CSE 373: Data Structures & Algorithms
Graph Traversals / Topological Sort

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Course Logistics

- HW4 out → graphs!

- Midterms back in section tomorrow. Regrade policy on the website.

Graphs Review from last time

What is some of the terminology for graphs and what do those terms mean?

- vertices and edges
- directed / undirected
- in-degree and out-degree
- connected and fully connected
- weighted / unweighted
- acyclic
- DAG: Directed Acyclic Graph

Graphs Applications Review

For each of the following examples:

- what are the vertices and what are the edges?
- would you use directed edges? Would they have self-edges?
- Are there 0-degree nodes? Is it strongly or weakly connected?
- Does it have weights? Do negative weights make sense?
- Does it have cycles? Is it a DAG?

- Web pages with links
- Facebook friends
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- Political donations to candidates

Graph Traversals

Graph Traversals

For an arbitrary graph and a starting node **v**, find all nodes *reachable* from **v** (i.e., there exists a path from **v**)

- Possibly "do something" for each node
- Examples: print to output, set a field, etc.

- **Also solves**: Is an undirected graph connected?
- **Related but different problem**: Is a directed graph strongly connected?

Basic idea of traversal:

- Keep following nodes
- But "mark" nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

Abstract Idea in Pseudocode

```
void traverseGraph(Node start) {
     Set pending = emptySet()
     pending.add(start)
     mark start as visited
     while(pending is not empty) {
       next = pending.remove()
       for each node u adjacent to next
          if (u is not marked visited) {
             mark u
             pending.add(u)
          }
     }
  }
```

Running Time and Options

- Assuming **`add`** and **`remove`** for pending set are $O(1)$, entire traversal is $O(|E|)$ using an adjacency list representation

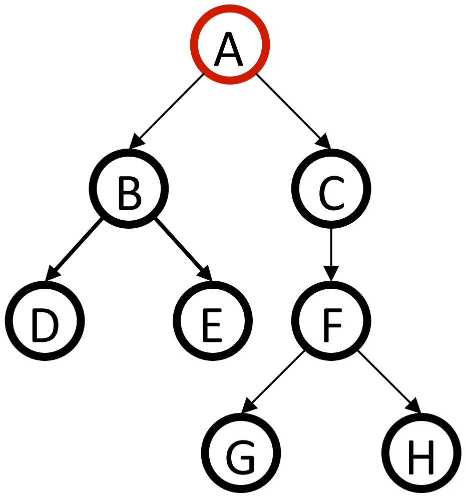
- The order we traverse depends entirely on **`add`** and **`remove`**
  - Popular choice: a stack "depth-first graph search" → **DFS**
  - Popular choice: a queue "breadth-first graph search" → **BFS**

- **DFS and BFS are "big ideas" in computer science**
  - Depth: recursively explore one part before going back to the other parts not yet explored
  - Breadth: explore areas closer to the start node first

Cool visualization: http://visualgo.net/dfsbfs.html

Example: trees

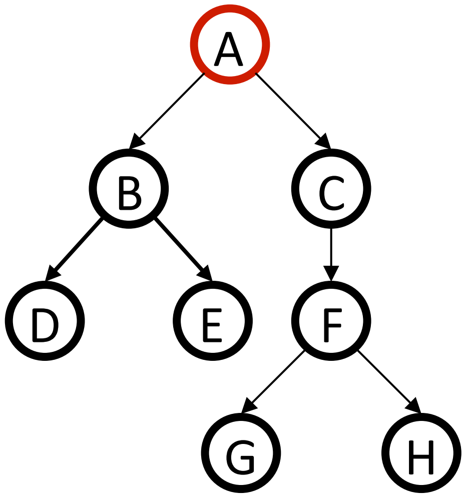
- A tree is a graph and make DFS and BFS are easier to "see"




```
DFS(Node start) {
  mark and process start
  for each node u adjacent to start
    if u is not marked
      DFS(u)
}
```

- A, B, D, E, C, F, G, H
- Exactly what we called a "pre-order traversal" for trees
  - The marking is because we support arbitrary graphs and we want to process each node exactly once

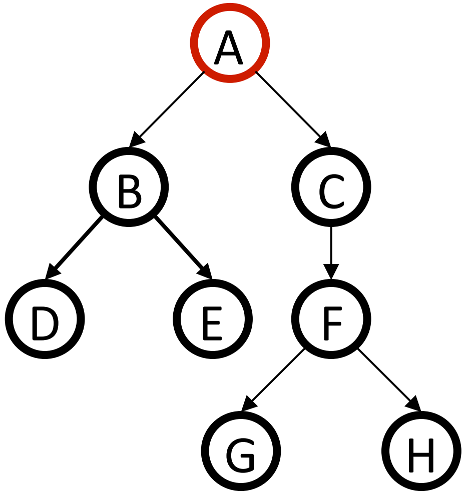
Example: trees




```
DFS2(Node start) {
   initialize stack s to hold start
   mark start as visited
   while(s is not empty) {
     next = s.pop() // and "process"
     for each node u adjacent to next
       if(u is not marked)
         mark u and push onto s
   }
}
```

- A, C, F, H, G, B, E, D
- A different but perfectly fine depth traversal

Example: trees




```
BFS(Node start) {
  initialize queue q to hold start
  mark start as visited
  while(q is not empty) {
    next = q.dequeue() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and enqueue onto q
  }
}
```

- A, B, C, D, E, F, G, H
- A "level-order" traversal

Comparison

- Breadth-first always finds shortest length paths, i.e., "optimal solutions"
  - Better for "what is the shortest path from **x** to **y**"
- But depth-first can use less space in finding a path
  - If *longest path* in the graph is **p** and highest out-degree is **d** then DFS stack never has more than **d*p** elements
  - But a queue for BFS may hold $O(|V|)$ nodes
- A third approach (useful in Artificial Intelligence)
  - *Iterative deepening (IDFS)*:
    - Try DFS but disallow recursion more than **K** levels deep
    - If that fails, increment **K** and start the entire search over
  - Like BFS, finds shortest paths. Like DFS, less space.

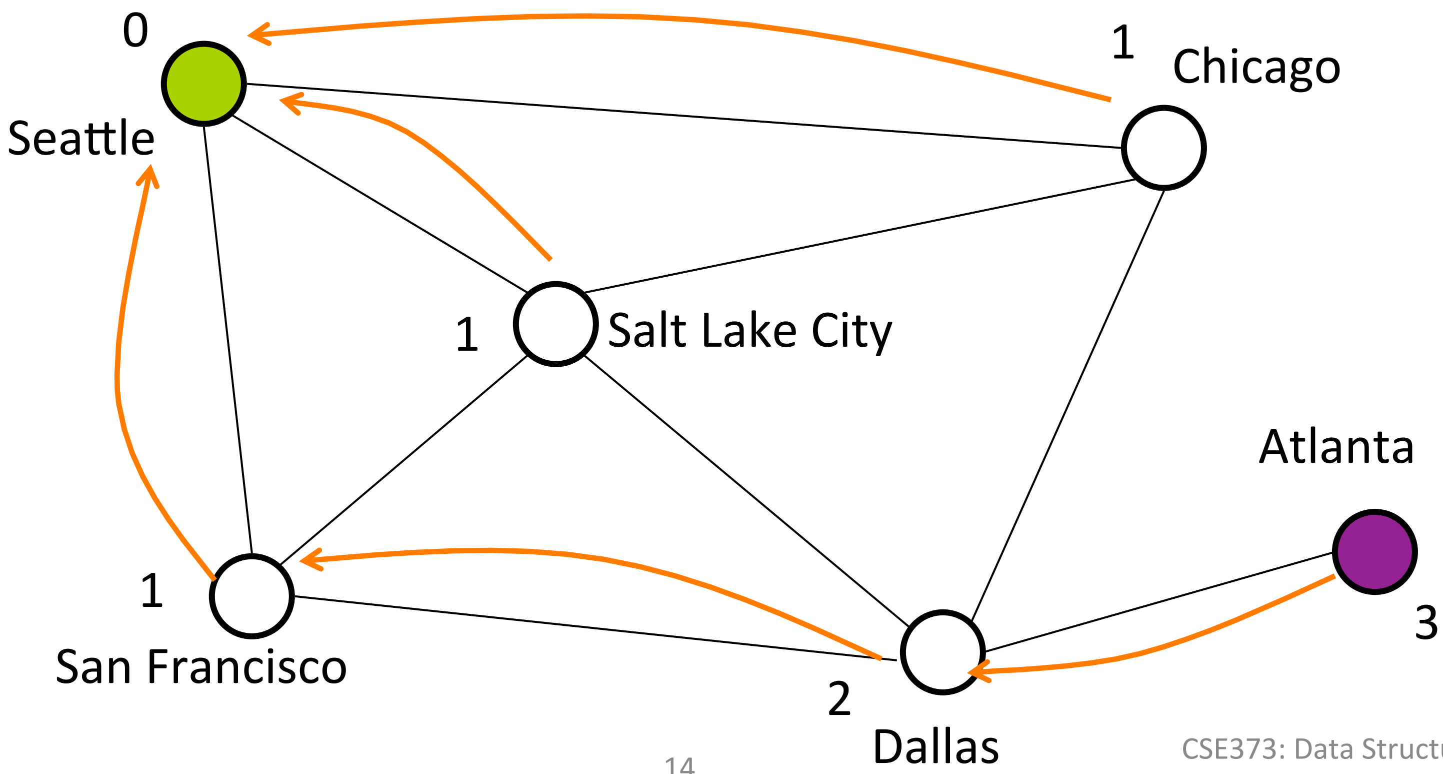
Saving the Path

- Our graph traversals can answer the reachability question:
  - "Is there a path from node x to node y?"
- But what if we want to actually output the path?
  - Like getting driving directions rather than just knowing it's possible to get there!
- How to do it:
  - Instead of just "marking" a node, store the previous node along the path (when processing **u** causes us to add **v** to the search, set **`v.path`** field to be **u**)
  - When you reach the goal, follow **`path`** fields back to where you started (and then reverse the answer)
  - If just wanted path *length*, could put the integer distance at each node instead

Example using BFS

What is a path from Seattle to Atlanta

- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique



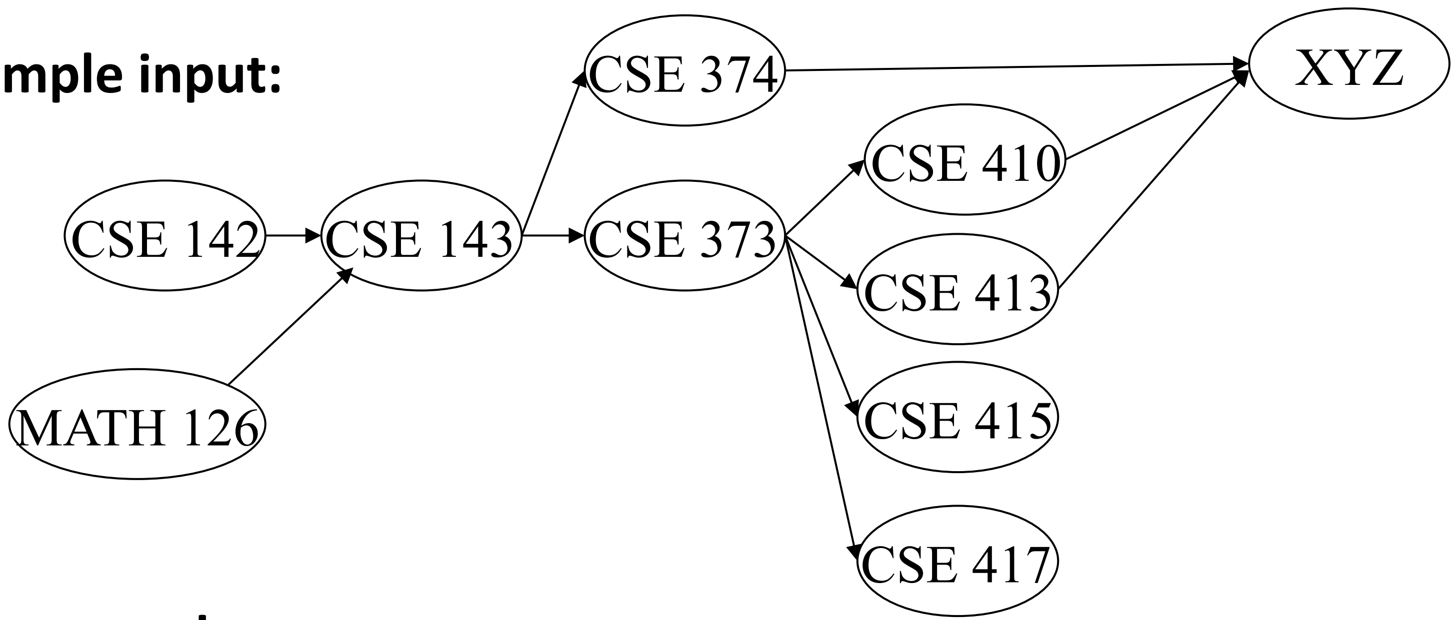
Topological Sort

Topological Sort

Disclaimer: Don't base your course schedules on this Material. Please…

Problem: Given a DAG **`G=(V,E)`**, output all vertices in an order such that no vertex appears before another vertex that has an edge to it

Example input:



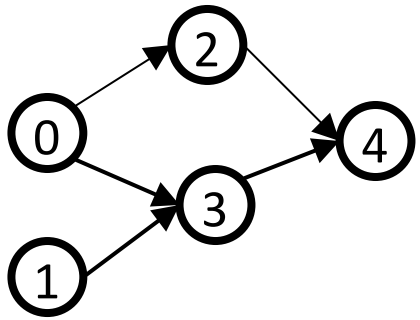
One example output:

126, 142, 143, 374, 373, 417, 410, 413, XYZ, 415

Questions and comments

- **Why do we perform topological sorts only on DAGs?**
  – Because a cycle means there is no correct answer

- **Is there always a unique answer?**
  – No, there can be 1 or more answers; depends on the graph
  – Graph with 5 topological orders:



- **Do some DAGs have exactly 1 answer?**
  – Yes, including all lists

- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Uses of Topological Sort

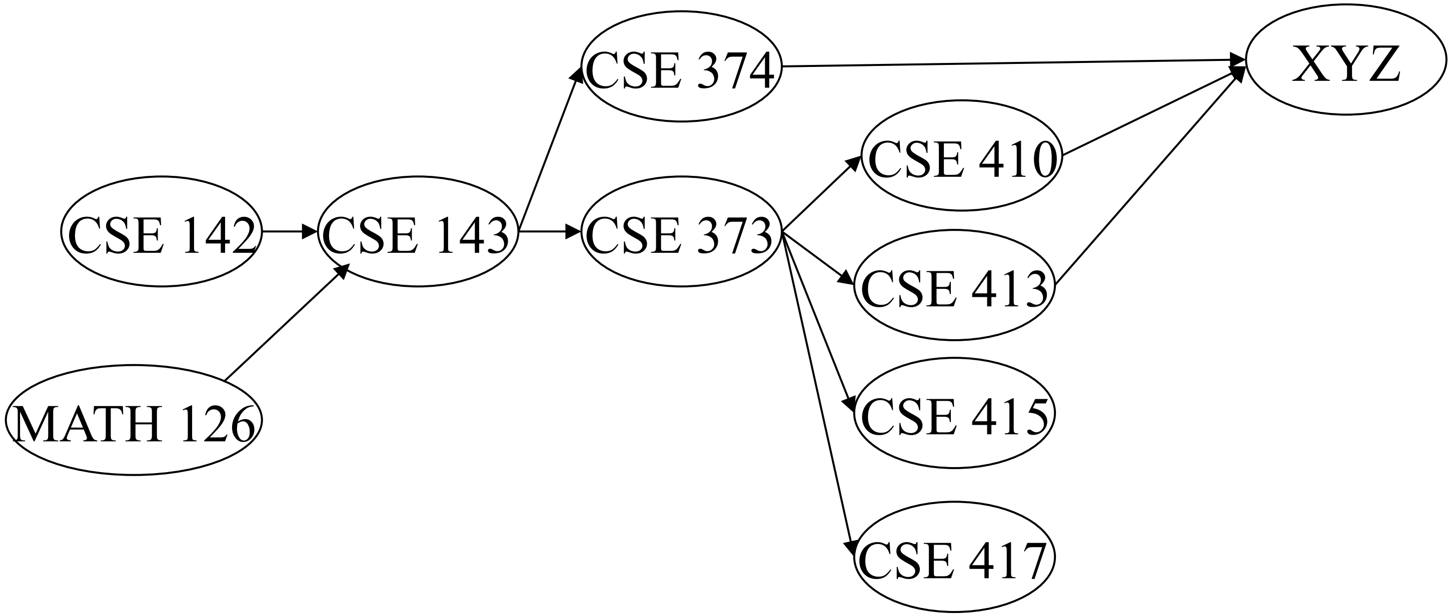
- Figuring out how to graduate
- Computing an order in which to recompute cells in a spreadsheet
- Determining an order to compile files using a Makefile
- In general, taking a dependency graph and finding an order of execution
- …

A First Algorithm for Topological Sort

1. Label ("mark") each vertex with its in-degree
   - Think "write in a field in the vertex"
   - Could also do this via a data structure (e.g., array) on the side

2. While there are vertices not yet output:
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **`E`**), decrement the in-degree of **u**

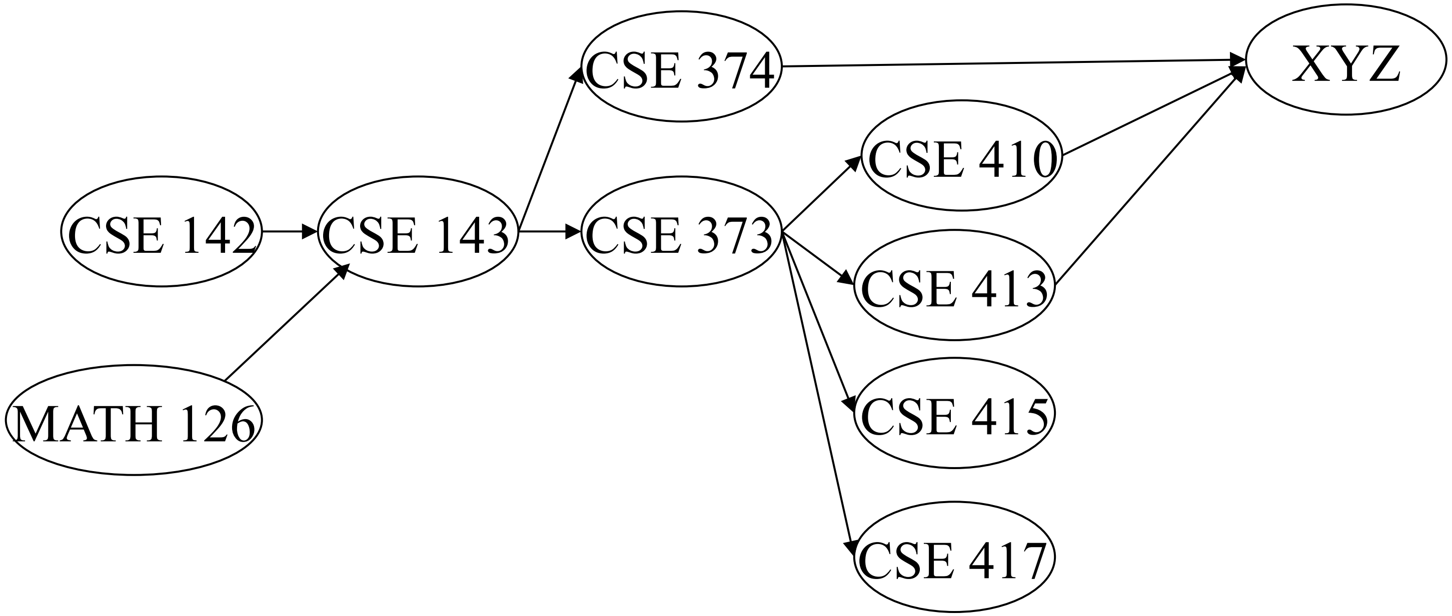
Example



Output:

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?										
In-degree:	0	0	2	1	1	1	1	1	1	3

Example

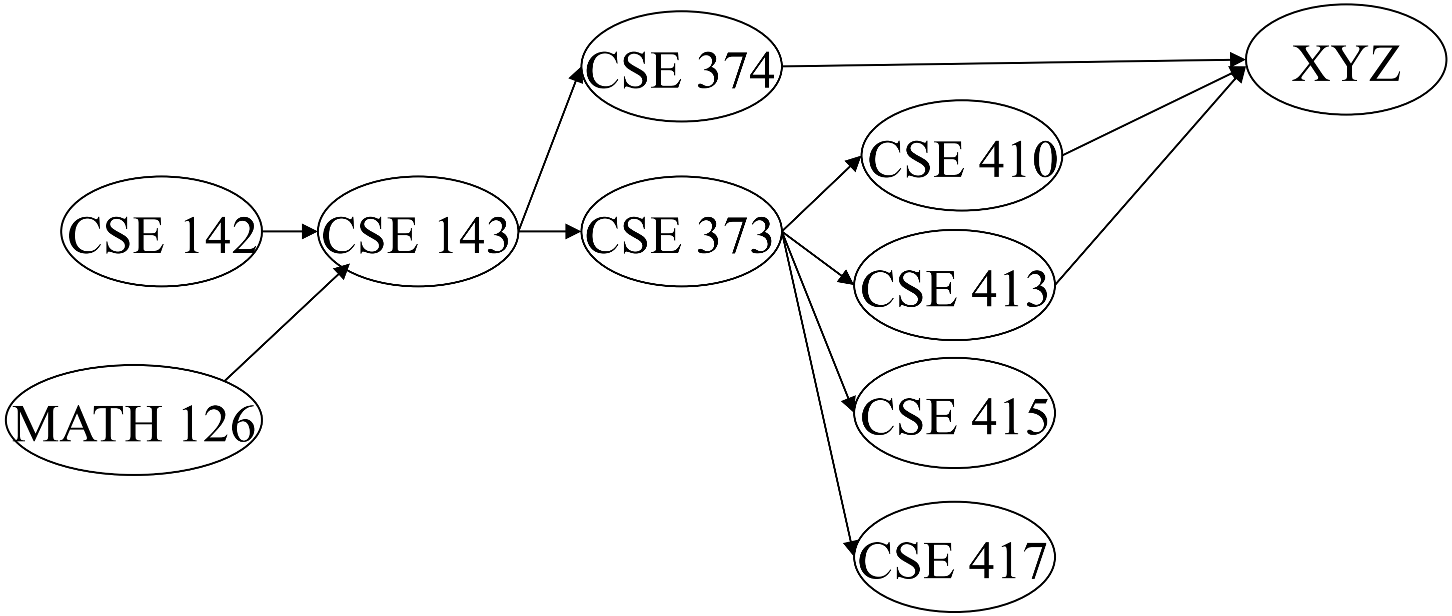


Output:

126

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x									
In-degree:	0	0	~~2~~ 1	1	1	1	1	1	1	3

Example



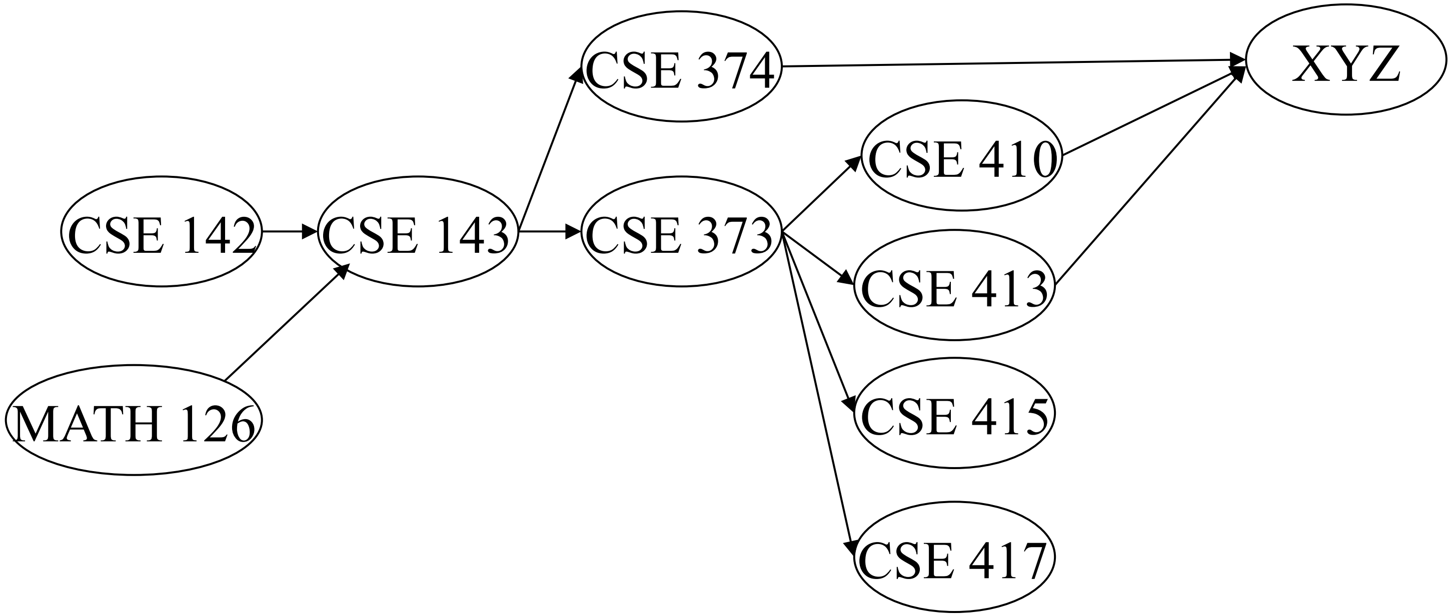
Output:

126

142

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x								
In-degree:	0	0	~~2~~ ~~1~~ 0	1	1	1	1	1	1	3

Example



Output:

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Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x							
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	1	1	1	1	3

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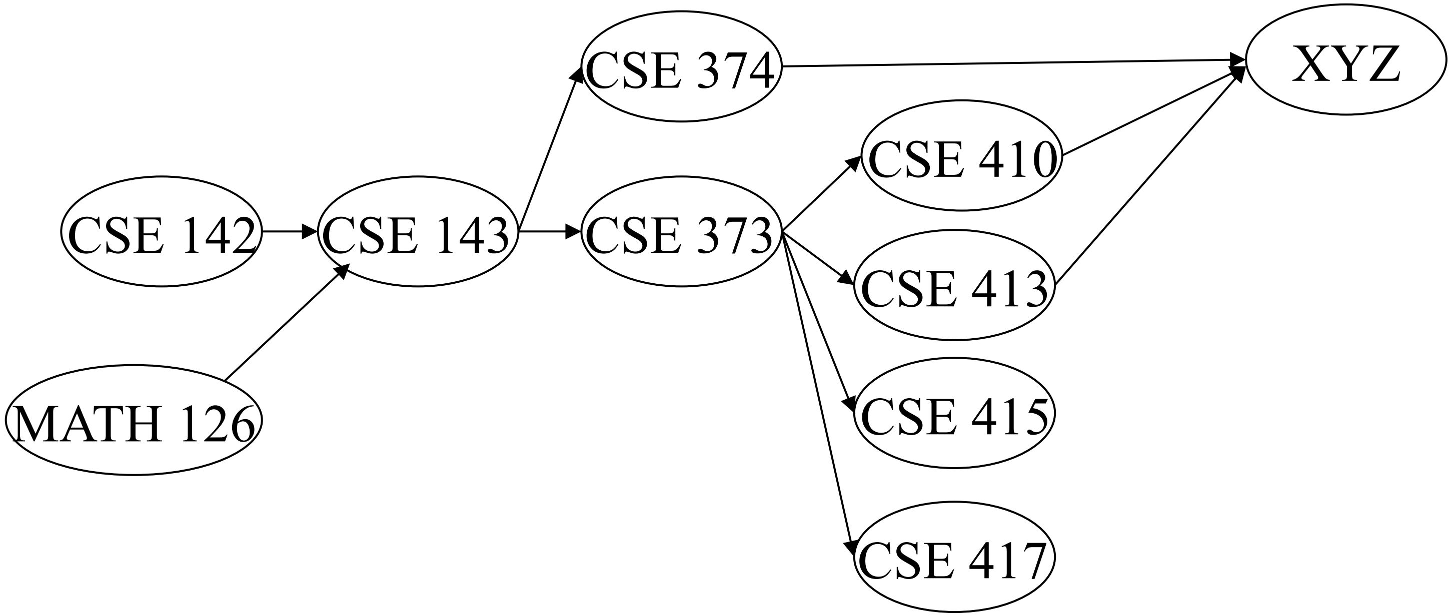

Output:

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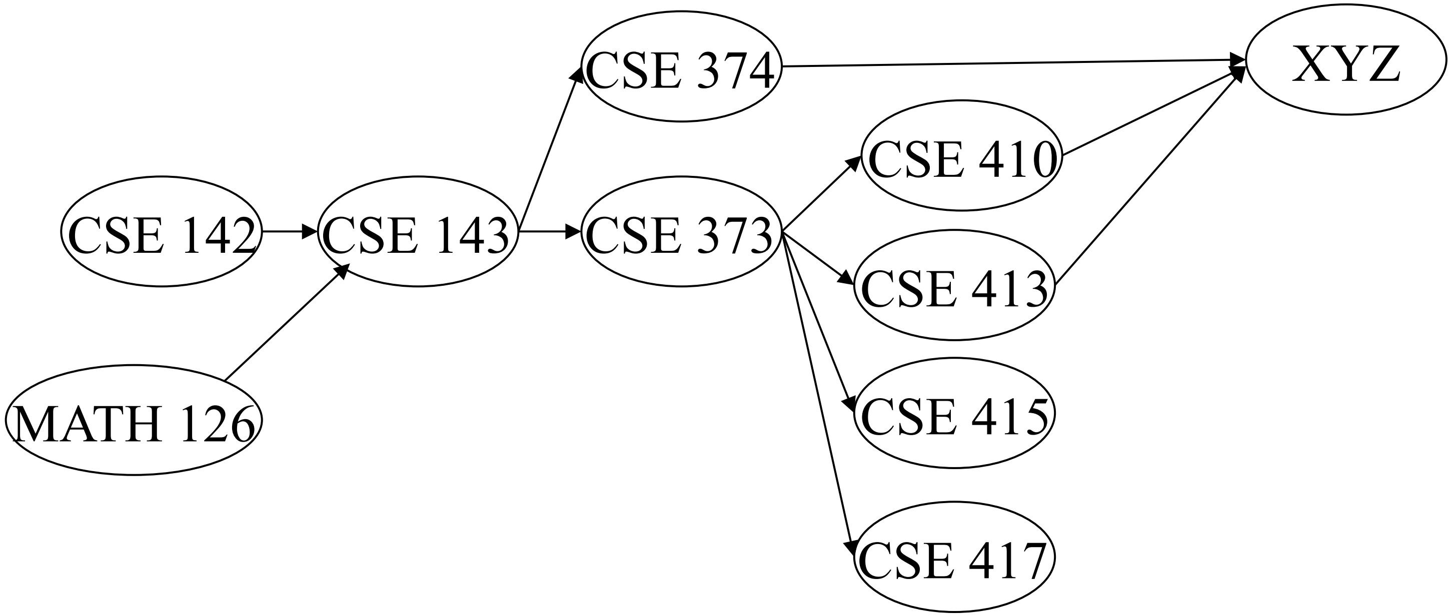
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x						
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	1	1	1	1	~~3~~ 2

Example



Output:

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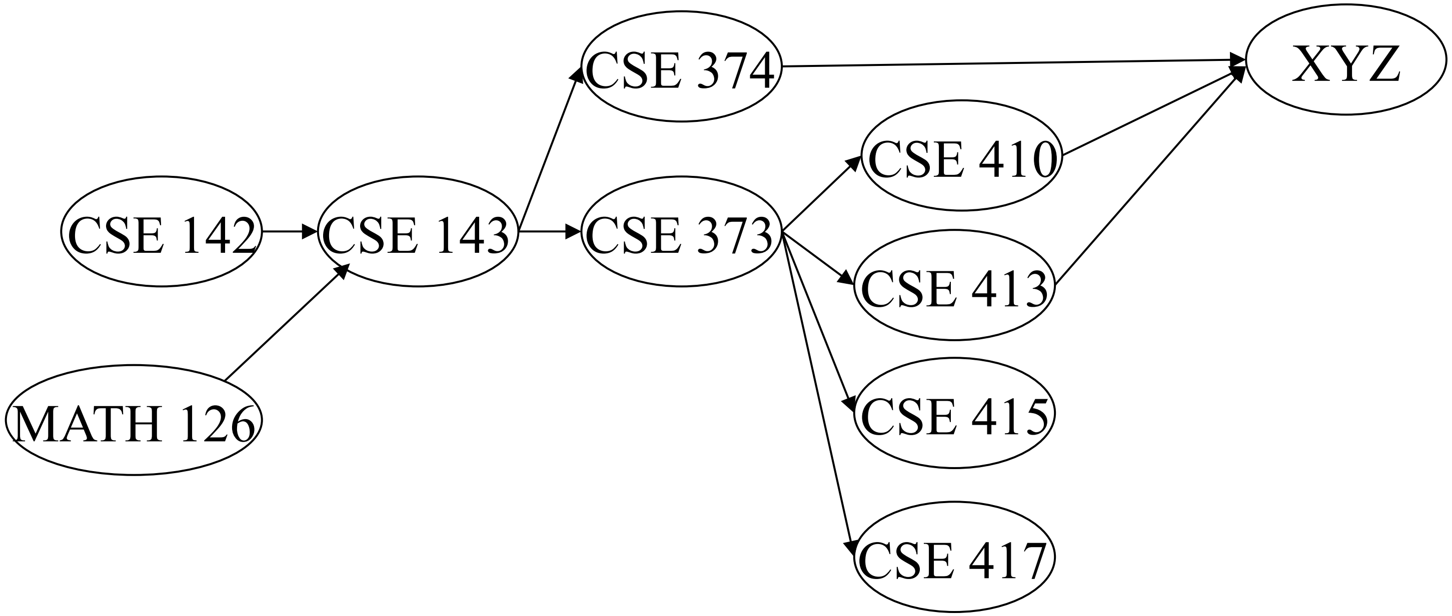
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x					
In-degree:	0	0	~~2~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~1~~	~~3~~
			~~1~~	0	0	0	0	0	0	2
			0							

Example



Output:

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Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x				x	
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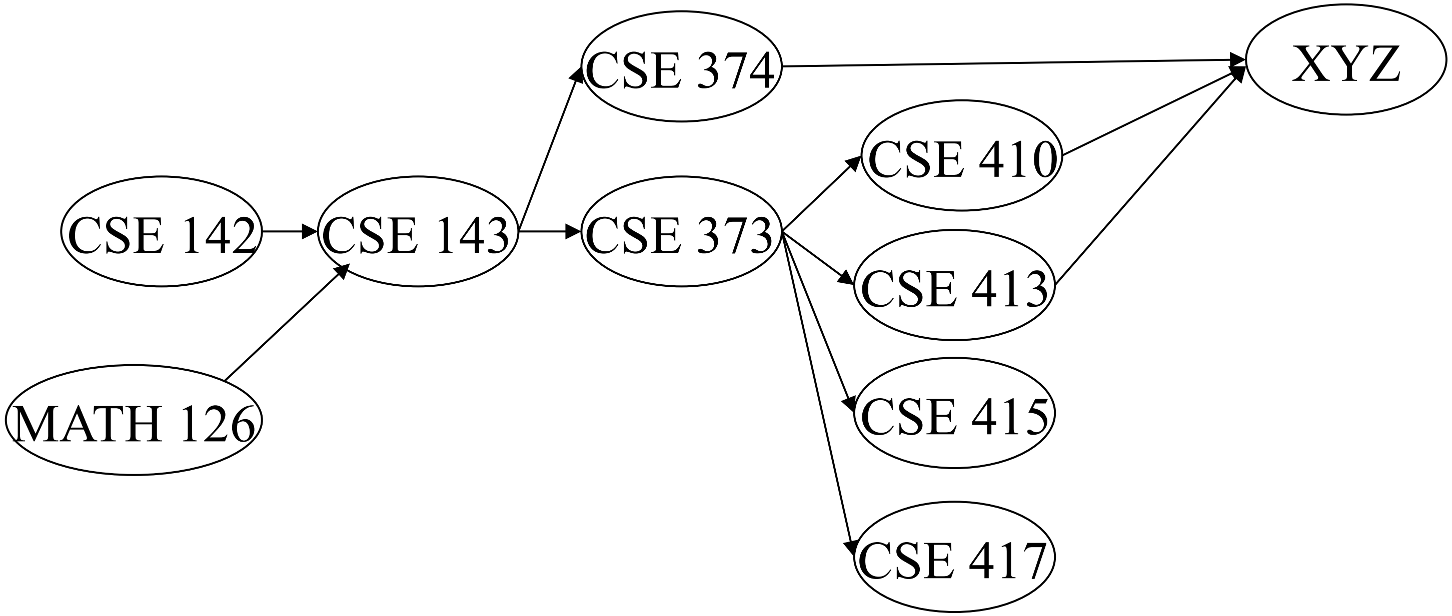

Output:

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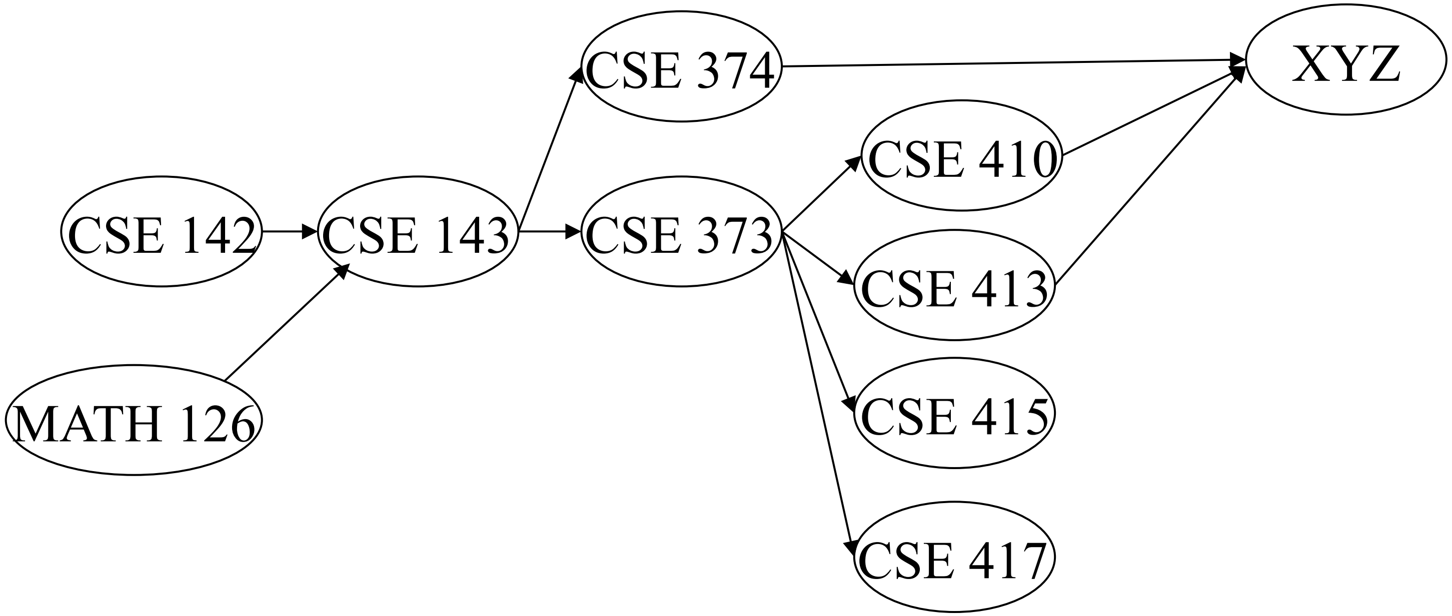
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x			x	
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ 1

Example



Output:
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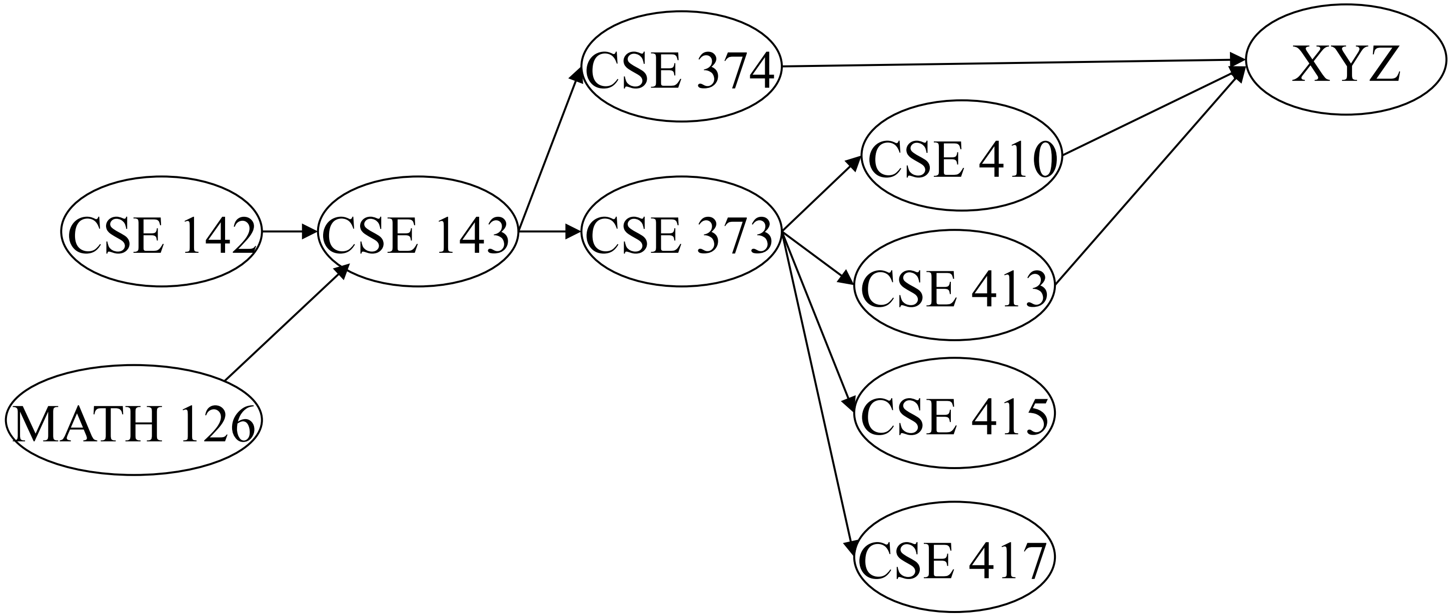
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x		x	
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ ~~1~~ 0

Example



Output:
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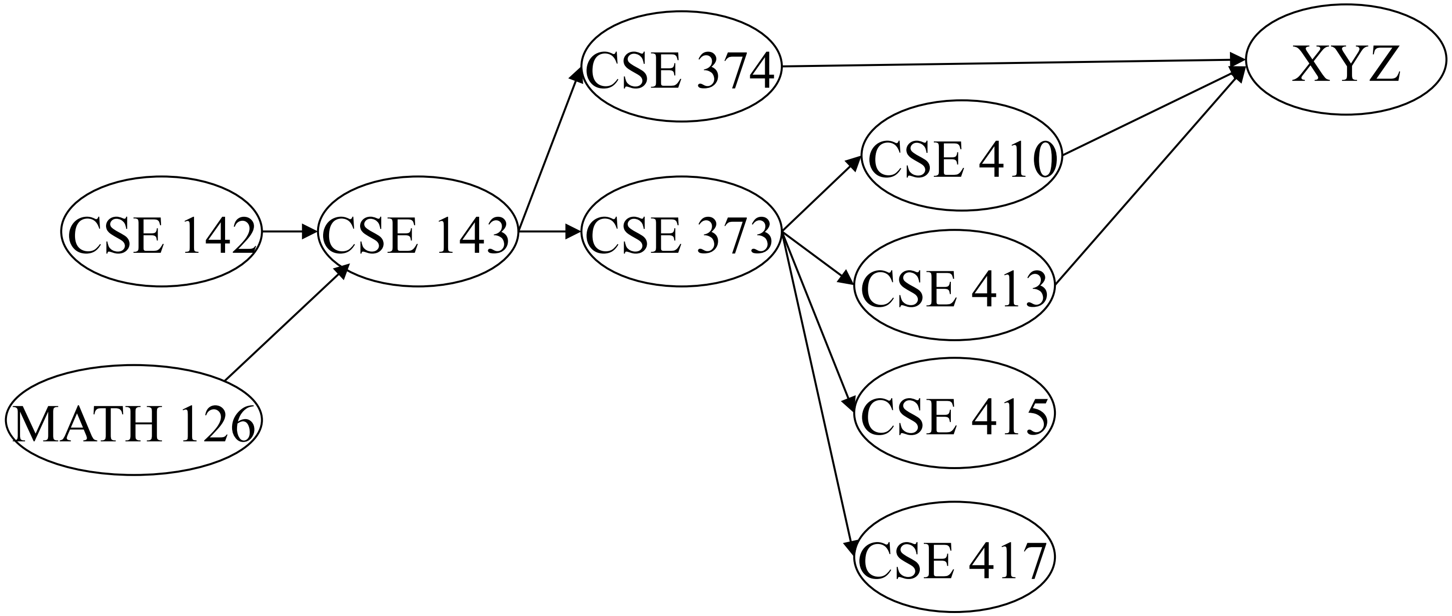
XYZ

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x		x	x
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ ~~1~~ 0

Example



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XYZ
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Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x	x	x	x
In-degree:	0	0	~~2~~ ~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~1~~ 0	~~3~~ ~~2~~ ~~1~~ 0

Notice

- Needed a vertex with in-degree 0 to start
  - Will always have at least 1 because no cycles

- Ties among vertices with in-degrees of 0 can be broken arbitrarily
  - Can be more than one correct answer, by definition, depending on the graph

Psuedocode Example

```
labelEachVertexWithItsInDegree();
for(ctr = 0; ctr < numVertices; ctr++){
    v = findNewVertexOfDegreeZero();
    put v next in output
    for each w adjacent to v
       w.indegree--;
}
```

- What is the worst-case running time?

Pseudocode Example

```
labelEachVertexWithItsInDegree();
for(ctr = 0; ctr < numVertices; ctr++){
    v = findNewVertexOfDegreeZero();
    put v next in output
    for each w adjacent to v
        w.indegree--;
}
```

- What is the worst-case running time?
  - Initialization $O(|V|+|E|)$ (assuming adjacency list)
  - Outer loop: runs |V| times
  - findNewVertex: $O(|V|)$
  - Sum of all decrements for the whole algorithm assuming adjacency list: $O(|E|)$ (each edge is *removed* once)
  - So total is $O(|V|^2)$ – not good for a sparse graph!

A better idea

The trick is to avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, bag, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both *O*(1)

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) **v** = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **E**), decrement the in-degree of **u**, if new degree is 0, enqueue it

Pseudocode Example 2

```
labelAllAndEnqueueZeros();
while queue not empty {
   v = dequeue();
   put v next in output
   for each w adjacent to v {
      w.indegree--;
      if (w.indegree==0)
         enqueue(v);
   }
}
```

- What is the worst-case running time?

Pseudocode Example 2

```
labelAllAndEnqueueZeros();
while queue not empty {
   v = dequeue();
   put v next in output
   for each w adjacent to v {
      w.indegree--;
      if (w.indegree==0)
         enqueue(v);
   }
}
```

- What is the worst-case running time?
  - Initialization: $O(|V|+|E|)$ (assuming adjacenty list)
  - Sum of all enqueues and dequeues: $O(|V|)$
  - Sum of all decrements: $O(|E|)$ (assuming adjacency list)
  - So total is $O(|E| + |V|)$ – much better for sparse graph!

Shortest Cost Path

Single source shortest paths

- Done: BFS to find the minimum path length from **v** to **u** in $O(|E|+|V|)$
- Actually, can find the minimum path length from **v** to *every node*
  – Still $O(|E|+|V|)$
  – No faster way for a "distinguished" destination in the worst-case
- Now: Weighted graphs

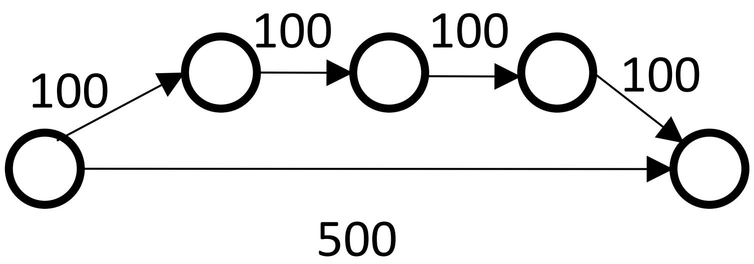
Given a weighted graph and node **v**,
find the minimum-cost path from **v** to every node

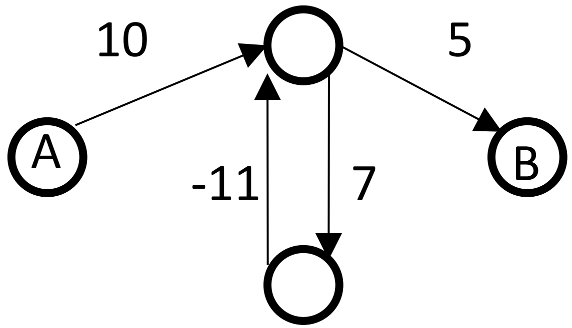
- As before, asymptotically no harder than for one destination
- Unlike before, BFS will not work -> only looks at path length.

Shortest Path: Applications

- Driving directions
- Cheap flight itineraries
- Network routing
- Critical paths in project management

Not as easy





Why BFS won't work: Shortest path may not have the fewest edges

- Annoying when this happens with costs of flights

We will assume there are no negative weights

- *Problem* is *ill-defined* if there are negative-cost *cycles*
- *Today's algorithm* is *wrong* if *edges* can be negative
  - There are other, slower (but not terrible) algorithms

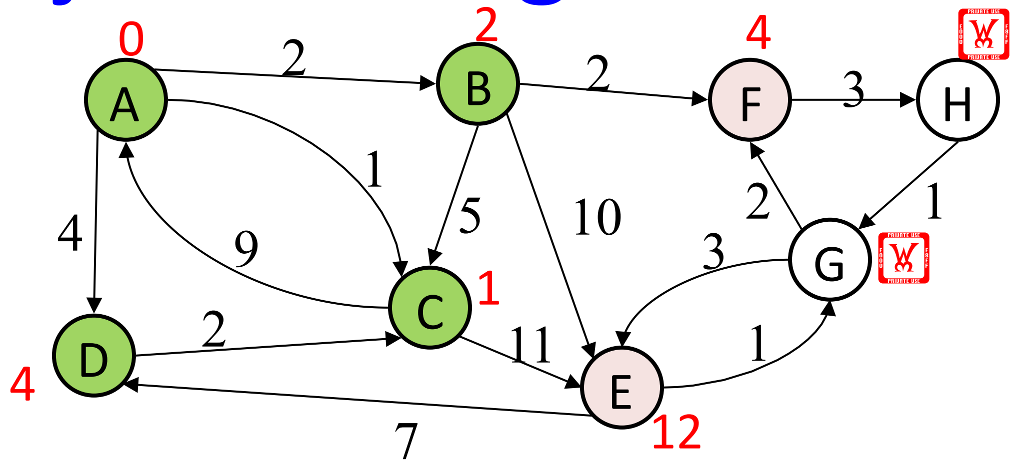
Dijkstra

- Algorithm named after its inventor Edsger Dijkstra (1930-2002)
  - Truly one of the "founders" of computer science; this is just one of his many contributions
  - My favorite Dijkstra quote: "computer science is no more about computers than astronomy is about telescopes"

Dijkstra's algorithm

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a "best distance so far"
  - A priority queue will turn out to be useful for efficiency

Dijkstra's Algorithm: Idea



- Initially, start node has cost 0 and all other nodes have cost ∞
- At each step:
  - Pick closest unknown vertex **v**
  - Add it to the "cloud" of known vertices
  - Update distances for nodes with edges from **v**
- That's it! (But we need to prove it produces correct answers)

The Algorithm

1. For each node **v**, set **v.cost = ∞** and **v.known = false**
2. Set **source.cost = 0**
3. While there are unknown nodes in the graph
   a) Select the unknown node **v** with lowest cost
   b) Mark **v** as known
   c) For each edge **(v,u)** with weight **w**,

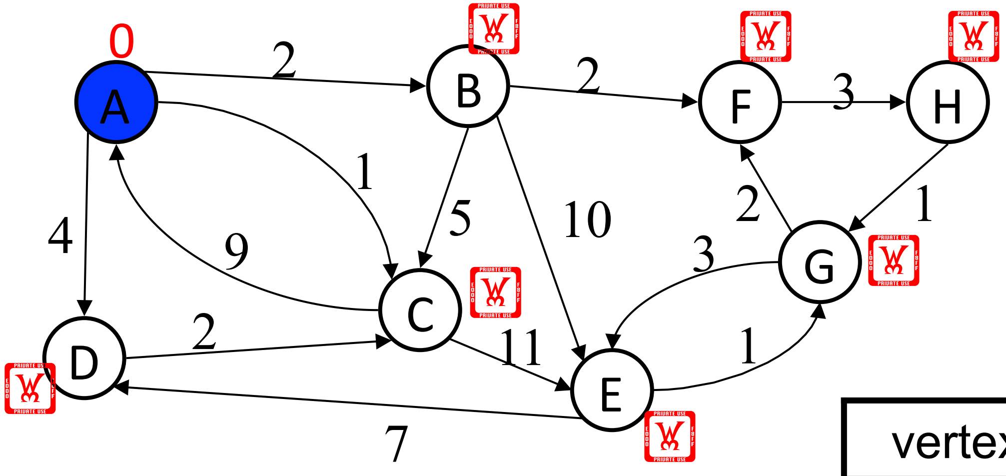
```
c1 = v.cost + w // cost of best path through v to u
c2 = u.cost     // cost of best path to u previously known
if(c1 < c2){ // if the path through v is better
  u.cost = c1
  u.path = v // for computing actual paths
}
```

Important features

- When a vertex is marked known, the cost of the shortest path to that node is known
  - The path is also known by following back-pointers

- While a vertex is still not known, another shorter path to it *might* still be found

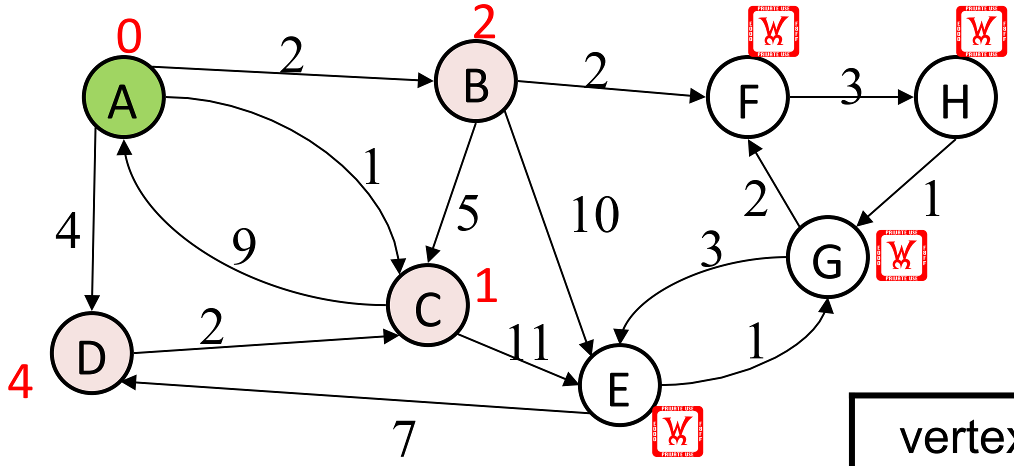
Example #1



<u>Order Added to Known Set:</u>

vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	
H		??	

Example #1

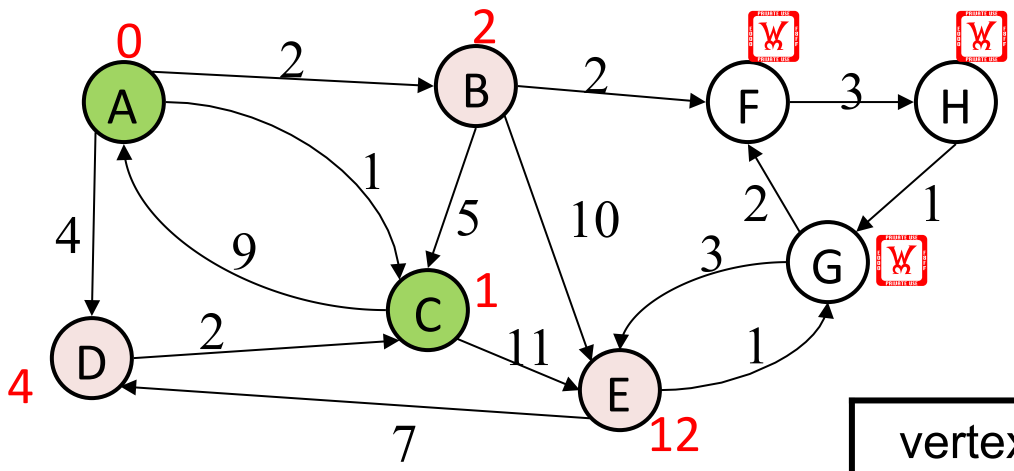


vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C		≤ 1	A
D		≤ 4	A
E		??	
F		??	
G		??	
H		??	

Order Added to Known Set:

A

Example #1

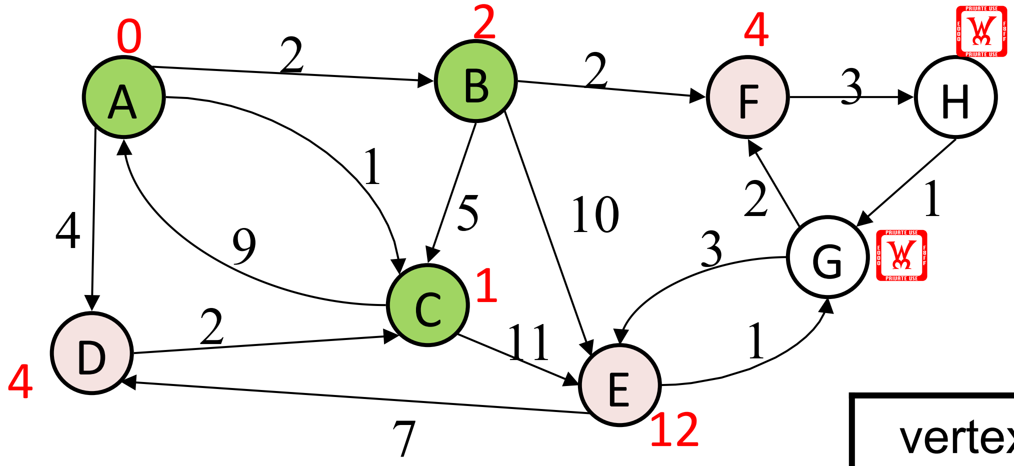


vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		??	
G		??	
H		??	

Order Added to Known Set:

A, C

Example #1

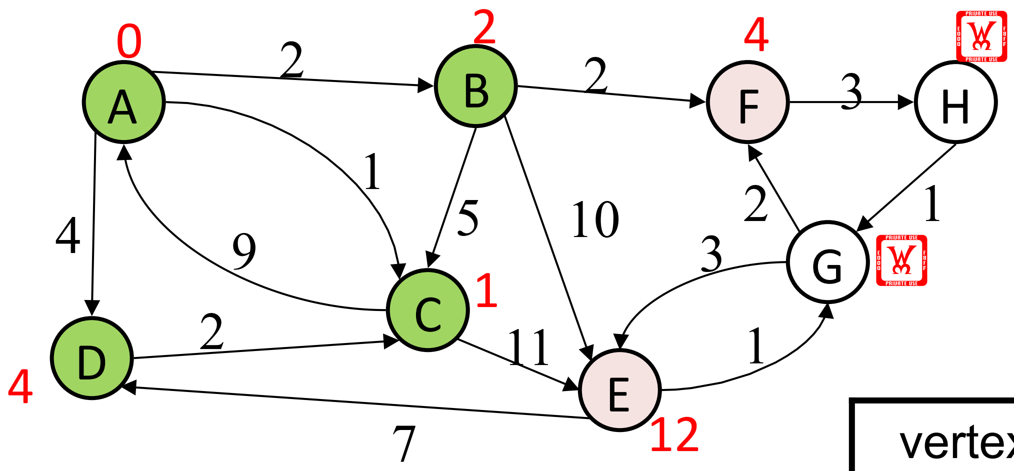


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Order Added to Known Set:

A, C, B

Example #1

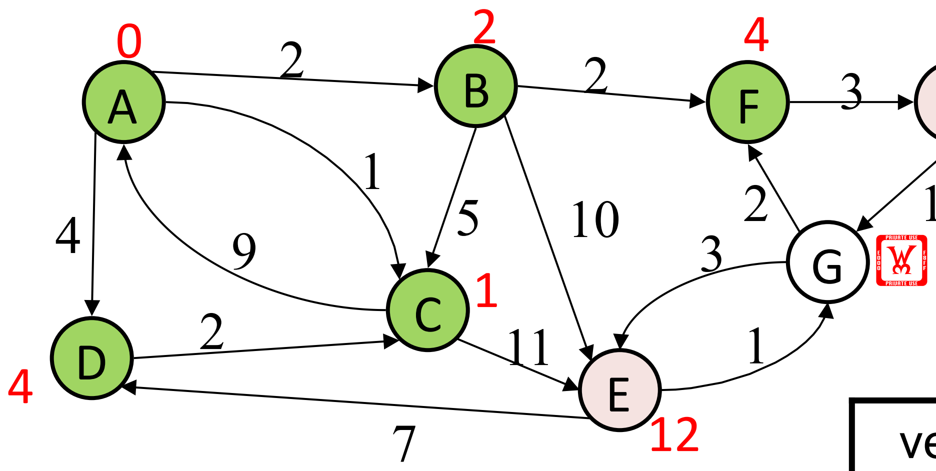


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Order Added to Known Set:

A, C, B, D

Example #1

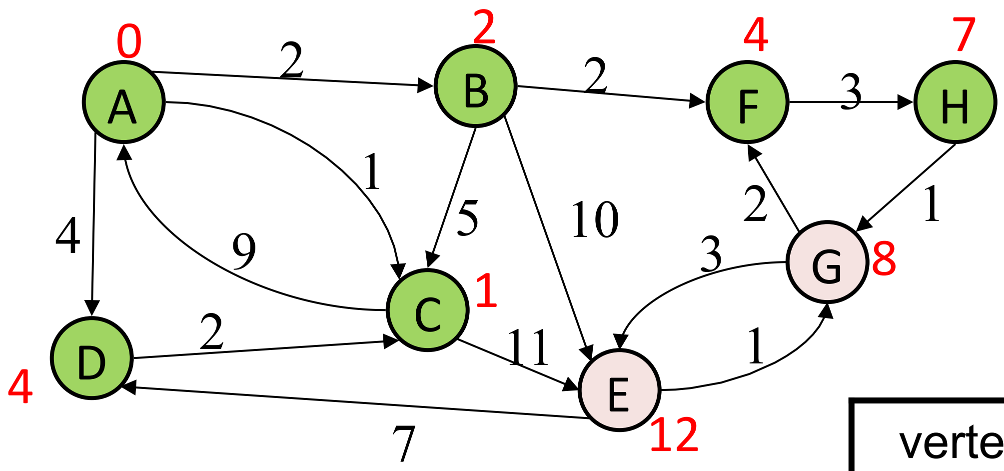


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		??	
H		≤ 7	F

Order Added to Known Set:

A, C, B, D, F

Example #1

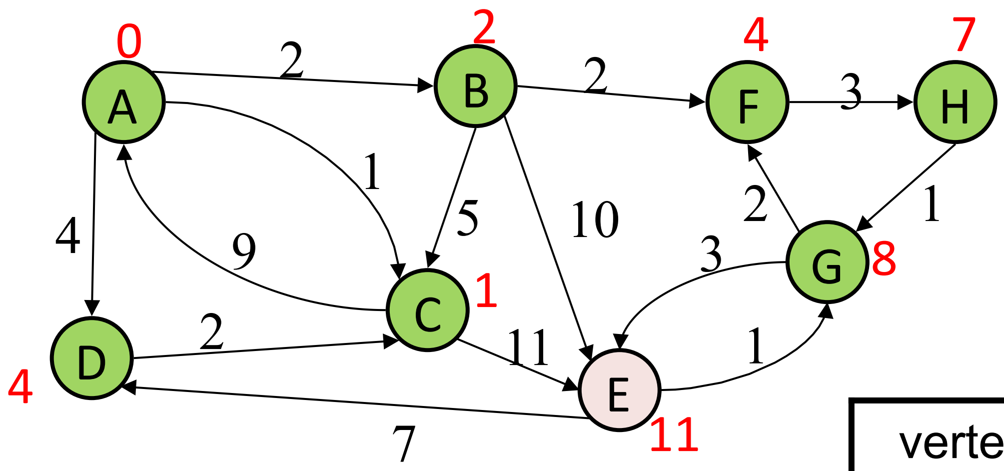


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		≤ 8	H
H	Y	7	F

<u>Order Added to Known Set:</u>

A, C, B, D, F, H

Example #1

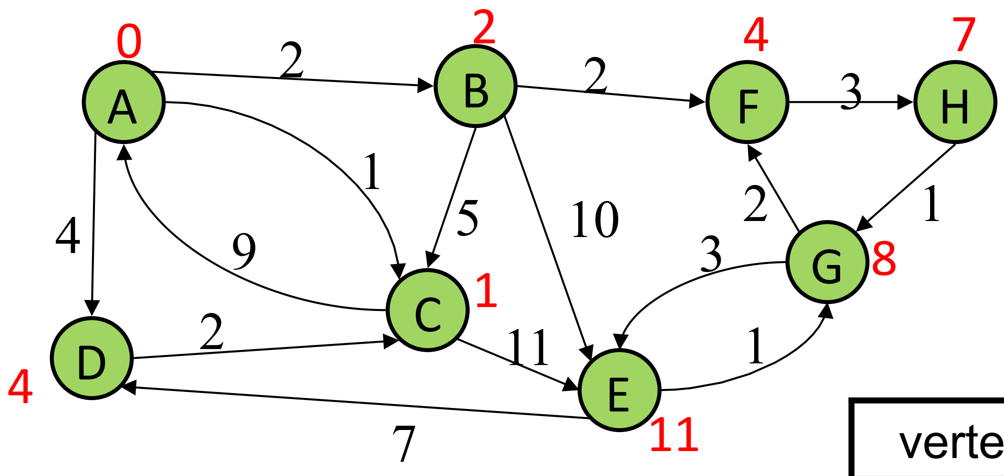


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H, G

Example #1



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Order Added to Known Set:

A, C, B, D, F, H, G, E

Features

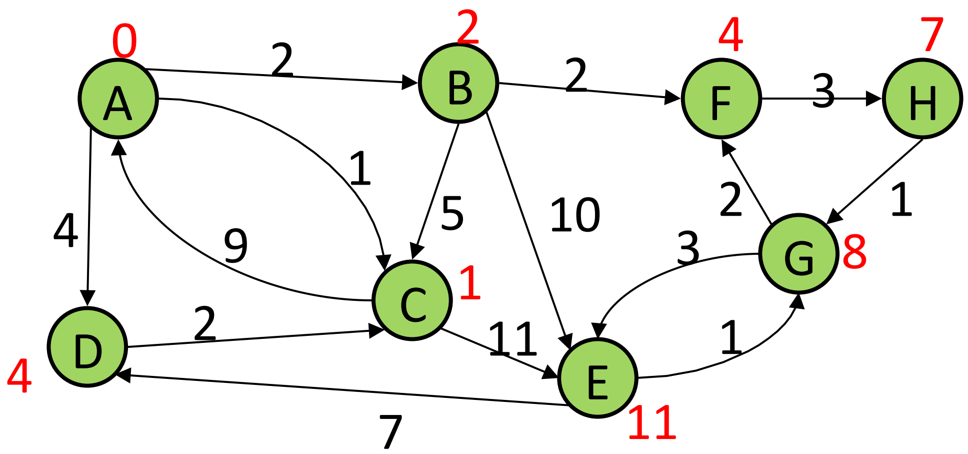
- When a vertex is marked known,
  the cost of the shortest path to that node is known
  - The path is also known by following back-pointers

- While a vertex is still not known,
  another shorter path to it might still be found

Note: The "Order Added to Known Set" is not important
- A detail about how the algorithm works (client doesn't care)
- Not used by the algorithm (implementation doesn't care)
- It is sorted by path-cost, resolving ties in some way
  - Helps give intuition of why the algorithm works

Interpreting the Results

- Now that we're done, how do we get the path from, say, A to E?



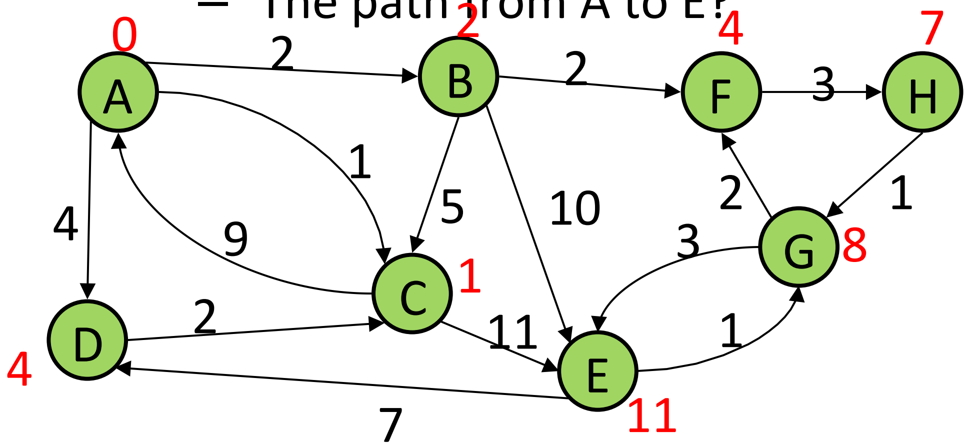
Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Stopping Short

- How would this have worked differently if we were only interested in:
  - The path from A to G?
  - The path from A to E?

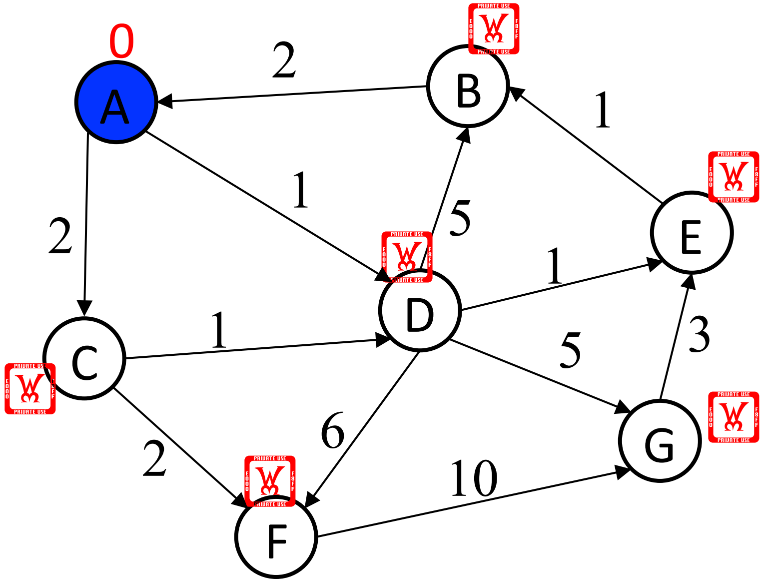


Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

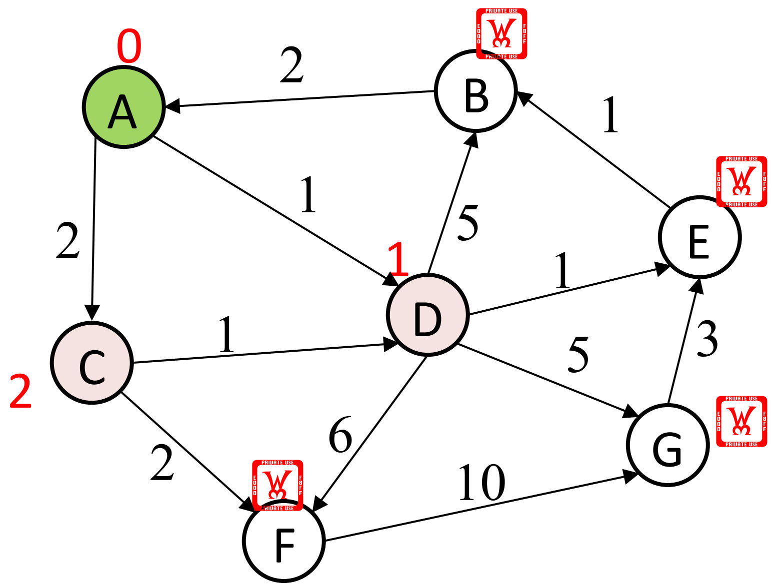
Example #2



Order Added to Known Set:

vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

Example #2

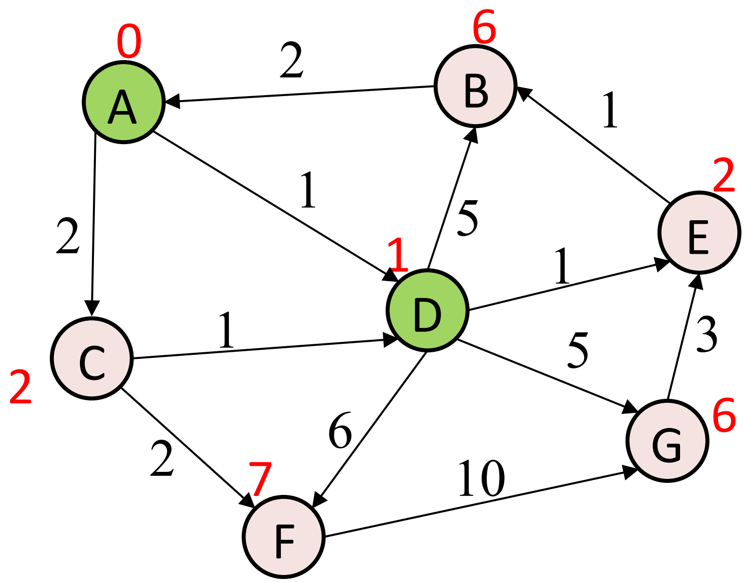


vertex	known?	cost	path
A	Y	0	
B		??	
C		≤ 2	A
D		≤ 1	A
E		??	
F		??	
G		??	

Order Added to Known Set:

A

Example #2

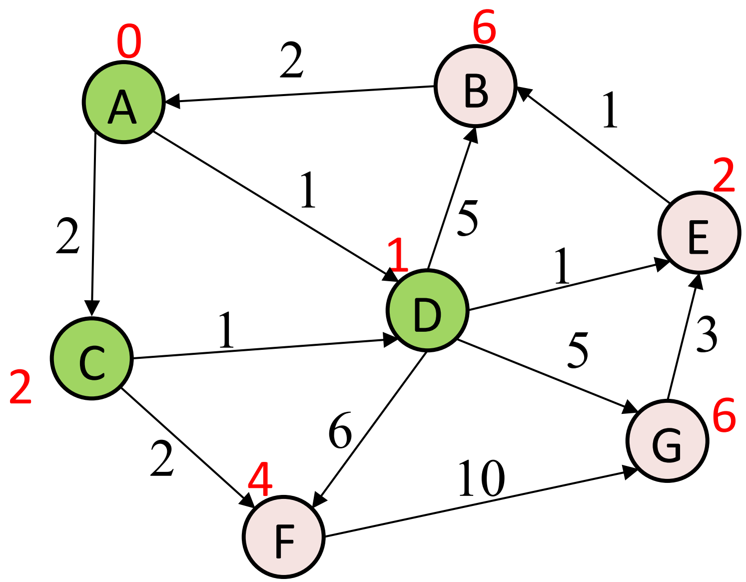


vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C		≤ 2	A
D	Y	1	A
E		≤ 2	D
F		≤ 7	D
G		≤ 6	D

Order Added to Known Set:

A, D

Example #2

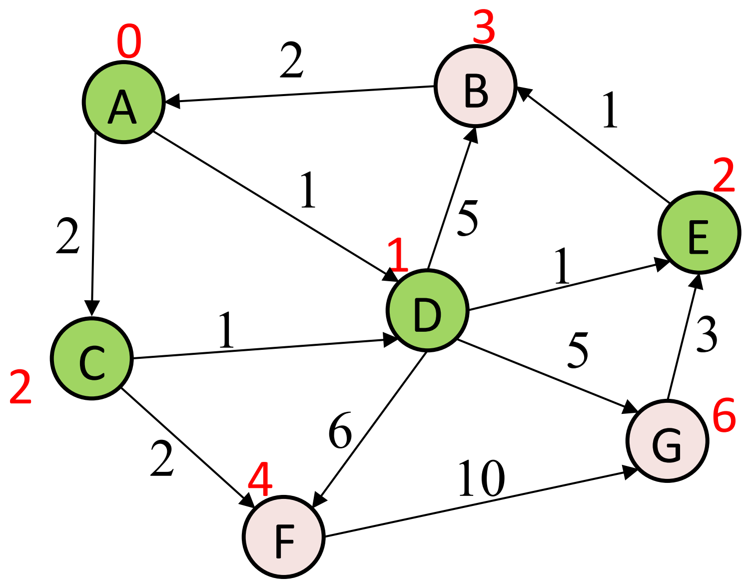


Order Added to Known Set:

A, D, C

vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C	Y	2	A
D	Y	1	A
E		≤ 2	D
F		≤ 4	C
G		≤ 6	D

Example #2

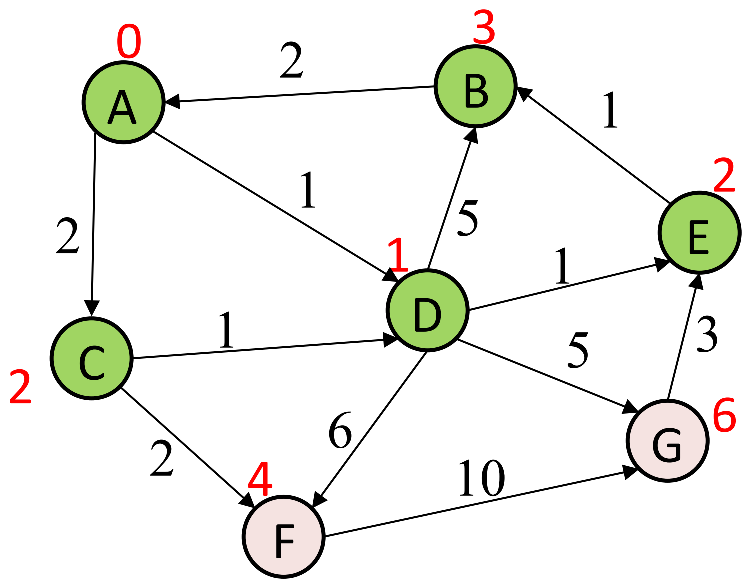


vertex	known?	cost	path
A	Y	0	
B		≤ 3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

<u>Order Added to Known Set:</u>

A, D, C, E

Example #2

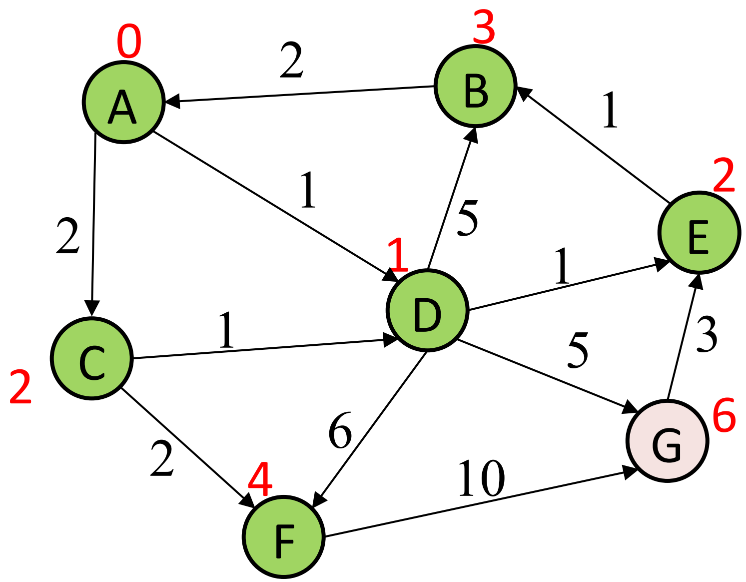


vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Order Added to Known Set:

A, D, C, E, B

Example #2

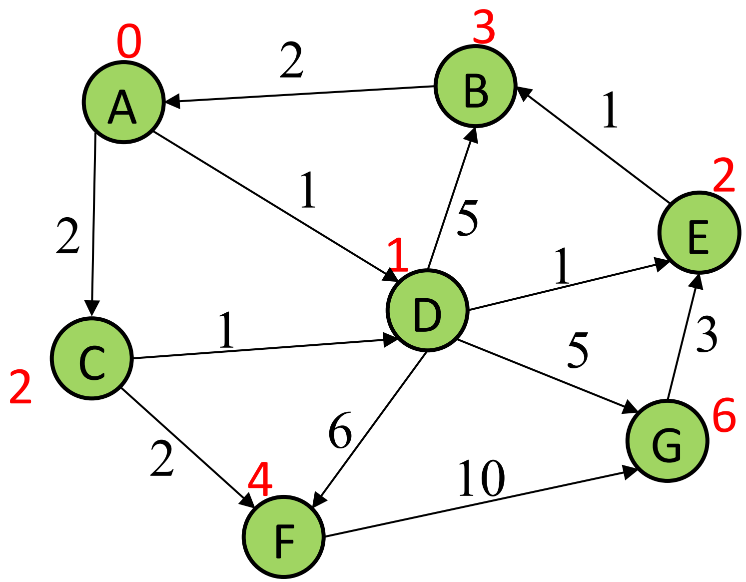


vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G		≤ 6	D

<u>Order Added to Known Set:</u>

A, D, C, E, B, F

Example #2

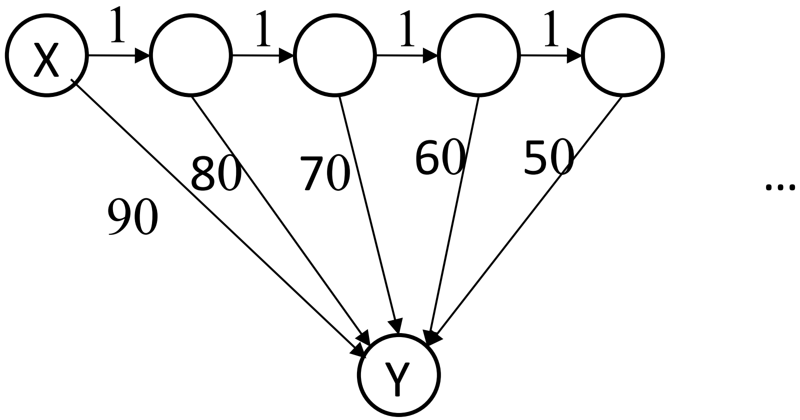


vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G	Y	6	D

Order Added to Known Set:

A, D, C, E, B, F, G

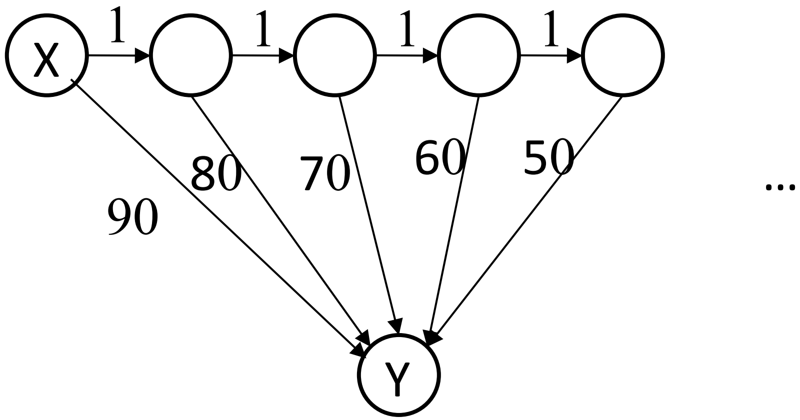
Example #3



How will the best-cost-so-far for Y proceed?

Is this expensive?

Example #3



How will the best-cost-so-far for Y proceed? 90, 81, 72, 63, 54, …

Is this expensive? No, each *edge* is processed only once

A Greedy Algorithm

- Dijkstra's algorithm
  - For single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges

- An example of a *greedy algorithm*:
  - At each step, irrevocably does what seems best at that step
    - A locally optimal step, not necessarily globally optimal
  - Once a vertex is known, it is not revisited
    - Turns out to be globally optimal

Where are We?

- Had a problem: Compute shortest paths in a weighted graph with no negative weights
- Learned an algorithm: Dijkstra's algorithm
- What should we do after learning an algorithm?
  - Prove it is correct
    - Not obvious!
    - We will sketch the key ideas
  - Analyze its efficiency
    - Will do better by using a data structure we learned earlier!

Correctness: Intuition

Rough intuition:

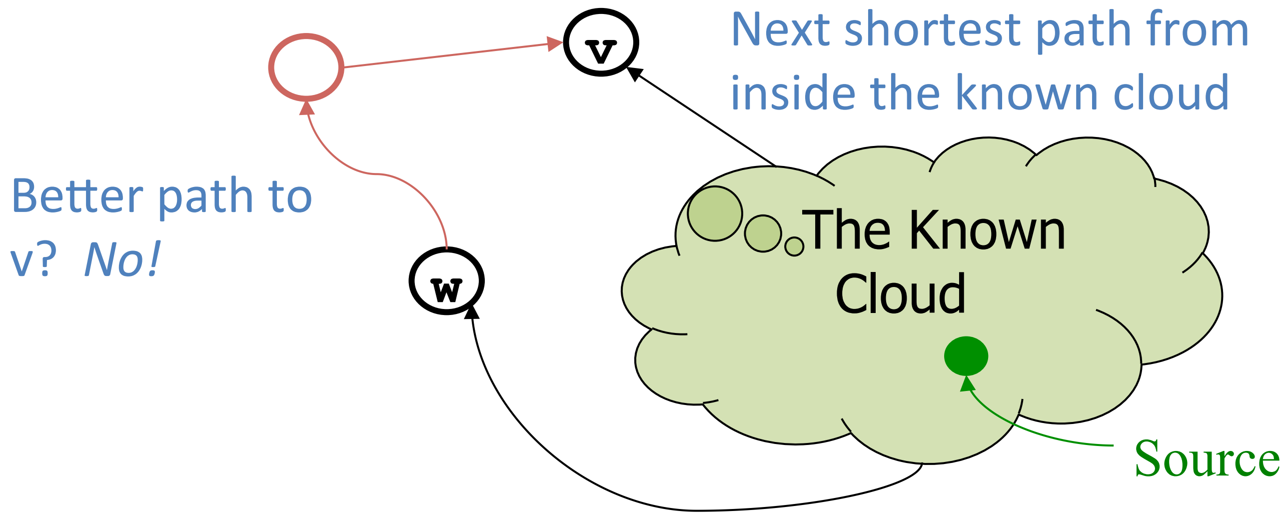
All the "known" vertices have the correct shortest path

- True initially: shortest path to start node has cost 0
- If it stays true every time we mark a node "known", then by induction this holds and eventually everything is "known"

Key fact we need: When we mark a vertex "known" we won't discover a shorter path later!

- This holds only because Dijkstra's algorithm picks the node with the next shortest path-so-far
- The proof is by contradiction…

Correctness: The Cloud (Rough Sketch)



Suppose v is the next node to be marked known ("added to the cloud")

- The best-known path to v must have only nodes "in the cloud"
  - Else we would have picked a node closer to the cloud than v
- Suppose the actual shortest path to v is different
  - It won't use only cloud nodes, or we would know about it
  - So it must use non-cloud nodes. Let w be the *first* non-cloud node on this path. The part of the path up to w is already known and must be shorter than the best-known path to v. So v would not have been picked. Contradiction.

Naïve asymptotic running time

- So far: $O(|V|^2)$

- We had a similar "problem" with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  - We solved it with a queue of zero-degree nodes
  - But here we need the lowest-cost node and costs can change as we process edges

- Solution?

Improving asymptotic running time

- So far: $O(|V|^2)$
- We had a similar "problem" with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  - We solved it with a queue of zero-degree nodes
  - But here we need the lowest-cost node and costs can change as we process edges
- Solution?
  - A priority queue holding all unknown nodes, sorted by cost
  - But must support **`decreaseKey`** operation
    - Must maintain a reference from each node to its current position in the priority queue
    - Conceptually simple, but can be a pain to code up

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  build-heap with all nodes
  while(heap is not empty) {
    b = deleteMin()
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
      if(b.cost + weight((b,a)) < a.cost){
        decreaseKey(a,"new cost – old cost")
        a.path = b
      }
}
```

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

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     if(!a.known)
       if(b.cost + weight((b,a)) < a.cost){
         decreaseKey(a,"new cost – old cost")
         a.path = b
       }
}
```

O(|V|)

O(|V|log|V|)

O(|E|log|V|)

O(|V|log|V|+|E|log|V|)