue to add each node we visit. As we dequeue each node, we will add it's children to the queue. We would get an $\mathcal{O}(n)$ runtime.

Challenge Problems

8. Recurrences

(a) For the following recurrence, use the unfolding method to first convert the recurrence into a summation. Then, find a big-Θ bound on the function in terms of n. Assume all division operations are integer division.

$$T(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2T(n/3) + n & \text{otherwise} \end{cases}$$

Solution:

In order to determine what this expression looks like as a summation, it helps to first partially unroll it. When unfolding a recurrence like this it helps to

(i) Distribute the 2 (coefficient of the recursive call) at each step, to avoid having too many parentheses within parentheses.

(ii)

$$\begin{aligned} T(n) &= n + 2T\left(\frac{n}{3}\right) \\ &= n + 2\left(2T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right) = n + \frac{2n}{3} 2^2 T\left(\frac{n}{3^2}\right) \\ &= n + \frac{2n}{3} + 2^2\left(2T\left(\frac{n}{3^3} + \frac{n}{3^2}\right)\right) = n + \frac{2n}{3} + \frac{2^2 n}{3^2} + 2^3 T\left(\frac{n}{3^3}\right) \\ &= n + \frac{2n}{3} + \frac{2^2 n}{3^2} + 2^3\left(\frac{n}{3^3} + T\left(\frac{n}{3^4}\right)\right) = n + \frac{2n}{3} + \frac{2^2 n}{3^2} + \frac{2^3 n}{3^3} + 2^4 T\left(\frac{n}{3^4}\right) \end{aligned}$$

We can start to see the pattern now: our summation is roughly of the form

$$n + \frac{2n}{3} + \frac{2^2 n}{3^2} + \cdots + \frac{2^{i-1} n}{3^{i-1}} + 2^i\left(\frac{n}{3^i}\right)$$

Choosing $i = \log_3(n)$ puts us in the base case, and we get:

$$T(n) = n + \frac{2n}{3} + \frac{2^2 n}{3^2} + \cdots + \frac{2^{\log_3(n)-1} n}{3^{\log_3(n)-1}} + 2^{\log_3(n)}$$

Notice that unlike with most of our previous recurrences, the term for the base case is not a constant. Because each of our recursive calls makes two other recursive calls, we hit the base case more than once (in fact $2^{\log_3(n)}$) times).

Rewriting to be in summation form:

$$\begin{aligned} T(n) &= \sum_{i=0}^{\log_3(n)-1} \frac{2^i n}{3^i} + 2^{\log_3 n} \\ &= n \sum_{i=0}^{\log_3(n)-1} \left(\frac{2}{3}\right)^i + 2^{\log_3 n} \end{aligned}$$

Let's handle the summation first. We could evaluate the summation exactly, but that would be a bit tedious. Instead, let's show that it evaluates to some constant (which we can supress in the $\Theta()$ at the end). Each term is positive, and the first is $2/3$, so it is at least some constant. On the other hand, if instead of stopping when $i = \log_3(n) - 1$, we kept going to $i = \infty$, we would get an infinite geometric series, whose closed

form is $\frac{1}{1-2/3} = 3$. Since this infinite series is larger than the finite one we care about, our series sums to at most 3. Thus it is between $2/3$ and 3, and definitely must be a constant.

Now let's figure out $2^{\log_3 n}$. Using logarithm identity 4 of the math review, we can change that number to $n^{\log_3(2)}$. So we have that

$$T(n) = \Theta(n) + n^{\log_3(2)}$$

.

Which of those is the dominating term? $\log_3(2)$ is the number which you raise 3 to to get 2. Since 3 is less than 2, $\log_3(2) < 1$. So the first term dominates and $\Theta(n)$ is our final answer.

9. Modeling recursive functions

Consider the following recursive function. You may assume that the input will be a multiple of 3.

```
public int test(int n) {
    if (n <= 6) {
        return 2;
    } else {
        int curr = 0;
        for (int i = 0; i < n * n; i++) {
            curr += 1;
        }
        return curr + test(n - 3);
    }
}
```

(a) Write a recurrence modeling the *worst-case runtime* of `test`.

Solution:

$$T(n) = \begin{cases} 1 & \text{When } n \leq 6 \\ n^2 + T(n-3) & \text{Otherwise} \end{cases}$$

(b) Unfold the recurrence into a summation (for $n \geq 6$).

Solution:

$$1 + \sum_{i=3}^{n/3} (3i)^2$$

Modeling this recurrence correctly is slightly challenging because we want to decrease n in increments of 3.

To do this, what we do is set the summation bounds to go up to $n/3$ instead of n, and multiply i on the inside by 3, simulating changing i in those increments.

We then also set the lower summation bound to be 3 instead of 0 or 1. That way, our summation will only consider numbers in the range 9 to n – if we set $i = 2$ or lower, our summation would double-cound $n \leq 6$, which should be handled by the base case.

Note: our model only works if n is a multiple of 3.

(c) Simplify the summation into a closed form (for $n \geq 6$).

Solution:

$$\begin{aligned}
1 + \sum_{i=3}^{n/3} (3i)^2 &= 1 + \sum_{i=0}^{n/3} (3i)^2 - \sum_{i=0}^{2} (3i)^2 && \text{Adjusting summation bounds} \\
&= 1 + 9 \sum_{i=0}^{n/3} i^2 - \sum_{i=0}^{2} (3i)^2 && \text{Pulling out a constant} \\
&= 1 + 9 \sum_{i=0}^{n/3} i^2 - (0 + 9 + 36) && \text{Evaluating the summation} \\
&= 9 \frac{\frac{n}{3}\left(\frac{n}{3}+1\right)\left(\frac{2n}{3}+1\right)}{6} - 44 && \text{Sum of squares}
\end{aligned}$$

A "closed form", within the context of this class, is just any expression that does not contain a summation or is recursive. This means we can stop here without needing to further simplify the expression.

That said, if you wanted to continue simplifying, we could:

$$\begin{aligned}
9 \frac{\frac{n}{3}\left(\frac{n}{3}+1\right)\left(\frac{2n}{3}+1\right)}{6} - 44 &= \frac{9}{6}\left(\frac{n}{3}\left(\frac{n}{3}+1\right)\left(\frac{2n}{3}+1\right)\right) - 44 \\
&= \frac{1}{2}\left(n\left(\frac{n}{3}+1\right)\left(\frac{2n}{3}+1\right)\right) - 44 \\
&= \frac{1}{2}\left(n\left(\frac{2}{9}n^2 + n + 1\right)\right) - 44 \\
&= \frac{1}{9}n^3 + \frac{1}{2}n^2 + \frac{1}{2}n - 44
\end{aligned}$$

10. AVL Trees

(a) Is there a relationship between an AVL tree's height, and its minimum or maximum number of nodes? If so, what is it?

Solution:

There is a relationship for the minimum number of nodes, and it's recursive:

$$S_{min}(h) = \begin{cases} 1 & \text{When } h = 0 \\ 2 & \text{When } h = 1 \\ 1 + S_{min}(h-2) + S_{min}(h-1) & \text{Otherwise} \end{cases}$$

The maximum number of nodes is a little more straightforward: level i of the tree can have up to 2^i nodes in it. Summing across all levels we get:

$$S_{max}(h) = 2^{h+1} - 1$$

(b) Write a method `isAVLTree` to check if a given tree (which is guaranteed to be a valid BST) is a valid AVL tree. If it helps, you may write this method for this tree class, `HeightTree`, which keeps track of the height of a tree at each node:

```
public class HeightTree {
    private IntHeightNode overallRoot;

    // constructors and other methods omitted for clarity

    private class IntHeightNode {
        public int data;
        public int height;
        public IntHeightNode left;
        public IntHeightNode right;

        // constructors omitted for clarity
    }
}
```

Solution:

```
public boolean isAVLTree() {
    return isAVLTree(overallRoot);
}

private boolean isAVLTree(IntHeightNode root) {
    if (root == null) {
        return true;
    } else if (Math.abs(height(root.left) - height(root.right)) > 1) {
        return false;
    } else {
        return isAVLTree(root.left) && isAVLTree(root.right);
    }
}

private int height(IntHeightNode root) {
    if (root == null) {
        return -1;
    } else {
        return root.height;
    }
}
```

(c) Now write `isAVLTree` *without* assuming that the tree is a valid BST.

Solution:

```
public boolean isAVLTree() {
    return isAVLTree(overallRoot, Integer.MIN_VALUE, Integer.MAX_VALUE);
}

private boolean isAVLTree(IntHeightNode root, int min, int max) {
    if (root == null) {
        return true;
    } else if (root.data > max || root.data < min) {
        return false;
    } else if (Math.abs(height(root.left) - height(root.right)) > 1) {
        return false;
    } else {
        return isAVLTree(root.left, min, root.data - 1) &&
            isAVLTree(root.right, root.data + 1, max);
    }
}

private int height(IntHeightNode root) {
    if (root == null) {
        return -1;
    } else {
        return root.height;
    }
}
```

(d) Now write `isAVLTree` for the `IntTree` class (you may assume again that the tree is guaranteed to be a valid BST).

Solution:

```
public boolean isAVLTree() {
    return isAVLTree(overallRoot);
}

private boolean isAVLTree(IntTreeNode root) {
    if (root == null) {
        return true;
    } else if (Math.abs(height(root.left) - height(root.right)) > 1) {
        return false;
    } else {
        return isAVLTree(root.left) && isAVLTree(root.right);
    }
}

private int height(IntTreeNode root) {
    if (root == null) {
        return -1;
    } else {
        return 1 + Math.max(height(root.left), height(root.right));
    }
}
```